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RICH DYNAMICS OF A DISCRETE-TIME PREY-PREDATOR MODEL

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Abstract. A newly-disclosed non-standard finite difference method has been used to discretize a prey-predator model to investigate the critical normal form coefficients of bifurcations for both one-parameter and two-parameter bifurcations. The discrete-time prey-predator model exhibits variety of local bifurcations such as period-doubling, Neimark-Sacker, and strong resonances. Critical normal form coefficients are determined to reveal dynamical scenario corresponding to each bifurcation point bifurcation. We also investigates the complex dynamics of the model numerically using by MATLAB package MATCONTM based on numerical continuation technique.

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1. INTRODUCTION

Let us consider the following continues prey-predator model

$$\begin{cases} \frac{dx_{pp}}{d\tau} = ax_{pp} - bx_{pp}y_{pp}, \\ \frac{dy_{pp}}{d\tau} = y_{pp}\left(c - d\,\frac{y_{pp}}{x_{pp}}\right), \end{cases} \tag{1.1}$$

where prey (predator) population at time τ is denoted by x_{pp} (y_{pp}). Furthermore, a represents the intrinsic growth rate of prey, b represents the functional response, c represents the growth rate of predator and d represents the number of prey required to support one predator at equilibrium, see [25].

In ecology, population dynamics are generally determined by both discrete-time and continuous-time dynamics. The study of discrete-time biology systems has received a great deal of attention in recent years, see [1, 3–8, 12, 13, 18]. Discrete models provide more realistic representations when the generation takes place in nonoverlapping spaces, and these models also provide more efficient computational models for numerical simulations when compared to continuous-time models, see [2, 9, 11, 16, 19–22, 24]. In the present paper, we are interested in examining the behavior of discrete-time version obtained from system (1.1). There is an expectation that the discrete-time model will be dynamically compatible with the continuous-time model. Zhang et [23] also point out that if there are no overlapping generations in a biological population, then one must obtain discrete-time systems from a dynamical model of continuous population dynamics. As a result, using a nonstandard scheme, the first one achieves the discrete form of (1.1). Follow these steps to discretize model (1.1). At first the model (1.1) is considered on a finite interval $[0,\Theta]$ and discretize this interval by a uniform mesh as follows

$$0=\tau_0<\tau_1<\cdots<\tau_{N-1}<\tau_N=\Theta,$$

and let $h=\tau_{j+1}-\tau_j$ for $j=1,2,\cdots,N$. Our non-standard finite difference (NSFD) schemes for model (1.1) are based on Mickens' theory as follows:

$$\begin{cases} \frac{x_{pp}^{n+1}-x_{pp}^{n}}{\varphi(h)}=ax_{pp}-bx_{pp}y_{pp}, \\ \frac{y_{pp}^{n+1}-y_{pp}^{n}}{\varphi(h)}=y_{pp}\left(c-d\,\frac{y_{pp}}{x_{pp}}\right), \end{cases} \tag{1.2}$$

where $\varphi(h)=1-e^{-h}$. The model

$$\begin{cases} x_{pp}\mapsto -b\left(1-\mathrm{e}^{-h}\right)x_{pp}y_{pp}+a\left(1-\mathrm{e}^{-h}\right)x_{pp}+x_{pp}, \\ y_{pp}\mapsto -\frac{x_{pp}y_{pp}}{c\left(1-\mathrm{e}^{-h}\right)x_{pp}-d\left(1-\mathrm{e}^{-h}\right)y_{pp}-x_{pp}}, \end{cases} \tag{1.3}$$

is the discrete model obtained from (1.2).

2. Existence and feasibility of fixed points

The system (1.2) can be considered as the following map

$$\begin{pmatrix} x_{pp} \\ y_{pp} \end{pmatrix} \mapsto \mathcal{M}^{pp}(\Psi,\Omega)=\begin{pmatrix} -b\left(1-\mathrm{e}^{-h}\right)x_{pp}y_{pp}+a\left(1-\mathrm{e}^{-h}\right)x_{pp}+x_{pp} \\ -\frac{x_{pp}y_{pp}}{c\left(1-\mathrm{e}^{-h}\right)x_{pp}-d\left(1-\mathrm{e}^{-h}\right)y_{pp}-x_{pp}} \end{pmatrix}, \tag{2.1}$$

where $\Psi=(x_{pp},y_{pp})^T$, $\Omega=(a,b,c,d,h)^T$.

In order to find the fixed points of the map (2.1), we solve the following system

$$\begin{cases} -b\left(1-\mathrm{e}^{-h}\right)x_{pp}y_{pp}+a\left(1-\mathrm{e}^{-h}\right)x_{pp}+x_{pp} &= x_{pp}, \\ -\frac{x_{pp}y_{pp}}{c\left(1-\mathrm{e}^{-h}\right)x_{pp}-d\left(1-\mathrm{e}^{-h}\right)y_{pp}-x_{pp}} &= y_{pp}. \end{cases}$$

The unique positive fixed point

$$\Psi_*^{pp}=\left(\frac{da}{cb},\frac{a}{b}\right),$$

are clearly visible in the map (2.1).

Let us consider

$$\mathcal{M}^{pp}(\Psi,\Omega)=\mathcal{J}_1(\Psi,\Omega)\Psi+\frac{1}{2!}\mathcal{J}_2(\Psi,\Psi)+\frac{1}{3!}\mathcal{J}_3(\Psi,\Psi,\Psi)+O(\|\Psi\|^4),$$

in which

$$\mathcal{J}_1(\Psi,\Omega)=\mathcal{M}_{\Psi}^{pp}(\Psi,\Omega)$$

$$= \begin{pmatrix} by_{pp}\mathrm{e}^{-h} - a\mathrm{e}^{-h} - by_{pp} + a + 1 & b\left(-1+\mathrm{e}^{-h}\right)x_{pp} \\ -\frac{y_{pp}{}^2 d\left(-1+\mathrm{e}^{-h}\right)}{\left(cx_{pp}\mathrm{e}^{-h} - dy_{pp}\mathrm{e}^{-h} - cx_{pp} + dy_{pp} + x_{pp}\right)^2} & \frac{x_{pp}{}^2\left(c\mathrm{e}^{-h} - c + 1\right)}{\left(cx_{pp}\mathrm{e}^{-h} - dy_{pp}\mathrm{e}^{-h} - cx_{pp} + dy_{pp} + x_{pp}\right)^2} \end{pmatrix},$$

$$\mathcal{J}_2(\Psi,\Psi) = \begin{pmatrix} \mathcal{J}_{21}(\Psi,\Psi) \\ \mathcal{J}_{22}(\Psi,\Psi) \end{pmatrix}, \quad \mathcal{J}_3(\Psi,\Psi,\Psi) = \begin{pmatrix} \mathcal{J}_{31}(\Psi,\Psi,\Psi) \\ \mathcal{J}_{32}(\Psi,\Psi,\Psi) \end{pmatrix},$$

and

$$\mathcal{J}_{2i}(\Gamma,\Sigma) = \sum_{j,k=1}^{2} \frac{\partial^2 \mathcal{M}_i^{pp}(\Psi,\Omega)}{\partial \Psi_j \partial \Psi_k} \gamma_j \sigma_k,$$

$$\mathcal{J}_{3i}(\Gamma,\Sigma,\Upsilon) = \sum_{j,k,l=1}^{2} \frac{\partial^3 \mathcal{M}_i^{pp}(\Psi,\Omega)}{\partial \Psi_j \partial \Psi_k \partial \Psi_l} \gamma_j \sigma_k \upsilon_l,$$

where

$$\Gamma = (\gamma_1,\gamma_2)^T,\ \Sigma = (\sigma_1,\sigma_2)^T,\ \Upsilon = (\upsilon_1,\upsilon_2)^T.$$

3. Bifurcations of positive fixed point Ψ_*^{pp}

3.1. *One parameter bifurcations*

The parameter h is considered in this part to be a bifurcation parameter.

Theorem 1. *The critical value*

$$a = a_{PD,*} = -2\,\frac{c\mathrm{e}^{-h} - c + 2}{c\left(\left(\mathrm{e}^{-h}\right)^2 - 2\,\mathrm{e}^{-h} + 1\right)},$$

causes a period-doubling bifurcation of Ψ_*^{pp}.

Proof. For $a = a_{PD,*}$ the matrix

$$\mathcal{J}_1(\Psi_{PD,*}^{pp},\Omega_{PD,*}) = \begin{pmatrix} 1 & -2\,\frac{\left(c\mathrm{e}^{-h} - c + 2\right)d}{\left(-1+\mathrm{e}^{-h}\right)c^2} \\ -\frac{\left(-1+\mathrm{e}^{-h}\right)c^2}{d} & c\mathrm{e}^{-h} - c + 1 \end{pmatrix},$$

$$\Omega_{PD,*} = (a_{PD,*},b,c,d,h)^T, \quad \Psi_{PD,*}^{pp} = \begin{pmatrix} -2\,\frac{\left(c\mathrm{e}^{-h} - c + 2\right)d}{c^2\left(\mathrm{e}^{-2h} - 2\,\mathrm{e}^{-h} + 1\right)b} \\ -2\,\frac{c\mathrm{e}^{-h} - c + 2}{c\left(\mathrm{e}^{-2h} - 2\,\mathrm{e}^{-h} + 1\right)b} \end{pmatrix},$$

has the multipliers

$$\lambda_{PD,*}^1 = -1, \qquad \lambda_{PD,*}^2 = c\mathrm{e}^{-h} - c + 3.$$

In the case where $\lambda_{PD,*}^2 \neq \pm 1$, a period-doubling bifurcation may occur on the curve

$$\mathcal{T}_{PD,*}^{pp} = \left\{(x_{pp},y_{pp},r,h,b,c,d);\, a = a_{PD,*}\right\}.$$

It is possible to consider map $\mathcal{M}^{pp}(\Psi^{pp}_{PD,*},\Omega_{PD,*})$ as follows:

$$\eta_{PD,*} \mapsto -\eta_{PD,*} + \frac{1}{6}\widehat{\beta^{pp}_{PD,*}}\eta^3_{PD,*} + O(\eta^4_{PD,*}).$$

We can derive the normal form coefficient $\widehat{\beta^{pp}_{PD,*}}$ as follows

$$\begin{aligned}\widehat{\beta^{pp}_{PD,*}} = \frac{1}{6}\langle w_{PD,*}, &\mathcal{J}_3(v_{PD,*},v_{PD,*},v_{PD,*})\\ &+3\mathcal{J}_2\Big(v_{PD,*},(I_2-\mathcal{J}_1(\Psi_{PD,*},\Omega_{PD,*}))^{-1}\mathcal{J}_2(v_{PD,*},v_{PD,*})\Big)\Big\rangle,\end{aligned}$$

where

$$\begin{aligned}&\mathcal{J}_1(\Psi_{PD,*},\Omega_{PD,*})v_{PD,*} = v_{PD,*},\\ &\mathcal{J}_1^T(\Psi_{PD,*},\Omega_{PD,*})w_{PD,*} = w_{PD,*}, \quad \langle w_{PD,*},v_{PD,*}\rangle = 1.\end{aligned}$$

As a result, we have

$$v_{PD,*} = \begin{pmatrix}\frac{\left(c\mathrm{e}^{-h}-c+2\right)d}{\left(-1+\mathrm{e}^{-h}\right)c^2}\\ 1\end{pmatrix}, \quad w_{PD,*} = \begin{pmatrix}\frac{\left(-1+\mathrm{e}^{-h}\right)c^2}{\left(c\mathrm{e}^{-h}-c+4\right)d}\\ 2\left(c\mathrm{e}^{-h}-c+4\right)^{-1}\end{pmatrix},$$

and

$$\widehat{\beta^{pp}_{PD,*}} = \frac{b^2\left(\mathrm{e}^{-3h}c-3\,\mathrm{e}^{-2h}c+\mathrm{e}^{-2h}+3\,c\mathrm{e}^{-h}-2\,\mathrm{e}^{-h}-c+1\right)}{c\mathrm{e}^{-h}-c+4}.$$

If $\widehat{\beta^{pp}_{PD,*}} \neq 0$ there is a generic period doubling bifurcation on the curve $\mathcal{T}^{pp}_{PD,*}$. When $\widehat{\beta^{pp}_{PD,*}} > 0$ ($\widehat{\beta^{pp}_{PD,*}} < 0$) the bifurcated period-2 cycles is stable (unstable) and the period doubling bifurcation is supercritical (subcritical). □

Theorem 2. *The critical value* $a = a_{NS,*} = -\left(-1+\mathrm{e}^{-h}\right)^{-1}$ *causes a Neimark-Sacker bifurcation of* Ψ^{pp}_*.

Proof. For $a = a_{NS,*}$ the matrix

$$\mathcal{J}_1(\Psi^{pp}_{NS,*},\Omega_{NS,*}) = \begin{pmatrix}1 & -\frac{d}{c}\\ -\frac{c^2\left(-1+\mathrm{e}^{-h}\right)}{d} & c\,\mathrm{e}^{-h}-c+1\end{pmatrix},$$

$$\Omega_{NS,*} = (a_{NS,*},b,c,d,h)^T, \qquad \Psi^{pp}_{NS,*} = \begin{pmatrix}-\frac{d}{c\left(-1+\mathrm{e}^{-h}\right)b}\\ -\frac{1}{\left(-1+\mathrm{e}^{-h}\right)b}\end{pmatrix},$$

has two complex multipliers

$$\lambda^{1,2}_{NS,*} = e^{\pm i\theta_0} = 1+1/2\,c\,\mathrm{e}^{-h}-c/2 \pm i/2\sqrt{-\left(\mathrm{e}^{-h}\right)^2c^2+2\,\mathrm{e}^{-h}c^2-4\,c\,\mathrm{e}^{-h}-c^2+4\,c}.$$

On the curve,

$$\mathcal{T}_{NS,*}^{pp} = \{(x_{pp}, y_{pp}, a, b, c, d, h); a = a_{NS,*}\}.$$

the Neimark-Sacker bifurcation will occur.

It is possible to write $\mathcal{M}^{pp}(\Psi_{NS,*}, \Omega_{NS,*})$ as follows:

$$\eta_{NS,*} \mapsto e^{i\theta_0}\eta_{NS,*} + \widehat{\delta_{NS,*}^{pp}}\eta_{NS,*}^2\overline{\eta_{NS,*}} + O(|\eta_{NS,*}|^4),$$

The normal form coefficient $\widehat{\delta_{NS,*}^{pp}}$ can be derived as

$$\begin{aligned}\widehat{\delta_{NS,*}^{pp}} = \frac{1}{2}\langle w_{NS,*}, &\mathcal{J}_3(v_{NS,*}, v_{NS,*}, \overline{v_{NS,*}}) \\ &+ 2\mathcal{J}_2\left(v_{NS,*}, (I_2 - \mathcal{J}_1(\Psi_{NS,*}, \Omega_{NS,*}))^{-1}\mathcal{J}_2(v_{NS,*}, \overline{v_{NS,*}})\right) \\ &+ \mathcal{J}_2\left(\overline{v_{NS,*}}, \left(e^{2i\theta_0}I_2 - \mathcal{J}_1(\Psi_{NS,*}, \Omega_{NS,*})\right)^{-1}\mathcal{J}_2(v_{NS,*}, v_{NS,*})\right)\rangle,\end{aligned}$$

where

$$\begin{aligned}&\mathcal{J}_1(\Psi_{NS,*}, \Omega_{NS})v_{NS,*} = e^{i\theta_0}v_{NS,*}, \qquad \mathcal{J}_1(\Psi_{NS,*}, \Omega_{NS,*})\overline{v_{NS,*}} = e^{-i\theta_0}\overline{v_{NS,*}}, \\ &\mathcal{J}_1^T(\Psi_{NS,*}, \Omega_{NS,*})w_{NS,*} = e^{-i\theta_0}w_{NS,*}, \qquad \mathcal{J}_1^T(\Psi_{NS,*}, \Omega_{NS,*})\overline{w_{NS,*}} = e^{i\theta_0}\overline{w_{NS,*}}, \\ &\langle w_{NS,*}, v_{NS,*}\rangle = 1,\end{aligned}$$

and

$$v_{NS,*} = \begin{pmatrix} -2\,\dfrac{d}{c\left(c\mathrm{e}^{-h} + i\sqrt{c\left(2c\mathrm{e}^{-h} - \mathrm{e}^{-2h}c - 4\mathrm{e}^{-h} - c + 4\right)} - c\right)} \\ 1 \end{pmatrix},$$

$$w_{NS,*} = \frac{1}{\overline{wv}}\begin{pmatrix} 2\,\dfrac{c^2\left(-1 + \mathrm{e}^{-h}\right)}{d\left(-c\mathrm{e}^{-h} + i\sqrt{c\left(2c\mathrm{e}^{-h} - \mathrm{e}^{-2h}c - 4\mathrm{e}^{-h} - c + 4\right)} + c\right)} \\ 1 \end{pmatrix},$$

where $wv =$

$$= 2\,\frac{c\left(\left(i\mathrm{e}^{-h} - i\right)\sqrt{-\left((-2c+4)\mathrm{e}^{-h} + \mathrm{e}^{-2h}c + c - 4\right)c} + (-2c+4)\mathrm{e}^{-h} + \mathrm{e}^{-2h}c + c - 4\right)}{\left(i\sqrt{-\left((-2c+4)\mathrm{e}^{-h} + \mathrm{e}^{-2h}c + c - 4\right)c} + c\left(-1 + \mathrm{e}^{-h}\right)\right)^2}.$$

If $\widehat{\delta_{NS,*}^{pp}} \neq 0$ there is a generic Neimark-Sacker on the curve $\mathcal{T}_{NS,*}^{pp}$. When

$$\widehat{\sigma_{NS,*}^{pp}} = \Re\left(e^{-i\theta_0}\widehat{\delta_{NS,*}^{pp}}\right) \neq 0,$$

a unique closed invariant curve for $\mathcal{M}^{pp}(\Psi_{NS,*}, \Omega_{NS,*})$ appears around the $\Psi_{NS,*}^{pp}$, when a crosses $a_{NS,*}$. The sign of $\widehat{\sigma_{NS,*}^{pp}}$ indicates the stability of the closed invariant curve. □

3.2. *Two parameters bifurcations*

Each bifurcation on the bifurcation curves mentioned in Section 3.1 is examined within a two-parameter space (a,h).

Theorem 3. *The critical values $a = a_{R_2,*} = c/4$ and $h = h_{R_2,*} = -\ln\left(\frac{-4+c}{c}\right)$ causes a strong resonance 1:2 bifurcation Ψ^{pp}_{*}*

Proof. For $r = r_{R_2,*}$ and $h = h_{R_2,*}$ the matrix

$$\mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*}) = \begin{pmatrix} 1 & -\frac{d}{c} \\ 4\,\frac{c}{d} & -3 \end{pmatrix},$$

$$\Psi^{pp}_{R_2,*} = \begin{pmatrix} 1/4\,\frac{d}{b} \\ 1/4\,\frac{c}{b} \end{pmatrix}, \quad \Omega_{R_2,*} = (a_{R_2,*},b,c,d,h_{R_2,*}),$$

has two repeated multipliers $\lambda^{1,2}_{R_2,*} = -1$. On the curve

$$\mathcal{T}^{pp}_{R_2,*} = \{(x_{pp},y_{pp},a,b,c,d,h), a = a_{R_2,*},\, h = h_{R_2,*}\}.$$

the resonance 1:2 bifurcation will occur.

The map $\mathcal{M}^{pp}(\Psi_{R_2,*},\Omega_{R_2,*})$ can be written as

$$\begin{pmatrix} \eta_{R_2,*} \\ \zeta_{R_2,*} \end{pmatrix} \mapsto \begin{pmatrix} -\eta_{R_2,*} + \zeta_{R_2,*} \\ \zeta_{R_2,*} + \widehat{\sigma^{pp}_{R_2,*}}\eta^3_{R_2,*} + \widehat{\delta^{pp}_{R_2,*}}\eta^2_{R_2,*}\zeta_{R_2,*} \end{pmatrix}.$$

The normal form coefficients $\widehat{\sigma^{pp}_{R_2,*}}$ and $\widehat{\delta^{pp}_{R_2,*}}$ can be derived as

$$\begin{aligned}
\widehat{\sigma^{pp}_{R_2,*}} &= \frac{1}{6}\langle w^0_{R_2}, \mathcal{J}_3(v^0_{R_2,*},v^0_{R_2,*},v^0_{R_2,*}) \\
&\quad + 3\mathcal{J}_2\left(v^0_{R_2,*},\left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})\right)^{-1}\right)\mathcal{J}_2(v^0_{R_2,*},v^0_{R_2,*})\rangle, \\
\widehat{\delta^{pp}_{R_2,*}} &= \frac{1}{2}\langle w^0_{R_2,*}, \mathcal{J}_3(v^0_{R_2,*},v^0_{R_2,*},v^1_{R_2,*}) + 2\mathcal{J}_2(v^0_{R_2,*},h^{11}_{R_2,*}) + \mathcal{J}_2(v^1_{R_2,*},h^{20}_{R_2,*})\rangle \\
&\quad + \frac{1}{2}\langle w^1_{R_2,*}, \mathcal{J}_3(v^0_{R_2,*},v^0_{R_2,*},v^0_{R_2,*}) + 2\mathcal{J}_2(v^0_{R_2,*},h^{20}_{R_2,*})\rangle, \\
h^{20}_{R_2,*} &= \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})\right)^{-1}\left(\mathcal{J}_2(v^0_{R_2,*},v^0_{R_2,*})\right), \\
h^{11}_{R_2,*} &= \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})\right)^{-1}\left(\mathcal{J}_2(v^0_{R_2,*},v^1_{R_2}) + h^{20}_{R_2,*}\right).
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})v^0_{R_2,*} = -v^0_{R_2,*}, && \mathcal{J}_1^T(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})w^1_{R_2,*} = -w^1_{R_2,*} + w^0_{R_2,*}, \\
&\mathcal{J}_1^T(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})w^0_{R_2,*} = -w^0_{R_2,*}, && \left\langle w^0_{R_2,*},v^1_{R_2,*}\right\rangle = \left\langle w^1_{R_2,*},v^0_{R_2,*}\right\rangle = 1, \\
&\mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})v^1_{R_2,*} = -v^1_{R_2,*} + v^0_{R_2,*}, && \left\langle w^0_{R_2,*},v^0_{R_2,*}\right\rangle = \left\langle w^1_{R_2,*},v^1_{R_2,*}\right\rangle = 0.
\end{aligned}$$

As a result, we have

$$v^0_{R_2,*} = \begin{pmatrix} 1/2\,\frac{d}{c} \\ 1 \end{pmatrix}, \qquad v^1_{R_2,*} = \begin{pmatrix} 1/4\,\frac{d^3}{c\,(9\,c^2+d^2)} \\ 3/4\,\frac{d^2}{9\,c^2+d^2} \end{pmatrix},$$

$$w^0_{R_2,*} = \begin{pmatrix} -8\,\frac{c\,(9\,c^2+d^2)}{d^3} \\ 4\,\frac{9\,c^2+d^2}{d^2} \end{pmatrix}, \qquad w^1_{R_2,*} = \begin{pmatrix} 6\,\frac{c}{d} \\ -2 \end{pmatrix}.$$

Consequently, we get

$$\widehat{\sigma^{pp}_{R_2,*}} = -96\,\frac{(9\,c^2+d^2)\,b^2}{d^2c^2}, \quad \widehat{\delta^{pp}_{R_2,*}} = -16\,\frac{b^2\,(360\,c^2+29\,d^2)}{d^2c^2}.$$

If $\widehat{\sigma^{pp}_{R_2,*}} \neq 0$ and $-2\widehat{\delta^{pp}_{R_2,*}} \neq -6\widehat{\sigma^{pp}_{R_2,*}}$ there is a generic resonance 1:2 bifurcation on the curve $\mathcal{T}^{pp}_{R_2,*}$. □

Theorem 4. *The critical values $a = a_{R_3,*} = c/3$ and $h = h_{R_3,*} = -\ln\left(\frac{-3+c}{c}\right)$ causes a strong resonance 1:3 bifurcation Ψ^{pp}_*.*

Proof. For $r = r_{R_3,*}$ and $h = h_{R_3,*}$ the matrix

$$\mathcal{J}_1(\Psi^{pp}_{R_3,*},\Omega_{R_3,*}) = \begin{pmatrix} 1 & -\frac{d}{c} \\ 3\,\frac{c}{d} & -2 \end{pmatrix},$$

$$\Psi^{pp}_{R_3,*} = \begin{pmatrix} 1/3\,\frac{d}{b} \\ 1/3\,\frac{c}{b} \end{pmatrix}, \quad \Omega_{R_3,*} = (a_{R_3,*},b,c,d,h_{R_3,*})^T,$$

has multipliers $\lambda^{1,2}_{R_3,*} = \cos(\frac{2\pi}{3}) \pm i\,\sin(\frac{2\pi}{3})$. The resonance 1:3 bifurcation will occur on the curve

$$\mathcal{T}^{pp}_{R_3,*} = \{(x_{pp},y_{pp},a,b,c,d,h), a = a_{R_3,*}, h = h_{R_3,*}\}.$$

The map $\mathcal{M}^{pp}(\Psi^{pp}_{R_3,*},\Omega_{R_3,*})$ can be written as

$$\eta_{R_3,*} \mapsto \left(\cos(\frac{2\pi}{3}) \pm i\,\sin(\frac{2\pi}{3})\right)\eta_{R_3,*} + \widehat{\beta^{pp}_{R_3,*}}\eta^2_{R_3,*}\overline{\eta_{R_3,*}} + \widehat{\sigma^{pp}_{R_3,*}}\overline{\eta_{R_3,*}}^3 + O(|\eta_{R_3,*}|^4).$$

The normal form coefficients $\widehat{\beta^{pp}_{R_3,*}}$ and $\widehat{\sigma^{pp}_{R_3,*}}$ can be derived as

$$\widehat{\beta^{pp}_{R_3,*}} = \frac{1}{2}\langle w_{R_3,*}, \mathcal{J}_2(\overline{v_{R_3,*}},\overline{v_{R_3,*}})\rangle,$$

$$\widehat{\sigma^{pp}_{R_3,*}} = \frac{1}{2}\langle w_{R_3,*}, \mathcal{J}_3(v_{R_3,*},v_{R_3,*},\overline{v_{R_3,*}}) + 2\mathcal{J}_2(v_{R_3,*},h^{11}_{R_3,*}) - \mathcal{J}_2\left(\overline{v_{R_3,*}},h^{20}_{R_3,*}\right)\rangle,$$

$$h^{11}_{R_3,*} = \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_3,*},\Omega_{R_3,*})\right)^{-1}\mathcal{J}_2(v_{R_3,*},\overline{v_{R_3,*}}),$$

$$h^{20}_{R_3,*} = \left(e^{\frac{4\pi}{3}i}I_2 - \mathcal{J}_1(\Psi^{pp}_{R_3,*},\Omega_{R_3,*})\right)^{-INV}\left(\overline{\widehat{\beta^{pp}_{R_3,*}}}\overline{v_{R_3,*}} - \mathcal{J}_2(v_{R_3,*},v_{R_3,*})\right),$$

where

$$\mathcal{J}_1(\Psi^{pp}_{R_3,*},\Omega_{R_3,*})v_{R_3,*} = e^{\frac{2\pi}{3}i}v_{R_3,*}, \qquad \mathcal{J}_1(\Psi^{pp}_{R_3,*},\Omega_{R_3,*})\overline{v_{R_3,*}} = e^{\frac{-2\pi}{3}i}\overline{v_{R_3,*}},$$
$$\mathcal{J}_1^T(\Psi^{pp}_{R_3,*},\Omega_{R_3,*})w_{R_3,*} = e^{\frac{-2\pi}{3}i}w_{R_3,*}, \qquad \mathcal{J}_1^T(\Psi^{pp}_{R_3,*},\Omega_{R_3,*})\overline{w_{R_3,*}} = e^{\frac{2\pi}{3}i}\overline{w_{R_3,*}},$$
$$\langle w_{R_3,*}, v_{R_3,*}\rangle = 1.$$

As a result, we have

$$v_{R_3,*} = \begin{pmatrix} -2\,\frac{d}{c\,(i\sqrt{3}-3)} \\ 1 \end{pmatrix}, \quad w_{R_3,*} = \begin{pmatrix} -12\,\frac{c}{(i\sqrt{3}+1)d\,(i\sqrt{3}+3)} \\ 2\,\left(i\sqrt{3}+1\right)^{-1} \end{pmatrix}.$$

Consequently, we get

$$\widehat{\beta^{pp}_{R_3,*}} = 3/2\,\frac{b\,\left(3\,i\sqrt{3}-1\right)}{c}, \quad \widehat{\sigma^{pp}_{R_3,*}} = -21\,\frac{b^2\left(2\,i\sqrt{3}+3\right)}{c^2}.$$

There is a generic resonance 1:3 bifurcation on the curve $\mathcal{T}^{pp}_{R_3,*}$, provided that $\widehat{\beta^{pp}_{R_3,*}} \neq 0$ and $\widehat{\sigma^{pp}_{R_3,*}} \neq 0$. □

Theorem 5. *The critical values $a = a_{R_4,*} = c/2$ and $h = h_{R_4,*} = -\ln\left(\frac{-2+c}{c}\right)$ causes a strong resonance 1:4 bifurcation Ψ^{pp}_**

Proof. For $a = a_{R_4,*}$ and $h = h_{r_4,*}$ the matrix

$$\mathcal{J}_1(\Psi^{pp}_{R_4,*},\Omega_{R_4,*}) = \begin{pmatrix} 1 & -\frac{d}{c} \\ 2\,\frac{c}{d} & -1 \end{pmatrix},$$
$$\Psi^{pp}_{R_4,*} = \begin{pmatrix} 1/2\,\frac{d}{b} \\ 1/2\,\frac{c}{b} \end{pmatrix}, \quad \Omega_{R_4,*} = (a_{R_4,*}, b.c.d, h_{R_4,*})^T,$$

has multipliers $\lambda^{1,2}_{R_4,*} = \cos(\frac{\pi}{2}) \pm i\,\sin(\frac{\pi}{2})$. The resonance 1:4 bifurcation will occur on the curve

$$\mathcal{T}^{pp}_{R_4,*} = \left\{(x_{pp}, y_{pp}, a, b, c, d, h),\, a = a_{R_4,*},\, h = h_{R_4,*}\right\}.$$

The map $\mathcal{M}^{pp}(\Psi^{pp}_{R_4,*},\Omega_{R_4,*})$ can be written as

$$\eta_{R_4,*} \mapsto \left(\cos(\frac{\pi}{2}) + i\,\sin(\frac{\pi}{2})\right)\eta_{R_4,*} + \widehat{\sigma^{pp}_{R_4,*}}\eta^2_{R_4,*}\overline{\eta_{R_4,*}} + \widehat{\delta^{pp}_{R_4,*}}\overline{\eta_{R_4,*}}^3 + O(|\eta_{R_4,*}|^4).$$

The normal form coefficients $\widehat{\sigma^{pp}_{R_4,*}}$ and $\widehat{\delta^{pp}_{R_4,*}}$ can be derived as

$$\widehat{\sigma^{pp}_{R_4,*}} = \frac{1}{2}\langle w_{R_4,*}, \mathcal{J}_3(v_{R_4,*}, v_{R_4,*}, \overline{v_{R_4,*}}) + 2\mathcal{J}_2\left(v_{R_4,*}, h^{11}_{R_4,*}\right) - \mathcal{J}_2\left(\overline{v_{R_4,*}}, h^{20}_{R_4,*}\right)\rangle,$$
$$\widehat{\sigma^{pp}_{R_4,*}} = \frac{1}{6}\langle w_{R_4,*}, \mathcal{J}_3(\overline{v_{R_4,*}}, \overline{v_{R_4,*}}, \overline{v_{R_4,*}}) - 3\mathcal{J}_2\left(\overline{v_{R_4,*}}, h^{02}_{R_4,*}\right)\rangle,$$
$$h^{11}_{R_4,*} = \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_4,*},\Omega_{R_4,*})\right)^{-1}\mathcal{J}_2\left(v_{R_4,*}, \overline{v_{R_4,*}}\right),$$

$$h^{20}_{R_4,*} = \left(I_2 + \mathcal{J}_1(\Psi^{pp}_{R_4,*},\Omega_{R_4,*})\right)^{-1} \mathcal{J}_2\left(v_{R_4,*}, v_{R_4,*}\right),$$
$$h^{02}_{R_4,*} = \left(I_2 + \mathcal{J}_1(\Psi^{pp}_{R_4,*},\Omega_{R_4,*})\right)^{-1} \mathcal{J}_2(\overline{v_{R_4,*}}, \overline{v_{R_4,*}}).$$

where

$$\mathcal{J}_1(\Psi^{pp}_{R_4,*},\Omega_{R_4,*})v_{R_4,*} = e^{\frac{\pi}{2}i}v_{R_4,*}, \qquad \mathcal{J}_1(\Psi^{pp}_{R_4,*},\Omega_{R_4,*})\overline{v_{R_4,*}} = e^{\frac{-\pi}{2}i}\overline{v_{R_4,*}},$$
$$\mathcal{J}_1^T(\Psi^{pp}_{R_4,*},\Omega_{R_4,*})w_{R_4,*} = e^{\frac{-\pi}{2}i}w_{R_4,*}, \qquad \mathcal{J}_1^T(\Psi^{pp}_{R_4,*},\Omega_{R_4,*})\overline{w_{R_4,*}} = e^{\frac{\pi}{2}i}\overline{w_{R_4,*}},$$
$$\langle w_{R_4,*}, v_{R_4,*}\rangle = 1.$$

As a result, we have

$$v_{R_4,*} = \begin{pmatrix} \frac{(1/2+i/2)d}{c} \\ 1 \end{pmatrix}, \quad w_{R_4,*} = \begin{pmatrix} \frac{ic}{d} \\ 1/2 - i/2 \end{pmatrix}.$$

Consequently, we get

$$\widehat{\sigma^{pp}_{R_4,*}} = \frac{(-24-14\,i)\,b^2}{c^2}, \quad \widehat{\delta^{pp}_{R_4,*}} = \frac{(6-8\,i)\,b^2}{c^2}.$$

There is a generic resonance 1:4 bifurcation on the curve $\mathcal{T}^{pp}_{R_4,*}$ provided that $\widehat{\sigma^{pp}_{R_4,*}} \neq 0$ and $\widehat{\delta^{pp}_{R_4,*}} \neq 0$.

The bifurcation scenario in the neighbourhood of the curve $\mathcal{T}^{pp}_{R_4,*}$ is determined by the coefficient $\widehat{A^{pp}_{R_4,*}} = -\frac{i\widehat{\sigma^{pp}_{R_4,*}}}{|\widehat{\delta^{pp}_{R_4,*}}|}$, provided that $\widehat{\delta^{pp}_{R_4,*}} \neq 0$. If $|\widehat{A^{pp}_{R_4,*}}| > 1$, we conclude there are two fold curves of cycles with four times the original period, see [15, 17]. □

4. NUMERICAL BIFURCATION ANALYSIS

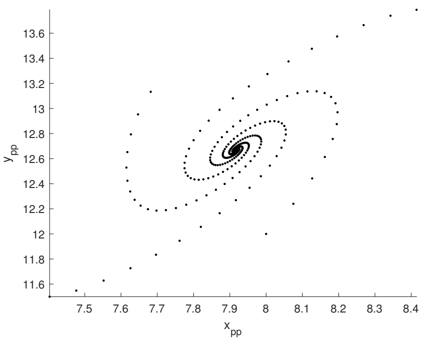
To confirm the obtained results in Section 3 and investigate further complex behaviour of $\mathcal{M}^{pp}(\Psi,\Omega)$, we use MATCONTM which is a MATLAB interactive toolbox for the numerical study of iterated discrete dynamical systems, [10, 14, 17].

In this case we consider fixed parameters $b = 0.3$, $c = 8$, $d = 5$ and $h = 0.3$ as a free parameter. By varying the free parameter a, the continuation method produces the following one parameter bifurcations:

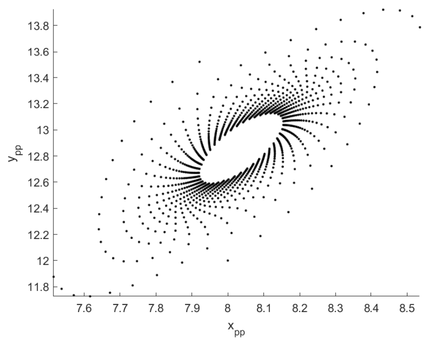
i) A Neimark-Sacker bifurcation (NS) occurs at the point $\Psi^{pp}_{NS,*} = (8.038116, 12.860986)$ for $a = a_{NS,*} = 3.858296$ with $\widehat{\sigma^{pp}_{NS,*}} = -2.225899 \times 10^{-2}$.

ii) A period-doubling bifurcation (PD) occurs at the point $\Psi^{pp}_{PD,*} = (0.569517, 0.911227)$ for $a = a_{PD,*} = 0.273368$ with $\widehat{\beta^{pp}_{PD,*}} = -3.366997 \times 10^{-3}$.

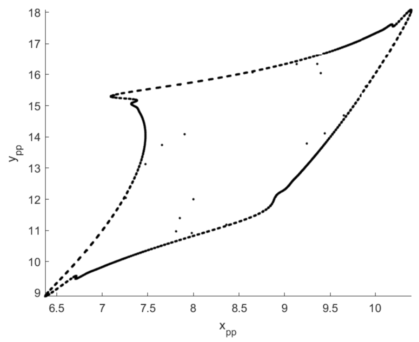
Since $\widehat{\sigma^{pp}_{NS,*}} < 0$ the bifurcation is supercritical and the bifurcated closed invariant is stable, see Fig. 6. The numerical simulation shown in these graphs indicates the dynamical behavior of the map $\mathcal{M}^{pp}(\Psi,\Omega)$ near $\Psi^{pp}_{NS,*}$.



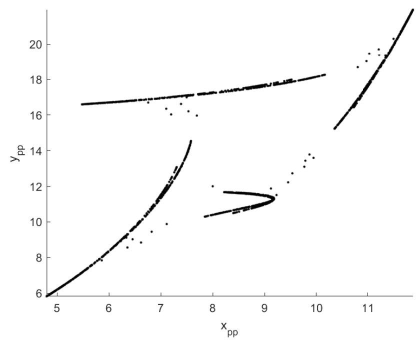
(A) A stable fixed point for $a = a_{NS,*} = 0.3.80$.



(B) A stable closed invariant for $a = a_{NS,*} = 3.8583$.



(C) The broken invariant closed curve after $\Psi^{pp}_{NS,*}$ for $a = a_{NS,*} = 4.2$.



(D) The chaotic attractor for $a = a_{NS,*} = 4.5$.

FIGURE 1. Behavior of the map $\mathcal{M}^{pp}(\Psi,\Omega)$ near the NS point.

Remark 1. Fig. 2 illustrates the maximum Lyapunov exponent for $a \succeq 4.45$, indicating that chaos exists. Positive Lyapunov exponents are generally considered to be a sign of chaos.

Since $\widehat{\beta^{pp}_{PD,*}} < 0$ the bifurcation is sub-critical and the bifurcated period-2 cycles is unstable.

In Fig. 3, we can see the stability region for the positive fixed point Ψ^{pp}_{*}.

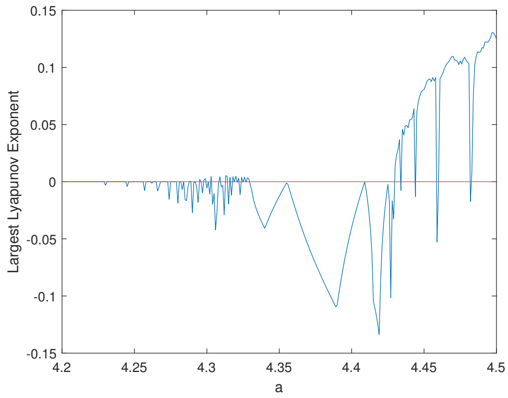


FIGURE 2. The maximum Lyapunov exponent corresponding to Fig. 6.

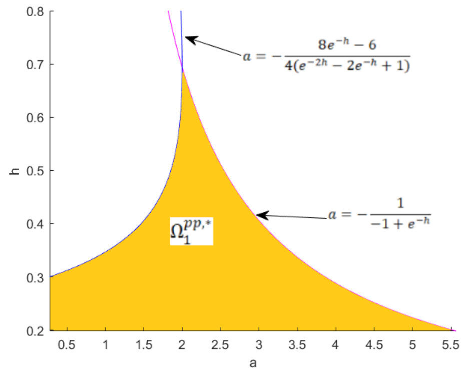


FIGURE 3. The stability region for the positive fixed point Ψ_*^{pp} in space (a,h).

The following two-parameter bifurcations can be obtained with the selected $\Psi_{NS,*}^{pp}$ and continuation with two free parameters, r, and h:

i) A resonance 1:3 bifurcation (R3) occurs at the point $\Psi_{R_3,*}^{pp} = (5.555624, 8.888998)$ for $a = a_{R_3,*} = 2.666699$ and $h = h_{R_3,*} = 0.0.470006$ with $\Re\left(\frac{1}{3}\left(e^{\frac{4\pi}{3}i}\widehat{\sigma_{R_3,*}^{pp}}/|\widehat{\beta_{R_3,*}^{pp}}|^2-1\right)\right) = -4.999943\times 10^{-1}$.

ii) A resonance 1:4 bifurcation (R4) occurs at the point $\Psi_{R_4,*}^{pp} = (8.333333, 13.333333)$ for $a = a_{R_4,*} = 4.000000$ and $h = h_{R_4,*} = 0.287682$ with $\widehat{A_{R_4,*}^{pp}} = -1.600000 + 6.000000\times 10^{-1}\,i$.

iii) A resonance 1:2 bifurcation (R2) occurs at the point $\Psi^{pp}_{R_2,*} = (4.166667, 6.666667)$ for $a = a_{R_2,*} = 2.000000$ and $h = h_{R_2,*}$ $= 0.0.693147$ with $\widehat{\sigma^{pp}_{R_2,*}} = 2.459786 \times 10^1$ and $\widehat{\delta^{pp}_{R_2,*}} = -1.639858$.

The following two-parameter bifurcation can be obtained with the selected PD point and continuation with two free parameters, r and h:

i) A resonance 1:2 bifurcation (R2) occurs at the point $\Psi^{pp}_{R_2,*} = (4.166667, 6.666667)$ for $a = a_{R_2,*} = 2.000000$ and $h = h_{R_2,*}$ $= 0.0.693147$ with $\widehat{\sigma^{pp}_{R_2,*}} = 2.459786 \times 10^1$ and $\widehat{\delta^{pp}_{R_2,*}} = -1.639858$.

Fig. 4 illustrates these results.

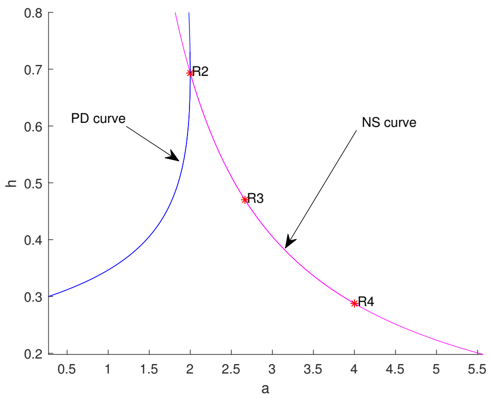


FIGURE 4. The Neimark-Sacker and period doubling bifurcations curves of $\mathcal{M}^{pp}(\Psi, \Omega)$ in the (r, h) space.

According to Theorem 5, since $|\widehat{A^{pp}_{R_4,*}}| > 1$, there are two fold curves of cycles with period 4, see Fig. 5.

For $a = 5.51841$ and $h = 0.240474$, $C^{pp}_4 = \{C^{pp}_{1,4}, C^{pp}_{2,4}, C^{pp}_{3,4}, C^{pp}_{4,4}\}$ gives a stable four-cycle where $C^{pp}_{1,4} = (8.846315, 22.650752)$, see Fig. 6a. In Fig. 6b we can see the stability region for C^{pp}_4.

5. DISCUSSION AND ECOLOGICAL IMPLICATIONS

A discrete-time prey-predator model using non-standard finite difference discretization method is examined in terms of its complex dynamics. We demonstrate that model (1.3) has unique interior fixed point (positive) Ψ^{pp}_*. Ψ^{pp}_* may bifurcate in many ways, as shown in Section (3). In Section (4), the curves of fixed points and one-parameter bifurcations of cycles up to the fourth order are computed. The analytical predictions and numerical observations obtained in Sections (3) and (4) are

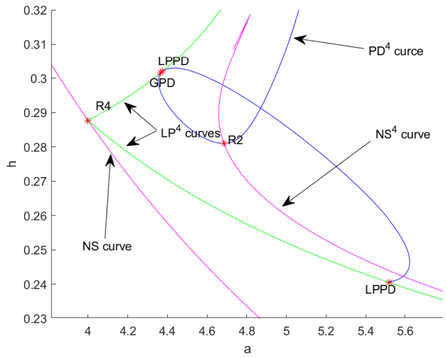
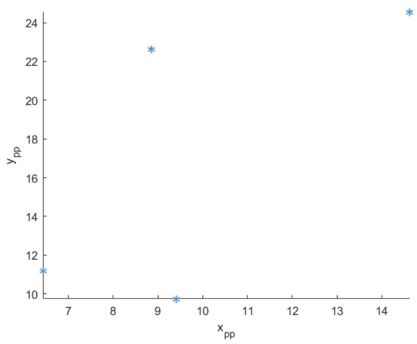
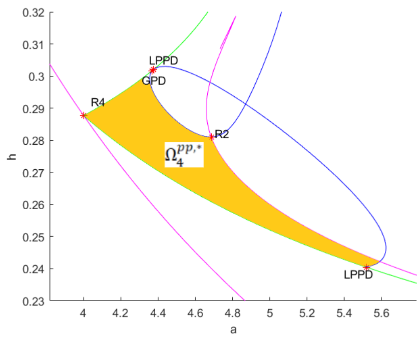


FIGURE 5. Two fold (LP) curves of the fourth iterate emanate from $\Psi^{pp}_{R_4,*}$ of $\mathcal{M}^{pp}(\Psi,\Omega)$ and the Period doubling and Neimark-Sacker bifurcations curves of the fourth iterate of $\mathcal{M}^{pp}(\Psi,\Omega)$ in the (r,h) space.



(A) Stable four-cycle C_4^{pp}.



(B) Stability region for C_4^{pp}.

FIGURE 6. The four-cycle with its stability region.

in excellent agreement. The complicated dynamics of model (1.3) are depicted via MatcontM and continuation technique. Period-doubling, Neimark-Sacker, and strong resonance bifurcation of Ψ^{pp}_* have been demonstrated.

A Neimark-Sacker bifurcation, implies that both prey and predator populations can oscillate around some mean values of intrinsic growth rate a and that these oscillations will continue as long as $\widehat{\sigma^{pp}_{NS,*}} < 0$ is constant. In ecology, an invariant closed curve is bifurcated, which means both predator and prey can live together and produce their own densities. A periodic or quasi-periodic dynamics may be present on the invariant curve. A period-doubling bifurcation is shown in the model, indicating that the prey and predator populations change over time. In this model, there are strong resonances bifurcating around some mean values of intrinsic growth rate and step size h when predator and prey coexist. If certain conditions are met, this coexistence is definitely possible.

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