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ON COMMON FIXED POINT RESULTS VIA IMPLICIT RELATIONS

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Abstract. There are certain contractive conditions (and contractions) available in the literature that ensure the existence of common fixed points of a couple and a family of mappings. However, to verify the validity of these conditions, each result must be checked separately. Thus, it becomes legitimate to obtain some contraction or contractive conditions that can bypass the computational difficulties of checking contraction and contractive conditions via individual results and ensure the existence of common fixed points simultaneously. In this article, by the virtue of implicit relations, we acquire some contraction and contractive conditions, christened A-contraction and $\mathcal{A}$-contractive conditions, which serve the desired purpose. The utility of the established conditions is exhibited through some typical fixed point results and concrete examples.

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1. Introduction

Throughout the last century, *metric fixed point theory* plays an important role in *nonlinear analysis* due to its simplicity and applicability in different fields. The main aim of this theory is to derive some adequate conditions on a mapping so that we can get the guaranty of existence of a (sometimes unique) fixed point of the mapping. The first one among these adequate conditions is the *contraction* condition which was taken into consideration by Banach [2] in 1922. After this, the contraction condition of Banach have been extended in a variety of ways and as a consequence, a lot of adequate conditions for existence of fixed points of a mapping have been established. Among these adequate conditions, the contraction conditions of Kannan [8], Chatterjea [4], Reich [12], Ćirić [5] are remarkable. In order to verify the validity of these contraction conditions, we need to prove and remember separate results for each contraction conditions. So one may naturally ask whether any contraction condition can be derived which can accommodate all of the above mentioned contractions. Fortunately, the answer is *positive* due to the introduction of A-contraction condition by Akram et al. [1]. The main result is given below:

Theorem 1 (cf. [1, p. 29, Theorem 5])**.** *Let a self map T on a complete metric space (X,d) satisfies the condition:*

$$d(Tx,Ty) \leq g(d(x,y),d(x,Tx),d(y,Ty))$$

for all $x,y \in X$ and some $g \in A$, where A is the collection of all functions $g\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ ($\mathbb{R}_+$ is the set of all non-negative real numbers) satisfying

(i) *g is continuous on the set $\mathbb{R}_+^3$ (with respect to the Euclidean metric on $\mathbb{R}^3$);*
(ii) *$u \leq kv$ for some $k \in [0,1)$ whenever $u \leq g(u,v,v)$ or $u \leq g(v,u,v)$ or $u \leq g(v,v,u)$ for all $u,v \in \mathbb{R}_+$.*

Then, T has a unique fixed point.

The mapping satisfying the contraction condition of this result is called an A-contraction mapping. It is to be noted that the contraction conditions of Banach, Kannan, Ćirić are particular cases of A-contraction for different g (see [1] for more details).

On the other hand, in *metric fixed point theory*, among the other type of adequate conditions for giving guaranty of existence of fixed point of a mapping, *contractive* condition is a remarkable one. This notion was initiated by Edelstein in 1962, see [6]. Afterwards, this contractive condition have been extended in many ways. Among all such extensions, $\mathcal{A}$-contractive and $\mathcal{A}'$-contractive conditions due to Garai et al. [7] are one of the most generalized ones, since these two contain a handful number of contractive conditions as particular cases.

Apart from all these, we know that in order to solve different kind of problems, the notion of fixed point has given arise some other notions. Such as in order to solve a pair or a family of equations, the notion of common fixed point arise; in order to find the approximate solution of an equation, the notion of best proximity point arise etc. In the literature, there are many contraction as well as contractive conditions which deal with the existence of common fixed points of a couple of mappings. But in order to verify the validity of these conditions, we need to prove separate results for each separate conditions. So it is now very natural to think about some contraction and contractive conditions which can accommodate a large number of contraction and contractive conditions. With the aim of thinking in this way, we find that if the notion of A-contraction and $\mathcal{A}$-contractive conditions can be established in case of two or more mappings, then our aim will be fulfilled. As a result, in this article, we first formulate the notions of A-contraction and $\mathcal{A}$-contractive conditions in case of a couple of mappings and also for a family of mappings. After this, we examine the validity of existence of common fixed points of mappings satisfying such contraction and contractive conditions. During this verification process, we show that completeness of the underlying metric space can give the guaranty of existence of common fixed point in case of A-contraction condition but can't give in case of $\mathcal{A}$-contractive condition and also we show that compactness of the underlying metric space can give such guaranty in case of $\mathcal{A}$-contractive condition.

Before going to our main findings, we now recall the definition of orbit of two mappings and the corresponding definition of orbital continuity.

If X is a non-empty set; $T,S\colon X\to X$ are two mappings and $x_0\in X$, then the orbit of (T,S), $O_{T,S}(x_0)$ is the set

$$O_{T,S}(x_0)=\{x_0,Tx_0,STx_0,TSTx_0,STSTx_0,\dots\}.$$

Moreover if X is a metric space, then the pair (T,S) of mappings is said to be orbitally continuous in pair if for any $x_0\in X$ and for any sequence $\{x_n\}\in O_{T,S}(x_0)$, $x_n\to y\in X$ implies $Tx_n\to Ty$ and $Sx_n\to Sy$ as $n\to\infty$.

2. A-CONTRACTION AND $\mathcal{A}$-CONTRACTIVE MAPPINGS

In this section, we give the formal definitions of A-contraction and $\mathcal{A}$-contractive conditions in case of a couple and a family of mappings. Before this, we consider two collection of mappings.

Let A be the collection of all functions $g\colon\mathbb{R}^3_+\to\mathbb{R}$ that satisfies the following conditions:

- (A_1) g is continuous on $\mathbb{R}^3_+$;
- (A_2) there exists $\gamma\in[0,1)$ such that if $u\le g(u,v,v)$ or $u\le g(v,u,v)$ or $u\le g(v,v,u)$, then $u\le\gamma v$;
- (A_3) $g(u,v,w)\le u+v+w$ for all $u,v,w\in\mathbb{R}_+$.

Let $\mathcal{A}$ be the collection of all functions $g\colon\mathbb{R}^3_+\to\mathbb{R}$ that satisfies the following conditions:

- (A_1) g is continuous on $\mathbb{R}^3_+$;
- (A_2) if $v>0$ and $u<g(u,v,v)$ or $u<g(v,u,v)$ or $u<g(v,v,u)$, then $u<v$;
- (A_3) $g(u,v,w)\le u+v+w$ for all $u,v,w\in\mathbb{R}_+$.

For examples of particular mappings 'g' belonging to the above two collections, the readers are referred to see [1, 7, 10, 11].

Now, we introduce the notion of A-contraction in case of a couple and a family of mappings.

Definition 1. Let (X,d) be a metric space and let $T,S\colon X\to X$ be two mappings. Then, the pair of mappings (T,S) is said to be A-contraction in pair if there exists $g\in A$ such that

$$d(Tx,Sy)\le g(d(x,y),d(x,Tx),d(y,Sy))\quad\text{for all }\ x,y\in X.$$

Definition 2. Let (X,d) be a metric space and let $S_i\colon X\to X$, $i=1,2,\dots,m$, be m number of mappings. Then, such collection of mappings is said to be A-contraction in pair if there exists $g_i \in A$, $i = 1,2,\ldots,m$, such that

$$d(S_i x, S_{i+1} y) \leq g_i(d(x,y), d(x,S_i x), d(y,S_{i+1} y)) \quad \text{for all } x,y \in X,$$

where we assume that $S_{m+1} = S_1$.

Next, we introduce the notion of $\mathcal{A}$-contractive condition in case of a couple and a family of mappings.

Definition 3. Let (X,d) be a metric space and let $T,S\colon X \to X$ be two mappings. Then, the pair of mappings (T,S) is said to be $\mathcal{A}$-contractive in pair if there exists $g \in \mathcal{A}$ such that

$$d(Tx,Sy) < g(d(x,y), d(x,Tx), d(y,Sy)) \quad \text{for all } x,y \in X \text{ with } x \neq y.$$

Definition 4. Let (X,d) be a metric space and let $S_i\colon X \to X$, $i = 1,2,\ldots,m$, be m number of mappings. Then, such collection of mappings is said to be $\mathcal{A}$-contractive in pair if there exists $g_i \in \mathcal{A}$, $i = 1,2,\ldots,m$, such that

$$d(S_i x, S_{i+1} y) < g_i(d(x,y), d(x,S_i x), d(y,S_{i+1} y)) \quad \text{for all } x,y \in X \text{ with } x \neq y,$$

where we assume that $S_{m+1} = S_1$.

3. Main results

In the beginning of this section, we prove the following two common fixed point results involving the A-contraction condition.

Theorem 2. *Let (X,d) be a complete metric space and $T,S\colon X \to X$ be two mappings such that (T,S) is A-contraction in pair. If (T,S) is orbitally continuous in pair, then T and S have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. We consider the sequence $\{x_n\}$ in X by setting $x_n = Tx_{n-1}$ if n is odd and $x_n = Sx_{n-1}$ if n is even. We show that $\{x_n\}$ is Cauchy. If n is even, then we have

$$\begin{aligned} & d(Tx_n, Sx_{n-1}) \leq g(d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Sx_{n-1})) \\ \Longrightarrow\ & d(x_{n+1},x_n) \leq g(d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n)) \\ \Longrightarrow\ & d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1}) \text{ where } \gamma \in [0,1). \end{aligned}$$

Again if n is odd, then we can similarly show that

$$d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1}).$$

Thus the infinite series $\sum_{n=0}^{\infty} d(x_n,x_{n+1})$ is convergent and hence the sequence $\{x_n\}$ is Cauchy. Since (X,d) is complete, there exists $\alpha \in X$ such that $\lim_{n\to\infty} x_n = \alpha$. Consequently the two subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converge to α also. By the orbital continuity of (T,S), it follows that $\{x_{2n+1}\}$ converge to $T\alpha$ and $\{x_{2n}\}$ converge to $S\alpha$. So we have $\alpha = T\alpha = S\alpha$, i.e., α is a common fixed point of T and S.

Next, we show the uniqueness of the common fixed point. Let α and α_1 be two common fixed points T and S. Then, we have

$$\begin{aligned} & d(T\alpha, S\alpha_1) \leq g(d(\alpha,\alpha_1), d(\alpha, T\alpha), d(\alpha_1, S\alpha_1)) \\ \Longrightarrow\ & d(\alpha,\alpha_1) \leq g(d(\alpha,\alpha_1), 0, 0) \\ \Longrightarrow\ & d(\alpha,\alpha_1) \leq 0, \end{aligned}$$

which implies that $\alpha = \alpha_1$. So α is the unique common fixed point of T and S. □

Theorem 3. *Let (X,d) be a complete metric space and $S_i\colon X \to X$, $i = 1,2,\dots,m$, be m number of mappings such that the collection of mappings is A-contraction in pair. If the collection of mappings is orbitally continuous in pair, then the collection has a unique common fixed point.*

Proof. From Theorem 2, it follows that the the pair of mappings S_i, S_{i+1} have a unique common fixed point. We denote this common fixed point of S_i and S_{i+1} by α_i. Then, from the definition of A-contraction condition, we have

$$\begin{aligned} & d(S_1\alpha_1, S_2\alpha_2) \leq g(d(\alpha_1,\alpha_2), d(\alpha_1, S_1\alpha_1), d(\alpha_2, S_2\alpha_2)) \\ \Longrightarrow\ & d(\alpha_1,\alpha_2) \leq g(d(\alpha_1,\alpha_2),0,0) \ \Longrightarrow\ d(\alpha_1,\alpha_2) \leq 0 \ \Longrightarrow\ \alpha_1 = \alpha_2. \end{aligned}$$

Similarly we can show that $\alpha_2 = \alpha_3$, $\alpha_3 = \alpha_4$, $\dots$, $\alpha_{m-1} = \alpha_m$. So α_1 is the unique common fixed point of the collection of mappings S_i, $i = 1,2,\dots,m$. □

By choosing a particular g in Theorem 2, we have the following remark:

Remark 1. If we choose $g(u,v,w) = \alpha u$, where $0 \leq \alpha < 1$; $g(u,v,w) = \alpha(v + w)$, where $0 \leq \alpha < \frac{1}{2}$; $g(u,v,w) = \alpha_1 u + \alpha_2 v + \alpha_3 w$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$; $g(u,v,w) = \alpha \max\{v,w\}$, where $0 \leq \alpha < 1$; $g(u,v,w) = \alpha\sqrt{vw}$, where $0 \leq \alpha < 1$ in Theorem 2, then we can obtain the the common fixed point results of corresponding contraction conditions of Banach [2], Kannan [8], Reich [12], Bianchini [3] and Khan [9] respectively as consequences.

Next, we show by an example that unlikely A-contraction mappings, $\mathcal{A}$-contractive mappings don't posses common fixed points if the underlying metric space is complete.

Example 1. Let us take $X = \{(x,y) : x > 0, y > 0\}$ and define a function $d\colon X \times X \to \mathbb{R}$ by

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} 0, & \text{if } (x_1,y_1) = (x_2,y_2) \\ 1 + \left|\frac{1}{x_1} - \frac{1}{x_2}\right| + \left|\frac{1}{y_1} - \frac{1}{y_2}\right|, & \text{if } (x_1,y_1) \neq (x_2,y_2). \end{cases}$$

Then, (X,d) is a complete metric space but (X,d) is not compact. Next, we define two mappings $T,S\colon X \to X$ by $T(x,y) = (2x,4y)$ and $S(x,y) = (4x,2y)$ for all $(x,y) \in X$. Let us choose $g \in \mathcal{A}$ defined by $g(u,v,w) = \frac{1}{3}u + \frac{1}{3}v + \frac{1}{3}w$.

Let $(x_1,y_1),(x_2,y_2) \in X$ be arbitrary with $(x_1,y_1) \neq (x_2,y_2)$. Then,

$$d(T(x_1,y_1),S(x_2,y_2)) = d((2x_1,4y_1),(4x_2,2y_2)) = 1+\left|\frac{1}{2x_1}-\frac{1}{4x_2}\right|+\left|\frac{1}{4y_1}-\frac{1}{2y_2}\right|$$

and

$$\begin{aligned}
&g(d((x_1,y_1),(x_2,y_2)),d((x_1,y_1),T(x_1,y_1)),d((x_2,y_2),S(x_2,y_2)))\\
&= g(d((x_1,y_1),(x_2,y_2)),d((x_1,y_1),(2x_1,4y_1)),d((x_2,y_2),(4x_2,2y_2)))\\
&= \frac{1}{3}\Bigg\{1+\left|\frac{1}{x_1}-\frac{1}{x_2}\right|+\left|\frac{1}{y_1}-\frac{1}{y_2}\right|+1+\left|\frac{1}{x_1}-\frac{1}{2x_1}\right|+\left|\frac{1}{y_1}-\frac{1}{4y_1}\right|\\
&\qquad +1+\left|\frac{1}{x_2}-\frac{1}{4x_2}\right|+\left|\frac{1}{y_2}-\frac{1}{2y_2}\right|\Bigg\}\\
&= 1+\frac{1}{3}\left|\frac{1}{x_1}-\frac{1}{x_2}\right|+\frac{1}{3}\left|\frac{1}{y_1}-\frac{1}{y_2}\right|+\frac{1}{6x_1}+\frac{1}{4y_1}+\frac{1}{4x_2}+\frac{1}{6y_2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(T(x_1,y_1),S(x_2,y_2)) < g(&d((x_1,y_1),(x_2,y_2)),d((x_1,y_1),T(x_1,y_1)),d((x_2,y_2),\\
&S(x_2,y_2))).
\end{aligned}$$

Thus (T,S) is $\mathcal{A}$-contractive in pair, but T and S have no common fixed point.

Next, we show that instead of completeness if the underlying space is compact, then $\mathcal{A}$-contractive mappings do posses common fixed points.

Theorem 4. *Let (X,d) be a compact metric space. Let $T,S\colon X \to X$ be two mappings such that (T,S) is $\mathcal{A}$-contractive in pair, commutative and also orbitally continuous in pair. Further assume that card $(\mathrm{Fix}(T))$, card $(\mathrm{Fix}(S)) \leq 1$. Then, T and S have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. We consider the sequence $\{x_n\}$ in X by setting $x_n = Tx_{n-1}$ if n is odd and $x_n = Sx_{n-1}$ if n is even. Since (X,d) is compact, $\{x_n\}$ has a convergent subsequence, say, $\{x_{n_k}\}$. Let $\lim_{k\to\infty} x_{n_k} = \alpha$.

Next, we consider a sequence of real numbers $\{s_n\}$, defined by $s_n = d(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. We prove that $s_n \to 0$ as $n \to \infty$. First we suppose that $x_n = x_{n+1}$ for some n. Without loss of generality, we assume that n is even. Then, by using the commutativity of T and S, we can show that $x_n,x_{n+1},x_{n+2},\dots$ all are fixed points of T. This together with the fact that card $(\mathrm{Fix}(T)) \leq 1$ proves that $x_n = x_{n+1} = x_{n+2} = \dots$. Then, clearly $s_n \to 0$. So we now assume that $x_n \neq x_{n+1}$ for all n. If n is odd, then we have

$$\begin{aligned}
&d(Tx_{n-1},Sx_n) < g(d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Sx_n))\\
\implies\ &d(x_n,x_{n+1}) < g(d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}))
\end{aligned}$$

$$\implies s_n < g(s_{n-1}, s_{n-1}, s_n) \implies s_n < s_{n-1}.$$

Again if n is even, then we can similarly show that $s_n < s_{n-1}$. Thus $\{s_n\}$ is a strictly decreasing sequence of non-negative real numbers. Hence $s_n \to b$ for some $b \in \mathbb{R}_+$. Therefore,

$$b = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}).$$

Now, we have three cases to consider.

Case I: Let all but finitely many n_k in the subsequence $\{x_{n_k}\}$ be even. Then, we have

$$b = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = \lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = d(\alpha, T\alpha).$$

Again

$$b = \lim_{k\to\infty} d(x_{n_k+1}, x_{n_k+2}) = \lim_{k\to\infty} d(Tx_{n_k}, STx_{n_k}) = d(T\alpha, ST\alpha).$$

If $b \neq 0$, then $\alpha \neq T\alpha$. Therefore,

$$\begin{aligned} & d(T\alpha, ST\alpha) < g(d(\alpha, T\alpha), d(\alpha, T\alpha), d(T\alpha, ST\alpha)) \\ \implies & d(T\alpha, ST\alpha) < d(\alpha, T\alpha) \implies b < b, \end{aligned}$$

which is a contradiction. So $b = 0$. Then, $\alpha = T\alpha$ and $T\alpha = ST\alpha \implies \alpha = S\alpha$. Therefore, α is a common fixed point of T and S.

Case II: Let all but finitely many n_k in the subsequence $\{x_{n_k}\}$ be odd. In this case, we can show that $b = d(\alpha, S\alpha) = (S\alpha, TS\alpha)$. Then, as in Case I, we can show that α is a common fixed point of T and S.

Case III: Let infinitely many n_k in the subsequence $\{x_{n_k}\}$ be even and infinitely many n_k be odd. Then, we can extract a subsequence $\{x_{n_{k_r}}\}$ from $\{x_{n_k}\}$ such that all n_{k_r} are even. Then we can show, as in Case I, that α is a common fixed point of T and S.

Next, we show the uniqueness of the common fixed point. Let $\alpha_1 (\neq \alpha)$ be another common fixed point T and S. Then, we have

$$\begin{aligned} & d(T\alpha, S\alpha_1) < g(d(\alpha, \alpha_1), d(\alpha, T\alpha), d(\alpha_1, S\alpha_1)) \\ \implies & d(\alpha, \alpha_1) < g(d(\alpha, \alpha_1), 0, 0) \leq d(\alpha, \alpha_1), \end{aligned}$$

which is a contradiction. So α is the unique common fixed point of T and S. □

Theorem 5. *Let (X,d) be a compact metric space. Let $S_i\colon X \to X$, $i = 1, 2, \ldots, m$, be m number of mappings such that the collection of mappings is $\mathcal{A}$-contractive in pair, orbitally continuous in pair and S_i, S_{i+1} are commutative for $i = 1, 2, \ldots, m$. Further assume that card $(\mathrm{Fix}(S_i)) \leq 1$ for $i = 1, 2, \ldots, m$. Then, the collection of mappings has a unique common fixed point.*

Proof. From Theorem 4, it follows that the the pair of mappings S_i, S_{i+1} have a unique common fixed point. We denote this common fixed point of S_i and S_{i+1} by α_i. If $\alpha_1 \neq \alpha_2$, then from the definition of $\mathcal{A}$-contractiveness, we have

$$\begin{aligned} & d(S_1\alpha_1, S_2\alpha_2) < g(d(\alpha_1,\alpha_2), d(\alpha_1, S_1\alpha_1), d(\alpha_2, S_2\alpha_2)) \\ \implies & d(\alpha_1,\alpha_2) < g(d(\alpha_1,\alpha_2),0,0) \leq d(\alpha_1,\alpha_2) \\ \implies & d(\alpha_1,\alpha_2) < d(\alpha_1,\alpha_2), \end{aligned}$$

which is a contradiction. So we must have $\alpha_1 = \alpha_2$. Similarly we can show that $\alpha_2 = \alpha_3$, $\alpha_3 = \alpha_4$, $\ldots$, $\alpha_{m-1} = \alpha_m$. So α_1 is the unique common fixed point of the collection of mappings S_i, $i = 1,2,\ldots,m$. □

Finally, we show that if we add some mild additional condition on $\mathcal{A}$-contractive mappings, then such mappings also do posses common fixed points in complete metric spaces.

Theorem 6. *Let (X,d) be a complete metric space. Let $T,S\colon X \to X$ be two mappings such that (T,S) is $\mathcal{A}$-contractive in pair, commutative and also orbitally continuous in pair. Further assume that card $(\mathrm{Fix}(T))$, card $(\mathrm{Fix}(S)) \leq 1$. If for any $\varepsilon > 0$ and for any $x_0 \in X$, there exists a positive $\delta = \delta(\varepsilon, x_0)$ such that*

$$g(d(x,y),d(x,Tx),d(y,Sy)) < \varepsilon + \delta \implies d(Tx,Sy) \leq \frac{\varepsilon}{4} \quad \text{for all } x,y \in O_{T,S}(x_0),$$

then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. We consider the sequences $\{x_n\}$ in X and $\{s_n\}$ in $\mathbb{R}_+$ which we have considered in Theorem 4. Then, $\{s_n\}$ is strictly decreasing and hence $\{s_n\}$ is convergent to some $b \in \mathbb{R}_+$. We now show that $b = 0$. If possible let $b > 0$. Then, there exists $\delta > 0$ such that

$$g(d(x,y),d(x,Tx),d(y,Sy)) < 4b + \delta \implies d(Tx,Sy) \leq b \text{ for all } x,y \in O_{T,S}(x_0).$$

Since $\{s_n\}$ converges to b, for the above $\delta > 0$, there exists $n \in \mathbb{N}$ such that

$$s_n < b + \frac{\delta}{4}.$$

Without loss of generality, we assume that n is odd. Therefore,

$$\begin{aligned} & g(d(x_{n+1},x_n),d(x_{n+1},x_{n+2}),d(x_n,x_{n+1})) \leq d(x_{n+1},x_n) + d(x_{n+1},x_{n+2}) + d(x_n,x_{n+1}) \\ & = s_n + s_{n+1} + s_n < 3s_n < 3\left(b + \frac{\delta}{4}\right) < 4b + \delta. \end{aligned}$$

This implies that

$$d(Tx_{n+1},Sx_n) \leq b \implies d(x_{n+2},x_{n+1}) \leq b \implies s_{n+1} \leq b, \quad \text{a contradiction.}$$

Therefore $b = 0$, i.e., $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Next, we show that $\{x_n\}$ is Cauchy. For this, let $\varepsilon > 0$ be arbitrary. Then, there exists $\delta > 0$ such that

$$g(d(x,y), d(x,Tx), d(y,Sy)) < \varepsilon + \delta \implies d(Tx,Sy) \leq \frac{\varepsilon}{4} \quad \text{for all } x, y \in O_{T,S}(x_0).$$

Without loss of generality, we show that $\delta \leq \varepsilon$. Since $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, we get $N_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\delta}{16} \quad \text{for all } n \geq N_1. \tag{3.1}$$

Let $n \geq N_1 + 1$ be arbitrary. Then, we show by induction that

$$d(x_n, x_{n+k}) \leq \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

The statement for $k = 1$, follows from (3.1). Let (3.2) be true for $k = 1, 2, \ldots, m$. Our next aim is to show that $d(x_n, x_{n+m+1}) \leq \varepsilon$. To show this, we first assume that n and $n+m+1$ both are even. Then, we have

$$d(x_{n-2}, x_{n+m}) \leq d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+m}) \leq \frac{\delta}{8} + \varepsilon.$$

Therefore,

$$\begin{aligned}
&g(d(x_{n-2}, x_{n+m}), d(x_{n-2}, Tx_{n-2}), d(x_{n+m}, Sx_{n+m})) \\
&\leq d(x_{n-2}, x_{n+m}) + d(x_{n-2}, x_{n-1}) + d(x_{n+m}, x_{n+m+1}) \\
&\leq \frac{\delta}{8} + \varepsilon + \frac{\delta}{16} + \frac{\delta}{16} < \varepsilon + \delta.
\end{aligned}$$

Therefore,

$$d(x_{n-1}, x_{n+m+1}) \leq \frac{\varepsilon}{4}$$

and so

$$d(x_n, x_{n+m+1}) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+m+1}) \leq \frac{\delta}{16} + \frac{\varepsilon}{4} \leq \varepsilon.$$

When both n and $n+m+1$ are odd or only one of n and $n+m+1$ is even, then by proceeding as in the above way, we can show that $d(x_n, x_{n+m+1}) \leq \varepsilon$. Thus by Principle of Mathematical induction, it follows that

$$d(x_n, x_{n+k}) \leq \varepsilon \text{ for all } n \geq N_1 + 1 \text{ and } k = 1, 2, \ldots.$$

Thus $\{x_n\}$ is a Cauchy sequence in X and hence there exists $\alpha \in X$ such that $x_n \to \alpha$ as $n \to \infty$. Then, by proceeding to the similar way of Theorem 4, we can show that α is the unique common fixed point of T and S. □

Theorem 7. *Let (X,d) be a complete metric space. Let $S_i \colon X \to X$, $i = 1, 2, \ldots, m$, be m number of mappings such that the collection of mapping is $\mathcal{A}$-contractive in pair, orbitally continuous in pair and S_i, S_{i+1} are commutative for $i = 1, 2, \ldots, m$.*

Further assume that card $(\text{Fix}(S_i)) \leq 1$ *for* $i = 1,2,\ldots,m$. *If for any* $\varepsilon > 0$ *and for any* $x_0 \in X$, *there exists a positive* $\delta = \delta(\varepsilon, x_0)$ *such that*

$$g(d(x,y), d(x,S_ix), d(y,S_{i+1}y)) < \varepsilon + \delta \implies d(S_ix, S_{i+1}y) \leq \frac{\varepsilon}{4}$$

for all $x,y \in O_{T,S}(x_0)$, *then the collection of mappings has a unique common fixed point.*

Proof. Follows from the proofs of Theorem 6 and Theorem 5. □

Again by picking out particular g in Theorem 4, we have the following remark.

Remark 2. If we choose $g(u,v,w) = u$; $g(u,v,w) = \frac{1}{2}(v+w)$; $g(u,v,w) = \alpha_1 u + \alpha_2 v + \alpha_3 w$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$; $g(u,v,w) = \max\{v,w\}$; $g(u,v,w) = \sqrt{vw}$ in Theorem 4, then we can obtain the the common fixed point results of corresponding contractive conditions of Banach [2], Kannan [8], Reich [12], Bianchini [3] and Khan [9] respectively as consequences.

We finish this section by demonstrating three supporting examples.

Example 2. Let us consider the set $\mathbb{R}^2$ equipped with the metric

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2.$$

Let $X = \{(x,y) \in \mathbb{R}^2 : 3 \leq x \leq y\}$. Then, (X,d) is a complete metric space. Let us define $T,S\colon X \to X$ by $T(x,y) = (3,3)$ and $S(x,y) = \left(\frac{3+x}{2}, 3\right)$. Then, (T,S) is orbitally continuous in pair. We choose $g \in A$ defined by $g(u,v,w) = \frac{1}{3}v + \frac{1}{3}w$.

Let $(x_1,y_1),(x_2,y_2) \in X$ be arbitrary. Then,

$$d(T(x_1,y_1), S(x_2,y_2)) = d\left((3,3), \left(\frac{3+x_2}{2}, 3\right)\right) = \frac{1}{2}(x_2 - 3)$$

and

$$\begin{aligned}
&\frac{1}{3}d((x_1,y_1),T(x_1,y_1)) + \frac{1}{3}d((x_2,y_2),S(x_2,y_2)) \\
&= \frac{1}{3}d((x_1,y_1),(3,3)) + \frac{1}{3}d\left((x_2,y_2), \left(\frac{3+x_2}{2}, 3\right)\right) \\
&= \frac{1}{3}\left\{x_1 - 3 + y_1 - 3 + \frac{1}{2}(x_2 - 3) + y_2 - 3\right\}.
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{1}{2}(x_2 - 3) - \frac{1}{3}\left\{x_1 - 3 + y_1 - 3 + \frac{1}{2}(x_2 - 3) + y_2 - 3\right\} \\
&= \frac{1}{6}\left\{3(x_2 - 3) - 2\left(x_1 + y_1 - 6 + \frac{x_2}{2} - \frac{9}{2} + y_2\right)\right\} \\
&= \frac{1}{6}\{3x_2 - 9 - 2x_1 - 2y_1 + 12 - x_2 + 9 - 2y_2\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}\{2x_2 - 2y_2 - 2x_1 - 2y_1 + 12\} \\
&\le \frac{1}{6}(2x_2 - 2y_2), \text{ since } x_1, y_1 \ge 3 \\
&\le 0, \text{ since } x_2 \le y_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}(x_2 - 3) \le \frac{1}{3}\left\{x_1 - 3 + y_1 - 3 + \frac{1}{2}(x_2 - 3) + y_2 - 3\right\} \\
\implies\ & d(T(x_1,y_1), S(x_2,y_2)) \le \frac{1}{3}d((x_1,y_1),T(x_1,y_1)) + \frac{1}{3}d((x_2,y_2),S(x_2,y_2)) \\
\implies\ & d(T(x_1,y_1), S(x_2,y_2)) \le g(d((x_1,y_1),(x_2,y_2)), d((x_1,y_1),T(x_1,y_1)), \\
& \qquad d((x_2,y_2),S(x_2,y_2))).
\end{aligned}
$$

Thus (T,S) is A-contraction in pair and so by Theorem 2, T and S have a unique common fixed point in X and $(3,3)$ is the unique common fixed point of T and S.

Example 3. Let us take $X = \{(x,y) : 0 \le x \le 2, 0 \le y \le 1\}$ and define a function $d\colon X \times X \to \mathbb{R}$ by $d((x_1,y_1),(x_2,y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for all $(x_1,y_1),(x_2,y_2) \in X$. Then, (X,d) is a compact metric space. Next, we define two mappings $T,S\colon X \to X$ by

$$
T(x,y) = \left(\frac{x}{3}, 0\right) \text{ and } S(x,y) = \left(0, \frac{y}{4}\right)
$$

for all $(x,y) \in X$. Also we choose $g \in \mathcal{A}$ defined by $g(u,v,w) = \frac{1}{2}v + \frac{1}{2}w$.

Let $(x_1,y_1),(x_2,y_2) \in X$ be arbitrary with $(x_1,y_1) \ne (x_2,y_2)$. Then,

$$
d(T(x_1,y_1), S(x_2,y_2)) = d\left(\left(\frac{x_1}{3},0\right),\left(0,\frac{y_2}{4}\right)\right) = \frac{x_1}{3} + \frac{y_2}{4}
$$

and

$$
\begin{aligned}
& g(d((x_1,y_1),(x_2,y_2)), d((x_1,y_1),T(x_1,y_1)), d((x_2,y_2),S(x_2,y_2))) \\
&= \frac{1}{2}d\left((x_1,y_1),\left(\frac{x_1}{3},0\right)\right) + \frac{1}{2}d\left((x_2,y_2),\left(0,\frac{y_2}{4}\right)\right) \\
&= \frac{1}{2}\left\{\left|x_1 - \frac{x_1}{3}\right| + y_1 + x_2 + \left|y_2 - \frac{y_2}{4}\right|\right\} = \frac{x_1}{3} + \frac{3y_2}{8} + \frac{1}{2}(x_2 + y_1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d(T(x_1,y_1), S(x_2,y_2)) < g(&d((x_1,y_1),(x_2,y_2)), d((x_1,y_1),T(x_1,y_1)), d((x_2,y_2), \\
&S(x_2,y_2))).
\end{aligned}
$$

Hence (T,S) is $\mathcal{A}$-contractive in pair and so by Theorem 4, T and S have a unique common fixed point in X. Indeed $(0,0)$ is the unique common fixed point of T and S.

Example 4. Let us consider the set $C\left[0,\frac{1}{12}\right]$ of all real valued continuous functions of real numbers defined on $\left[0,\frac{1}{12}\right]$ and consider the sup metric d on $C\left[0,\frac{1}{12}\right]$. Then, $\left(C\left[0,\frac{1}{12}\right],d\right)$ is a complete metric space. Also we take $g\in\mathcal{A}$ defined by $g(u,v,w)=\frac{1}{2}u+\frac{1}{2}\max\{v,w\}$. After this, we consider two self-mappings T and S on X defined by

$$T(x(t))=\frac{t}{2}x(t)\ \text{ and }\ S(x(t))=tx(t)$$

for all $x\in C\left[0,\frac{1}{12}\right]$. Then, for $x,y\in C\left[0,\frac{1}{12}\right]$ and $t\in\left[0,\frac{1}{12}\right]$, we have

$$\begin{aligned}
|T(x(t))-S(y(t))| &= \left|\frac{t}{2}x(t)-ty(t)\right| \le \left|\frac{t}{2}x(t)-\frac{t}{2}y(t)\right|+\left|\frac{t}{2}y(t)\right|\\
&= \frac{t}{2}|x(t)-y(t)|+\frac{t}{2}|y(t)| \le \frac{t}{2}|x(t)-y(t)|+\frac{1}{2}(1-t)|y(t)|\\
&= \frac{t}{2}|x(t)-y(t)|+\frac{1}{2}|y(t)-ty(t)|\\
&= \frac{t}{2}|x(t)-y(t)|+\frac{1}{2}|y(t)-Sy(t)|\\
&\le \frac{t}{2}d(x,y)+\frac{1}{2}d(y,Sy)\\
&\le \frac{1}{4}d(x,y)+\frac{1}{2}\max\{d(x,Tx),d(y,Sy)\}\\
\implies d(Tx,Sy) &\le \frac{1}{4}d(x,y)+\frac{1}{2}\max\{d(x,Tx),d(y,Sy)\}\\
&< \frac{1}{2}d(x,y)+\frac{1}{2}\max\{d(x,Tx),d(y,Sy)\}\\
\implies d(Tx,Sy) &< g(d(x,y),d(x,Tx),d(y,Sy)).
\end{aligned}$$

Thus (T,S) is $\mathcal{A}$-contractive in pair.

Now, let $x\in C\left[0,\frac{1}{12}\right]$ and $\varepsilon>0$ be arbitrary. Then, for any $y\in O_{T,S}(x)$, y will look like $y(t)=t\alpha(t)x(t)$ for some α, where α is a function of t. We choose $\delta=\frac{\varepsilon}{2}$. Then, for $y\in O_{T,S}(x)$, we have

$$\begin{aligned}
g(d(x,y),d(x,Tx),d(y,Sy)) &< \varepsilon+\frac{\varepsilon}{2}\\
\implies \frac{1}{2}d(x,y)\le g(d(x,y),d(x,Tx),d(y,Sy)) &< \frac{3\varepsilon}{2}\\
\implies (1-t\alpha(t))|x(t)| &< 3\varepsilon.
\end{aligned}$$

Therefore,

$$d(Tx,Sy)=d\left(\frac{t}{2}x(t),t^2\alpha(t)x(t)\right)=\sup_{t\in\left[0,\frac{1}{12}\right]}\left(\frac{t}{2}-t^2\alpha(t)\right)|x(t)|$$

$$\leq \frac{1}{12} \sup_{t\in\left[0,\frac{1}{12}\right]} \left(\frac{1}{2} - t\alpha(t)\right) |x(t)| \leq \frac{1}{12} \sup_{t\in\left[0,\frac{1}{12}\right]} (1 - t\alpha(t)) |x(t)|$$
$$\leq \frac{3\varepsilon}{12} = \frac{\varepsilon}{4}.$$

Thus all the conditions of Theorem 6 hold good. So by the same theorem, T and S have a unique common fixed point in X. Indeed $x \in C\left[0, \frac{1}{12}\right]$ defined by $x(t) = 0$ for all t, is the unique common fixed point of T and S.

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