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NEW GENERALIZATIONS OF SOME IMPORTANT INEQUALITIES FOR SARıKAYA FRACTIONAL INTEGRALS

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Abstract. In this research paper, we investigate some new identifies for Sarıkaya fractional integrals which introduced by Sarıkaya and Ertuğral in [20]. The fractional integral operators also have been applied to Hermite-Hadamard type integral inequalities to provide their generalized properties. Furthermore, as special cases of our main results, we present several known inequalities such as Simpson, Bullen, trapezoid for convex functions.

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1. INTRODUCTION

The convex functions for inequalities are introduced firstly by C. Hermite and J. Hadamard. Let $\mathcal{F} : I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $\sigma, \rho \in I$ with $\sigma < \rho$. Then, the following double inequality

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\varkappa)d\varkappa \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2} \tag{1.1}$$

is valid for all convex functions, which is known in the literature as the Hermite-Hadamard inequality. If $\mathcal{F}$ is concave, then both inequalities in (1.1) hold to the reverse direction. With the help of the inequality (1.1), many researchers have considered the Hermite-Hadamard inequality and related inequalities such as trapezoid, midpoint, and Simpson's inequality.

Over the years, considerable number of studies have been focused on obtaining trapezoid and midpoint type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1.1), respectively. Dragomir and Agarwal first introduced trapezoid inequalities for convex functions in [5]. Kirmacı first established midpoint inequalities for convex functions in the paper [14]. In addition to these, Hwang and Tseng prove some new Hermite-Hadamard type inequalities for fractional integrals and introduce several applications for the beta function in [11] and some new Fejer-type inequalities for convex functions are proved in the paper

[23]. Moreover, Hwang et al. establish some Hermite-Hadamard type, Bullen-type, and Simpson-type inequalities for fractional integrals and some applications for the beta function are also given in the paper [10]. Some fractional trapezoid and midpoint type inequalities for convex functions are proved in [22] and [12], respectively. Some generalized midpoint type inequalities for Riemann-Liouville fractional integrals are given in [1] and [4].

The results of Simpson-type for convex functions have been studied extensively by many mathematicians. To be more precise, some inequalities of Simpson's type for s-convex functions are investigated by using differentiable functions [15]. By using the differentiable convex function, the new variants of Simpson's type inequalities are proved in the papers [16,21]. For further information about Simpson type inequalities for various convex classes, we refer the reader to Refs. [2,3,6,7,9,13,17–19] and the references therein.

Now, we introduce some definitions and notations which are used frequently in throughout main section.

Definition 1. [20] Let us consider that a function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the condition $\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < \infty$. Then, the following left-sided and right-sided Sarıkaya fractional integral operators are described as

$$_{\sigma+}I_{\varphi}F(\varkappa) = \int_{\sigma}^{\varkappa} \frac{\varphi(\varkappa-\tau)}{\varkappa-\tau} F(\tau)d\tau, \quad \varkappa > \sigma \tag{1.2}$$

and

$$_{\rho-}I_{\varphi}F(\varkappa) = \int_{\varkappa}^{\rho} \frac{\varphi(\tau-\varkappa)}{\tau-\varkappa} F(\tau)d\tau, \quad \varkappa < \rho, \tag{1.3}$$

respectively.

The most significant feature of Sarıkaya fractional integrals is that they generalize some important types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Hadamard fractional integrals, conformable fractional integral, Katugampola fractional integrals etc. These important special cases of the integral operators (1.2) and (1.3) are given as follows:

i. Let us consider $\varphi(\tau) = \tau$. Then, the operators (1.2) and (1.3) reduce to the Riemann integral.
ii. If we assign $\varphi(\tau) = \frac{\tau^{\alpha}}{\Gamma(\alpha)}$ and $\alpha > 0$, then the operators (1.2) and (1.3) reduce to the Riemann-Liouville fractional integrals $J_{\sigma+}^{\alpha}F(\varkappa)$ and $J_{\rho-}^{\alpha}F(\varkappa)$, respectively. Here, Γ is Gamma function defined by the integral formula

$$\Gamma(\varkappa) := \int_0^{\infty} \tau^{\varkappa-1} e^{-\tau} d\tau, \; \varkappa \in \mathbb{R}^{+}.$$

iii. Suppose $\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ and $\alpha, k > 0$. The operators (1.2) and (1.3) reduce to the k-Riemann-Liouville fractional integrals $J^{\alpha}_{\sigma+,k}F(\varkappa)$ and $J^{\alpha}_{\rho-,k}F(\varkappa)$, respectively. Here, Γ_k is k-Gamma function defined by

$$\Gamma_k(\alpha) = \int_0^{\infty} \tau^{\alpha-1} e^{-\frac{\tau^k}{k}} d\tau, \qquad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \qquad \mathcal{R}(\alpha) > 0; k > 0.$$

2. Some identities for Sarikaya fractional integrals

Throughout this paper for brevity, we define

$$\Delta(\varkappa) = \int_{\sigma}^{\varkappa} \frac{\varphi(\tau-\sigma)}{\tau-\sigma} d\tau \qquad \text{and} \qquad \Lambda(\varkappa) = \int_{\varkappa}^{\rho} \frac{\varphi(\rho-\tau)}{\rho-\tau} d\tau.$$

Lemma 1. *Let $F : [\sigma,\rho] \to \mathbb{R}$ be an absolutely continuous mapping (σ,ρ) such that $F' \in L_1([\sigma,\rho])$. Then, the following equality*

$$\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma)\right] - \frac{F(c)+F(d)}{2} = \frac{1}{2\Lambda(\sigma)}\int_{\sigma}^{\rho} \Psi(c,d;\varkappa)F'(\varkappa)d\varkappa$$

is valid. Here,

$$\Psi(c,d;\varkappa) = \begin{cases} \Lambda(\varkappa) - \Delta(\varkappa) - \Lambda(\sigma), & \varkappa \in [\sigma,c], \\ \Lambda(\varkappa) - \Delta(\varkappa), & \varkappa \in (c,d), \\ \Lambda(\varkappa) - \Delta(\varkappa) + \Lambda(\sigma), & \varkappa \in [d,\rho]. \end{cases}$$

Proof. By using integration by parts, we obtain

$$\begin{aligned} \int_{\sigma}^{\rho} \Psi(c,d;\varkappa)F'(\varkappa)d\varkappa &= \int_{\sigma}^{\rho} [\Lambda(\varkappa) - \Delta(\varkappa)]F'(\varkappa)d\varkappa \\ &\quad - \Lambda(\sigma)\int_{\sigma}^{c} F'(\varkappa)d\varkappa + \Lambda(\sigma)\int_{d}^{\rho} F'(\varkappa)d\varkappa \\ &= [\Lambda(\varkappa) - \Delta(\varkappa)]F(\varkappa)|_{\sigma}^{\rho} - \int_{\sigma}^{\rho}\left[-\frac{\varphi(\rho-\varkappa)}{\rho-\varkappa} - \frac{\varphi(\varkappa-\sigma)}{\varkappa-\sigma}\right]F(\varkappa)d\varkappa \\ &\quad - \Lambda(\sigma)[F(c) - F(\sigma)] + \Lambda(\sigma)[F(\rho) - F(d)] \\ &= -\Delta(\rho)F(\rho) - \Lambda(\sigma)F(\sigma) + \left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma)\right] \\ &\quad - \Lambda(\sigma)[F(c) - F(\sigma)] + \Lambda(\sigma)[F(\rho) - F(d)] \\ &= \left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma)\right] - \Lambda(\sigma)[F(c) + F(d)]. \end{aligned}$$

This ends the proof of Lemma 1. □

Corollary 1. *If we choose $\varphi(\tau)=\tau$ in Lemma 1, then the following equality holds:*

$$\frac{1}{(\rho-\sigma)}\int_{\sigma}^{\rho}F(\tau)\,d\tau-\frac{F(c)+F(d)}{2}=\frac{1}{2(\rho-\sigma)}\left[2\int_{\sigma}^{c}(\sigma-\varkappa)F'(\varkappa)\,d\varkappa+\int_{c}^{d}(\sigma+\rho-2\varkappa)F'(\varkappa)\,d\varkappa+2\int_{d}^{\rho}(\rho-\varkappa)F'(\varkappa)\,d\varkappa\right].$$

Corollary 2. *Let us consider $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Lemma 1. Then, the following equality*

$$\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}F(\rho)+J_{\rho-}^{\alpha}F(\sigma)\right]-\frac{F(c)+F(d)}{2}=\frac{1}{2(\rho-\sigma)^{\alpha}}\int_{\sigma}^{\rho}\Psi_1(\alpha,\varkappa)F'(\varkappa)\,d\varkappa$$

is valid. Here,

$$\Psi_1(\alpha,\varkappa)=\begin{cases}(\rho-\varkappa)^{\alpha}-(\varkappa-\sigma)^{\alpha}-(\rho-\sigma)^{\alpha}, & \varkappa\in[\sigma,c],\\(\rho-\varkappa)^{\alpha}-(\varkappa-\sigma)^{\alpha}, & \varkappa\in(c,d),\\(\rho-\varkappa)^{\alpha}-(\varkappa-\sigma)^{\alpha}+(\rho-\sigma)^{\alpha}, & \varkappa\in[d,\rho].\end{cases}$$

Corollary 3. *If we assign $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ in Lemma 1, then the following inequality holds:*

$$\frac{\Gamma_k(\alpha+k)}{2(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J_{\sigma+,k}^{\alpha}F(\rho)+J_{\rho-,k}^{\alpha}F(\sigma)\right]-\frac{F(c)+F(d)}{2}=\frac{1}{2(\rho-\sigma)^{\frac{\alpha}{k}}}\int_{\sigma}^{\rho}\Psi_2(\alpha,k;\varkappa)F'(\varkappa)\,d\varkappa,$$

where

$$\Psi_2(\alpha,k;\varkappa)=\begin{cases}(\rho-\varkappa)^{\frac{\alpha}{k}}-(\varkappa-\sigma)^{\frac{\alpha}{k}}-(\rho-\sigma)^{\frac{\alpha}{k}}, & \varkappa\in[\sigma,c],\\(\rho-\varkappa)^{\frac{\alpha}{k}}-(\varkappa-\sigma)^{\frac{\alpha}{k}}, & \varkappa\in(c,d),\\(\rho-\varkappa)^{\frac{\alpha}{k}}-(\varkappa-\sigma)^{\frac{\alpha}{k}}+(\rho-\sigma)^{\frac{\alpha}{k}}, & \varkappa\in[d,\rho].\end{cases}$$

Lemma 2. *Suppose $F:[\sigma,\rho]\to\mathbb{R}$ is an absolutely continuous mapping (σ,ρ) such that $F'\in L_1([\sigma,\rho])$. Then, the following equality*

$$\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}=\frac{1}{2\Lambda(\sigma)}\int_{\sigma}^{\rho}\Phi(\varkappa)F'(\varkappa)\,d\varkappa$$

is valid. Here, $\Omega(\alpha,\beta) = (1-\beta)^{\alpha} - \beta^{\alpha}$ *with* $\alpha > 0$, $0 \le \beta \le \frac{1}{2}$ *and*

$$\Phi(\varkappa) = \begin{cases} \Lambda(\varkappa) - \Delta(\varkappa) - \Omega(\alpha,\beta)\Lambda(\sigma), & \varkappa \in \left[\sigma, \frac{\sigma+\rho}{2}\right), \\ \Lambda(\varkappa) - \Delta(\varkappa) + \Omega(\alpha,\beta)\Lambda(\sigma), & \varkappa \in \left[\frac{\sigma+\rho}{2}, \rho\right). \end{cases}$$

Proof. With the help of the integration by parts, we get

$$\begin{aligned}
&\int_{\sigma}^{\rho} \Phi(\varkappa) F'(\varkappa)\, d\varkappa \\
&= \int_{\sigma}^{\rho} [\Lambda(\varkappa) - \Delta(\varkappa)] F'(\varkappa)\, d\varkappa \\
&\quad - \left((1-\beta)^{\alpha} - \beta^{\alpha}\right)\Lambda(\sigma) \int_{\sigma}^{\frac{\sigma+\rho}{2}} F'(\varkappa)\, d\varkappa + \left((1-\beta)^{\alpha} - \beta^{\alpha}\right)\Lambda(\sigma) \int_{\frac{\sigma+\rho}{2}}^{\rho} F'(\varkappa)\, d\varkappa \\
&= [\Lambda(\varkappa) - \Delta(\varkappa)] F(\varkappa)\big|_{\sigma}^{\rho} - \int_{\sigma}^{\rho} \left[-\frac{\varphi(\rho-\varkappa)}{\rho-\varkappa} - \frac{\varphi(\varkappa-\sigma)}{\varkappa-\sigma}\right] F(\varkappa)\, d\varkappa \\
&\qquad - \left((1-\beta)^{\alpha} - \beta^{\alpha}\right)\Lambda(\sigma)\left[F\left(\frac{\sigma+\rho}{2}\right) - F(\sigma)\right] \\
&\qquad + \left((1-\beta)^{\alpha} - \beta^{\alpha}\right)\Lambda(\sigma)\left[F(\rho) - F\left(\frac{\sigma+\rho}{2}\right)\right] \\
&= -\Delta(\rho)F(\rho) - \Lambda(\sigma)F(\sigma) + \left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma)\right] \\
&\qquad + \left((1-\beta)^{\alpha} - \beta^{\alpha}\right)\Lambda(\sigma)\left[F(\sigma) - 2F\left(\frac{\sigma+\rho}{2}\right) + F(\rho)\right] \\
&= \left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma)\right] - 2\Omega(\alpha,\beta)\Lambda(\sigma)F\left(\frac{\sigma+\rho}{2}\right) \\
&\qquad - (1-\Omega(\alpha,\beta))\Lambda(\sigma)\left(F(\sigma) + F(\rho)\right).
\end{aligned}$$

This ends the proof of Lemma 2. □

Corollary 4. *If we choose* $\varphi(\tau) = \tau$ *in Lemma 2, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{(\rho-\sigma)} \int_{\sigma}^{\rho} F(\tau)\, d\tau - \Omega(\alpha,\beta) F\left(\frac{\sigma+\rho}{2}\right) - (1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2} \\
&\quad = \frac{1}{2(\rho-\sigma)} \int_{\sigma}^{\rho} \Phi_1(\varkappa) F'(\varkappa)\, d\varkappa.
\end{aligned}$$

Here,

$$\Phi_1(\varkappa)=\begin{cases}\sigma+\rho-2\varkappa-\Omega(\alpha,\beta)(\rho-\sigma), & \varkappa\in\left[\sigma,\frac{\sigma+\rho}{2}\right),\\ \sigma+\rho-2\varkappa+\Omega(\alpha,\beta)(\rho-\sigma), & \varkappa\in\left[\frac{\sigma+\rho}{2},\rho\right).\end{cases}$$

Corollary 5. *Let us consider* $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ *in Lemma 2. Then, we obtain*

$$\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}F(\rho)+J^{\alpha}_{\rho-}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)$$
$$-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}=\frac{1}{2(\rho-\sigma)^{\alpha}}\int_{\sigma}^{\rho}\Phi_2(\alpha,\varkappa)F'(\varkappa)\,d\varkappa,$$

where

$$\Phi_2(\alpha,\varkappa)=\begin{cases}(\rho-\varkappa)^{\alpha}-(\varkappa-\sigma)^{\alpha}-\Omega(\alpha,\beta)(\rho-\sigma)^{\alpha}, & \varkappa\in\left[\sigma,\frac{\sigma+\rho}{2}\right),\\ \Lambda(\varkappa)-\Delta(\varkappa)+\Omega(\alpha,\beta)\Lambda(\sigma), & \varkappa\in\left[\frac{\sigma+\rho}{2},\rho\right).\end{cases}$$

3. New inequalities for Sarikaya fractional integrals

In this Section we present some new inequalities for convex function by using Sarıkaya fractional integrals.

Theorem 1. *Let us consider that the assumptions of Lemma 1 are valid. Let us also consider that the mapping* $|F'|$ *is convex on* $[\sigma,\rho]$. *Then, we have the following inequality*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(c)+F(d)}{2}\right|\le\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\omega_1(c,d)|F'(\sigma)|+\omega_2(c,d)|F'(\rho)|}{2}\right],\tag{3.1}$$

where ω_1 *and* ω_2 *are defined by*

$$\omega_1(c,d)=\int_{\sigma}^{\rho}|\Psi(c,d;\varkappa)|(\rho-\varkappa)\,d\varkappa \text{ and } \omega_2(c,d)=\int_{\sigma}^{\rho}|\Psi(c,d;\varkappa)|(\varkappa-\sigma)\,d\varkappa.$$

Proof. By taking modulus in Lemma 1, we get

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(c)+F(d)}{2}\right|\le\frac{1}{2\Lambda(\sigma)}\int_{\sigma}^{\rho}|\Psi(c,d;\varkappa)|\left|F'(\varkappa)\right|d\varkappa.$$

By using convexity of $|F'|$, we obtain

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(c)+F(d)}{2}\right|$$

$$\leq \frac{1}{2\Lambda(\sigma)} \int_{\sigma}^{\rho} |\Psi(c,d;\varkappa)| \left[\frac{\rho-\varkappa}{\rho-\sigma} |F'(\sigma)| + \frac{\varkappa-\sigma}{\rho-\sigma} |F'(\rho)| \right] d\varkappa$$

$$= \frac{1}{2\Lambda(\sigma)(\rho-\sigma)} \left[|F'(\sigma)| \int_{\sigma}^{\rho} |\Psi(c,d;\varkappa)| (\rho-\varkappa) d\varkappa \right.$$

$$\left. + |F'(\rho)| \int_{\sigma}^{\rho} |\Psi(c,d;\varkappa)| (\varkappa-\sigma) d\varkappa \right].$$

This finishes the proof of Theorem 1. □

Corollary 6. *In Theorem 1, suppose $c = (1-\beta)\sigma + \beta\rho$ and $d = \beta\sigma + (1-\beta)\rho$ with $0 \leq \beta \leq \frac{1}{2}$. Then, the inequality (3.1) reduces to the inequality*

$$\left| \frac{1}{2\Lambda(\sigma)} \left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma) \right] - \frac{F((1-\beta)\sigma+\beta\rho) + F(\beta\sigma+(1-\beta)\rho)}{2} \right|$$

$$\leq \frac{1}{\Lambda(\sigma)(\rho-\sigma)} \left[\frac{\omega_1((1-\beta)\sigma+\beta\rho, \beta\sigma+(1-\beta)\rho)|F'(\sigma)|}{2} \right.$$

$$\left. + \frac{\omega_2((1-\beta)\sigma+\beta\rho, \beta\sigma+(1-\beta)\rho)|F'(\rho)|}{2} \right],$$

where ω_1 and ω_2 are defined as in Theorem 1.

Corollary 7. *In Corollary 6, let us consider $\beta = 0$. Then, we have*

$$\left| \frac{1}{2\Lambda(\sigma)} \left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma) \right] - \frac{F(\sigma)+F(\rho)}{2} \right|$$

$$\leq \frac{1}{\Lambda(\sigma)(\rho-\sigma)} \left[\frac{\omega_1(\sigma,\rho)|F'(\sigma)| + \omega_2(\sigma,\rho)|F'(\rho)|}{2} \right],$$

where

$$\begin{cases} \omega_1(\sigma,\rho) = \int_{\sigma}^{\rho} |\Psi(\sigma,\rho;\varkappa)| (\rho-\varkappa) d\varkappa = \int_{\sigma}^{\rho} |\Lambda(\varkappa) - \Delta(\varkappa)| (\rho-\varkappa) d\varkappa, \\ \omega_2(\sigma,\rho) = \int_{\sigma}^{\rho} |\Psi(\sigma,\rho;\varkappa)| (\varkappa-\sigma) d\varkappa = \int_{\sigma}^{\rho} |\Lambda(\varkappa) - \Delta(\varkappa)| (\varkappa-\sigma) d\varkappa. \end{cases}$$

Remark 1. If we choose $\varphi(\tau) = \tau$ in Corollary 7, then Corollary 7 reduces to [5, Theorem 2.2].

Remark 2. Let us consider $\varphi(\tau) = \frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 7. Then, Corollary 7 reduces to [22, Theorem 3].

Corollary 8. *Let us note $\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ in Corollary 7. Corollary 7 reduces to [8, Theorem 2.4].*

Corollary 9. *In Corollary 6, let $\beta=\frac{1}{2}$. Then, we obtain*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-F\left(\frac{\sigma+\rho}{2}\right)\right|$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\omega_1\left(\frac{\sigma+\rho}{2},\frac{\sigma+\rho}{2}\right)|F'(\sigma)|+\omega_2\left(\frac{\sigma+\rho}{2},\frac{\sigma+\rho}{2}\right)|F'(\rho)|}{2}\right].$$

Remark 3. If we assign $\varphi(\tau)=\tau$ in Corollary 9, then Corollary 9 reduces to [14, Theorem 2.2].

Remark 4. Consider $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 9. Then, Corollary 9 reduces to [10, Theorem 2.1].

Corollary 10. *For $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ in Corollary 9, we have*

$$\left|\frac{\Gamma_k(\alpha+k)}{2(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\sigma+,k}F(\rho)+J^{\alpha}_{\rho-,k}F(\sigma)\right]-F\left(\frac{\sigma+\rho}{2}\right)\right| \tag{3.2}$$
$$\leq\frac{(\rho-\sigma)}{4\left(\frac{\alpha+k}{k}\right)}\left(\frac{\alpha}{k}-1+\frac{1}{2^{\left(\frac{\alpha-k}{k}\right)}}\right)\left(|F'(\sigma)|+|F'(\rho)|\right).$$

Corollary 11. *In Corollary 6, let us consider $\beta=\frac{1}{4}$. Then, we get*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{1}{2}\left[F\left(\frac{3\sigma+\rho}{4}\right)+F\left(\frac{\sigma+3\rho}{4}\right)\right]\right|$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\omega_1\left(\frac{3\sigma+\rho}{4},\frac{\sigma+3\rho}{4}\right)|F'(\sigma)|+\omega_2\left(\frac{3\sigma+\rho}{4},\frac{\sigma+3\rho}{4}\right)|F'(\rho)|}{2}\right].$$

Corollary 12. *If we choose $\varphi(\tau)=\tau$ in Corollary 11, then we have*

$$\left|\frac{1}{(\rho-\sigma)}\int_{\sigma}^{\rho}F(\varkappa)\,d\varkappa-\frac{1}{2}\left[F\left(\frac{3\sigma+\rho}{4}\right)+F\left(\frac{\sigma+3\rho}{4}\right)\right]\right|$$
$$\leq\frac{(\rho-\sigma)}{16}\left(|F'(\sigma)|+|F'(\rho)|\right).$$

Remark 5. Let us consider $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 11. Then, Corollary 11 reduces to [11, Theorem G].

Corollary 13. *Suppose $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ in Corollary 11. Then, we have*

$$\left|\frac{\Gamma_k(\alpha+k)}{2(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\sigma+,k}F(\rho)+J^{\alpha}_{\rho-,k}F(\sigma)\right]-\frac{1}{2}\left[F\left(\frac{3\sigma+\rho}{4}\right)+F\left(\frac{\sigma+3\rho}{4}\right)\right]\right|$$

$$\leq (\rho-\sigma)\left[\frac{1}{8}+\frac{3^{\left(\frac{\alpha+k}{k}\right)}-2^{\left(\frac{\alpha+k}{k}\right)}+1}{4^{\left(\frac{\alpha+k}{k}\right)}\left(\frac{\alpha+k}{k}\right)}-\frac{1}{2\left(\frac{\alpha+k}{k}\right)}\right]\left(\left|F'(\sigma)\right|+\left|F'(\rho)\right|\right).$$

Theorem 2. *Let us consider that the assumptions of Lemma 2 are valid. Let us also consider that the mapping $|F'|$ is convex on $[\sigma,\rho]$. Then, we get the following inequality*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)\right.$$
$$\left.-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}\right|$$
$$\leq \frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\Upsilon_1(\alpha,\beta;\varkappa)|F'(\sigma)|+\Upsilon_2(\alpha,\beta;\varkappa)|F'(\rho)|}{2}\right],$$

where $\Upsilon_1(\alpha,\beta;\varkappa)$ and $\Upsilon_2(\alpha,\beta;\varkappa)$ are defined by

$$\begin{cases}\Upsilon_1(\alpha,\beta;\varkappa)=\int_{\sigma}^{\rho}|\Phi(\varkappa)|(\rho-\varkappa)\,d\varkappa,\\ \Upsilon_2(\alpha,\beta;\varkappa)=\int_{\sigma}^{\rho}|\Phi(\varkappa)|(\varkappa-\sigma)\,d\varkappa.\end{cases}$$

Proof. By taking modulus in Lemma 2, we obtain

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)\right.$$
$$\left.-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}\right|$$
$$\leq \frac{1}{2\Lambda(\sigma)}\int_{\sigma}^{\rho}|\Phi(\varkappa)|\left|F'(\varkappa)\right|d\varkappa.$$

By using convexity of $|F'|$, we have

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)\right.$$
$$\left.-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}\right|$$
$$\leq \frac{1}{2\Lambda(\sigma)}\int_{\sigma}^{\rho}|\Phi(\varkappa)|\left[\frac{\rho-\varkappa}{\rho-\sigma}\left|F'(\sigma)\right|+\frac{\varkappa-\sigma}{\rho-\sigma}\left|F'(\rho)\right|\right]d\varkappa$$
$$=\frac{1}{2\Lambda(\sigma)(\rho-\sigma)}\left[\left|F'(\sigma)\right|\int_{\sigma}^{\rho}|\Phi(\varkappa)|(\rho-\varkappa)\,d\varkappa\right.$$

$$+\left|F'(\rho)\right|\int_{\sigma}^{\rho}|\Phi(\varkappa)|(\varkappa-\sigma)\,d\varkappa\Bigg].$$

This completes the proof of Theorem 2. □

Corollary 14. *In Theorem 2, let us note that* $\beta=0$. *Then, we have*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-F\left(\frac{\sigma+\rho}{2}\right)\right|$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\Upsilon_1(\alpha,0;\varkappa)\left|F'(\sigma)\right|+\Upsilon_2(\alpha,0;\varkappa)\left|F'(\rho)\right|}{2}\right].$$

Remark 6. Let us consider that $\varphi(\tau)=\tau$ in Corollary 14, Then, Corollary 14 reduces to [14, Theorem 2.2].

Remark 7. If we select $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 14, then Corollary 14 reduces to [10, Theorem 2.1].

Remark 8. Let us now note that $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ in Corollary 14. Then, Corollary 14 reduces to inequality (3.2).

Corollary 15. *In Theorem 2, let us choose* $\beta=\frac{1}{2}$. *Then, we obtain*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(\sigma)+F(\rho)}{2}\right|$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\Upsilon_1\left(\alpha,\frac{1}{2};\varkappa\right)\left|F'(\sigma)\right|+\Upsilon_2\left(\alpha,\frac{1}{2};\varkappa\right)\left|F'(\rho)\right|}{2}\right].$$

Remark 9. For $\varphi(\tau)=\tau$ in Corollary 15, Corollary 15 reduces to [5, Theorem 2.2].

Remark 10. If we take $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 15, then Corollary 15 reduces to [22, Theorem 3].

Corollary 16. *Let us consider that* $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ *in Corollary 15, Then, Corollary 15 reduces to [8, Theorem 2.4].*

4. Conclusion

In this paper, we establish some new refinements of Hermite-Hadamard inequalities for convex functions by using special choices of parameters. Furthermore, we prove that our results generalize the inequalities obtained by Hwang and Tseng [11]. In the future work of the authors, generalization or improvement of our results can be examined by using different kind of convex function classes or other type fractional integral operators.

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