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ON TRACES OF PERMUTING n-DERIVATIONS ON PRIME IDEALS

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Abstract. In this article we investigate some properties of permuting n-derivations acting on a prime ideal. More precisely, let $n \geq 2$ be a fixed positive integer, P be a prime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If there exists a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ satisfies $\overline{[[\delta(x),x],x]} \in Z(R/P)$ for all $x \in R$, then $\Delta(R^n) \subseteq P$ or R/P is commutative.

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1. Introduction

Throughout this article, R will represent an associative ring with center $Z(R)$. Recall that an ideal P of R is said to be prime if $P \neq R$ and for all $x,y \in R$, $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. Therefore, R is called a prime ring if and only if (0) is the only minimal prime ideal of R. R is n-torsion free if whenever $nx = 0$, with $x \in R$ implies $x = 0$. For any $x,y \in R$, the symbol $[x,y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. A mapping $f\colon R \longrightarrow R$ is said to be centralizing on a subset S of R if $[f(s),s] \in Z(R)$ for all $s \in S$. In particular, if $[f(s),s] = 0$ for all $s \in S$, then f is commuting on S. A map $d\colon R \longrightarrow R$ is a derivation of a ring R if d is additive and satisfies $d(xy) = d(x)y + xd(y)$ for all $x,y \in R$.

Suppose n is a fixed positive integer and $R^n = R \times R \times \cdots \times R$, a map $\Delta\colon R^n \longrightarrow R$ is n-additive if it satisfies

$$\Delta(x_1,x_2,\ldots,x_i+x_i',\ldots,x_n) = \Delta(x_1,x_2,\ldots,x_i,\ldots,x_n) + \Delta(x_1,x_2,\ldots,x_i',\ldots,x_n)$$

for all $x_i,x_i' \in R$, $i = 1,2,\ldots,n$. A map $\Delta\colon R^n \longrightarrow R$ is said to be permuting if $\Delta(x_1,x_2,x_3,\ldots,x_n) = \Delta(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)},\ldots,x_{\pi(n)})$ for all $x_i \in R$ and for every permutation $\pi \in S_n$; where S_n is the symmetric group on n symbols $\{1,2,3,\ldots,n\}$. A map $\delta\colon R \longrightarrow R$ is called the trace of Δ if $\delta(x) = \Delta(x,x,x,\ldots,x)$ for all $x \in R$. It is obvious to verify that if $\Delta\colon R^n \longrightarrow R$ is a permuting and n-additive map, then the trace δ of Δ satisfies the relation

$$\delta(x+y)=\delta(x)+\delta(y)+\sum_{i=1}^{n-1}\binom{n}{i}\Delta(x,x,\ldots,x,y,y,\ldots,y)$$

where x appears $(n-i)$-times and y appears i-times.

Park [4] introduced the notion of permuting n-derivation as follows: a permuting map $\Delta\colon R^n \longrightarrow R$ is said to be a permuting n-derivation if Δ is n-additive and

$$\Delta(x_1,x_2,\ldots,x_ix_i^{'},\ldots,x_n)=x_i\Delta(x_1,x_2,\ldots,x_i^{'},\ldots,x_n)+\Delta(x_1,x_2,\ldots,x_i,\ldots,x_n)x_i^{'}$$

for all $x_1,x_2,\ldots,x_i,x_i^{'},\ldots,x_n\in R$. Clearly, a 1-derivation is a usual derivation and a 2-derivation is a symmetric bi-derivation. However, in the case of $n=3$ we get the concept of symmetric tri-derivation.

Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive maps defined on R. A well known result due to Posner [1] states that a prime ring R which admits a nonzero centralizing derivation is commutative. Since then, several authors have done a great deal of work concerning commutativity of prime and semi-prime rings admitting different kinds of maps which are centralizing on some appropriate subsets of R ([6, 7] and [5] for a further references). More recently several authors have studied various identities involving trace of permuting n-derivations and have obtained interesting theorems. Indeed, motivated by the results due to Posner [1], Vukman obtained some results concerning the trace of symmetric bi-derivation in prime ring (see [2] and [3]). Ashraf [8] proved similar results for semi-prime ring. In [11], Jung and Park obtained the similar results to Posner's and Vukman's ones for permuting 3-derivations on prime and semiprime rings. In the year 2009, Park [4] introduced the concept of symmetric permuting n-derivation and obtained some results related to the commuting traces of permuting n-derivations in rings. Further, Ashraf and Jamal [10] obtained commutativity of rings admitting n-derivations whose traces satisfy certain polynomial conditions. In 2018, the authors in [9] generalize the result of Park by considering more general identities rather than centralizing mappings. More precisely, they proved that if R is $(n+1)!$-torsion free semi-prime ring admitting a permuting n-derivation Δ such that the trace δ of Δ satisfies $[[\delta(x),x],x]\in Z(R)$ for all $x\in R$. Then δ is commuting on R.

The present paper is motivated by the previous results and we here continue this line of investigation by considering a more general concept rather than commuting and centralizing traces of permuting n-derivations. In fact, we define the concepts of S-commuting maps and S-centralizing maps as follows:

Definition 1. Let R be a ring, S a subset of R and $f\colon R\longrightarrow R$ a map. Then

1. f is called S-commuting if $[f(x),x]\in S$ for all $x\in R$;
2. f is called S-centralizing if $[[f(x),x],y]\in S$ for all $x,y\in R$.

1.1. *Preliminary results*

The following fact is essential for the proofs of our results.

Lemma 1 ([4], Lemma 2.4). *Let n be a fixed positive integer and let R be a $n!-$torsion free ring. Suppose that $y_1, y_2, \ldots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0$ (or $\in Z(R)$) for $\lambda = 1, 2, \ldots, n$. Then $y_i = 0$ (or $y_i \in Z(R)$) for all i.*

In ([4], Theorem 2.3) the author proved that if $n \geq 2$ is a fixed positive integer and R is a non-commutative $n!$-torsion free prime ring provided with a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ is commuting on R, then $\Delta = 0$. In the following lemma we prove the same result by considering a more general situation.

Lemma 2. *Let $n \geq 2$ be a fixed positive integer, P be a prime ideal of a ring R such that R/P is noncommutative $n!-$torsion free. If there exists a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ is P-commuting, then we have $\Delta(R^n) \subseteq P$.*

Proof. We are given that

$$\overline{[\delta(x), x]} = \overline{0} \quad \text{for all } x \in R. \tag{1.1}$$

An easier computation shows that the trace δ of Δ satisfies the relation

$$\delta(x+y) = \delta(x) + \delta(y) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(x,y) \quad \text{for all } x, y \in R, \tag{1.2}$$

where $h_k(x,y) = \Delta(x,x,x,\ldots,x,y,y,\ldots,y)$; y appears k times and x appears n-k times. Let λ $(1 \leq \lambda \leq n)$ be any integer. By replacing x by $x + \lambda y$ in (1.1) and using (1.1), we obtain

$$\begin{aligned} &\lambda\Big(\overline{[\delta(x), y]} + {}^nC_1\overline{[h_1(x,y), x]}\Big) + \lambda^2\Big({}^nC_1\overline{[h_1(x,y), y]} + {}^nC_2\overline{[h_2(x,y), x]}\Big) + \ldots \\ &\quad + \lambda^n\Big(\overline{[\delta(y), x]} + {}^nC_{n-1}\overline{[h_{n-1}(x,y), y]}\Big) = \overline{0} \end{aligned} \tag{1.3}$$

for all $x, y \in R$. From Lemma 1 and equation (1.3) we conclude that

$$\overline{[\delta(x), y]} + n\overline{[h_1(x,y), x]} = \overline{0} \quad \text{for all } x, y \in R. \tag{1.4}$$

Writing xy instead of y in (1.4), one can see that

$$\overline{x}\Big(\overline{[\delta(x), y]} + n\overline{[h_1(x,y), x]}\Big) + n\overline{\delta(x)[y,x]} = \overline{0} \quad \text{for all } x, y \in R. \tag{1.5}$$

Invoking (1.4), equation (1.5) implies that

$$n\overline{\delta(x)[y,x]} = \overline{0} \quad \text{for all } x, y \in R.$$

Since n divides $n!$, we find that R/P is n-torsion free and thus

$$\overline{\delta(x)[y,x]} = \overline{0} \quad \text{for all } x, y \in R. \tag{1.6}$$

Substituting ry for y in (1.6), we get

$$\overline{\delta(x)}(R/P)\overline{[y,x]} = \overline{0} \quad \text{for all } x,y \in R. \tag{1.7}$$

Since P is a prime ideal, then (1.7) implies $[y,x] \in P$ or $\delta(x) \in P$ for all $x,y \in R$. It follows that $\delta(x) \in P$ for all $x \in R$ such that $\bar{x} \notin Z(R/P)$.

Now, let $x \in R$ such that $\bar{x} \in Z(R/P)$ and let $y \in R$ with $\bar{y} \notin Z(R/P)$ $(\overline{\delta(y)} = 0)$. Then $\bar{y} + \lambda\bar{x} \notin Z(R/P)$. Thus we obtain

$$\overline{\delta(y+\lambda x)} = \overline{\delta(y)} + \overline{\delta(x)}\lambda^n + \sum_{k=1}^{n-1} {}^nC_k\overline{h_k(y,x)}\lambda^k = \overline{0}.$$

Therefore

$$\sum_{k=1}^{n-1} {}^nC_k\overline{h_k(y,x)}\lambda^k + \overline{\delta(x)}\lambda^n = \overline{0}. \tag{1.8}$$

Applying Lemma 1, equation (1.8) yields $\delta(x) \in P$ for all $x \in R$ $(\bar{x} \in Z(R/P))$. Hence, we conclude that

$$\delta(x) \in P \quad \text{for all } x \in R. \tag{1.9}$$

Let μ $(1 \leq \mu \leq n-1)$ be any integer. By (1.9) we have

$$\overline{\delta(\mu x + x_n)} = \overline{0} \quad \text{for all } x, x_n \in R. \tag{1.10}$$

For each $k = 1,2,\ldots,n$, let

$$P_k(x) = \Delta(x,x,\ldots,x,x_{k+1},x_{k+2},\ldots,x_n),$$

where $x, x_i \in R, i = k+1, k+2, \ldots, n$.

By viewing of (1.9), equation (1.10) yields

$$\sum_{k=1}^{n-1} {}^nC_k\overline{P_k(x)}\mu^k = \overline{0} \quad \text{for all } x \in R. \tag{1.11}$$

Thus Lemma 1 and (1.11) imply

$$n\overline{P_{n-1}(x)} = \overline{0} \quad \text{for all } x \in R. \tag{1.12}$$

Since R/P is n-torsion free ring, then equation (1.12) forces

$$\overline{P_{n-1}(x)} = \overline{0} \quad \text{for all } x \in R. \tag{1.13}$$

Let ν $(1 \leq \nu \leq n-2)$ be any integer. By (1.13) the relation

$$\overline{P_{n-1}(\nu x + x_{n-1})} = \nu^{n-1}\overline{P_{n-1}(x)} + \overline{P_{n-1}(x_{n-1})} + \sum_{k=1}^{n-2} {}^nC_k\overline{P_k(x)}\nu^k = \overline{0}$$

holds for all $x, x_{n-1} \in R$, and hence one can see that

$$\sum_{k=1}^{n-2} {}^nC_k\overline{P_k(x)}\nu^k = \overline{0} \quad \text{for all } x \in R. \tag{1.14}$$

Using Lemma 1 and (1.14), we get

$$^{n}C_{n-2}\overline{P_{n-2}(x)} = \overline{0} \quad \text{for all } x \in R,$$

in such a way that

$$\overline{P_{n-2}(x)} = \overline{0} \quad \text{for all } x \in R.$$

Now if we continue with the same method as above, we finally obtain

$$^{n}C_{1}\overline{P_{1}(x)} = \overline{0} \quad \text{for all } x \in R,$$

which leads to

$$\overline{P_{1}(x)} = \overline{0} \quad \text{for all } x \in R.$$

Thus we obtain

$$\Delta(x_1, x_2, \ldots, x_n) \in P$$

for all $x_i \in R$. This completes the proof of the lemma. □

2. Main results

In [[9], Theorem 1] it is proved that if $n \geq 2$ is a fixed positive integer and R is a $(n+1)!$-torsion free semiprime ring provided with a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ satisfies $[[\delta(x), x], x] = 0$ for all $x \in R$, then δ is commuting. Motivated by this result we continue our investigation of permuting n-derivations with the next theorem which extends the result of [[9], Theorem 1], ([4], Theorem 2.5) to the case of any semiprime ideals rather than the zero ideal. The following result shows that if the trace δ of a permuting n-derivation Δ is P-centralizing then it is P-commuting. In fact, we prove rather a more general result:

Theorem 1. *Let $n \geq 2$ be a fixed positive integer, P be a semiprime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If R admits a permuting n-derivation Δ such that the trace δ of Δ satisfies $[[\delta(x), x], x] \in P$ for all $x \in R$, then δ is P-commuting.*

Proof. We are given that

$$\big[[\delta(x), x], x\big] \in P \quad \text{for all } x \in R. \tag{2.1}$$

Consider a positive integer k, $1 \leq k \leq n+1$. Replacing x by $x + ky$ in (2.1), we obtain

$$kQ_1(x,y) + k^2 Q_2(x,y) + \cdots + k^{n+1} Q_{n+1}(x,y) \in P \quad \text{for all } x, y \in R, \tag{2.2}$$

where $Q_i(x,y)$ denotes the sum of the terms in which y appears i times. Using (2.2) together with Lemma 1 we find that

$$\big[[\delta(x), x], y\big] + \big[[\delta(x), y], x\big] + n\big[[\Delta(x, x, x, \ldots, y), x], x\big] \in P \tag{2.3}$$

for all $x, y \in R$. Replacing y by xy in (2.3), we get

$$\begin{aligned} x\big[[\delta(x), x], y\big] + \big[[\delta(x), x], x\big]y + x\big[[\delta(x), y], x\big] + \big[[\delta(x), x], x\big]y + [\delta(x), x][y, x] \\ + nx\big[[\Delta(x, x, x, \ldots, y), x], x\big] + n\big[[\delta(x), x], x\big]y + n[\delta(x), x][y, x] \\ n[\delta(x), x][y, x] + n\delta(x)[[y, x], x] \in P. \end{aligned} \tag{2.4}$$

Using (2.4) and (2.3) we find that

$$(2n+1)[\delta(x),x][y,x]+n\delta(x)\big[[y,x],x\big]\in P. \tag{2.5}$$

Similarly, replacing y by yx in (2.3), one can get

$$(2n+1)[y,x][\delta(x),x]+n\big[[y,x],x\big]\delta(x)\in P. \tag{2.6}$$

Substituting yz for y in (2.5), we have

$$(2n+1)\big([\delta(x),x]y[z,x]+[\delta(x),x][y,x]z\big)+n\delta(x)y\big[[z,x],x\big]+$$
$$n\delta(x)[y,x][z,x]+n\delta(x)[y,x][z,x]+n\delta(x)\big[[y,x],x\big]z\in P.$$

Using equation 2.5, we obtain

$$(2n+1)[\delta(x),x]y[z,x]+n\delta(x)y[[z,x],x]+2n\delta(x)[y,x][z,x]\in P.$$

Replacing y by $\delta(x)$ in the above relation we find that

$$(2n+1)[\delta(x),x]\delta(x)[z,x]+n\delta(x)^2[[z,x],x]+2n\delta(x)[\delta(x),x][z,x]\in P. \tag{2.7}$$

From (2.5) we have

$$(2n+1)\delta(x)[\delta(x),x][y,x]+n\delta(x)^2[[y,x],x]\in P.$$

Now using this relation in (2.7) we get

$$(2n+1)[\delta(x),x]\delta(x)[z,x]-(2n+1)\delta(x)[\delta(x),x][z,x]+2n\delta(x)[\delta(x),x][z,x]\in P.$$

Therefore

$$\Big((2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]\Big)[z,x]\in P. \tag{2.8}$$

Puting $A=(2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]$ and replacing z by yz in (2.8) we get

$$Ay[z,x]\in P \quad \text{for all } x,y,z\in R. \tag{2.9}$$

Taking $z=\delta(x)$ in (2.9) and multiplying by $(2n+1)\delta(x)$ we obtain

$$Ay(2n+1)[\delta(x),x]\delta(x)\in P \quad \text{for all } x,y\in R. \tag{2.10}$$

Substituting $y\delta(x)$ for y and $\delta(x)$ for z in (2.9) we find that

$$Ay\delta(x)[\delta(x),x]\in P \quad \text{for all } x,y\in R. \tag{2.11}$$

Substracting (2.11) from (2.10) one can see that

$$(2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]\in P \quad \text{for all } x\in R. \tag{2.12}$$

Similarly, (2.6) gives

$$(2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x)\in P \quad \text{for all } x\in R. \tag{2.13}$$

Combining (2.12) and (2.13), we obtain

$$2n\Big(\delta(x)[\delta(x),x]+[\delta(x),x]\delta(x)\Big)\in P \quad \text{for all } x\in R.$$

Since $2n$ divides $(n+1)!$, then R is $2n$-torsion free and hence for all $x\in R$,

$$\delta(x)[\delta(x),x]+[\delta(x),x]\delta(x)\in P \quad \text{for all } x\in R. \tag{2.14}$$

Using (2.14) together with (2.13) one can see that $(2n+2)[\delta(x),x]\delta(x)\in P$ for all $x\in R$. Since $2(n+1)$ divides $(n+1)!$, then R is $2(n+1)$-torsion free and hence

$$[\delta(x),x]\delta(x)\in P \quad \text{for all } x\in R. \tag{2.15}$$

Similarly, from (2.13) and (2.14) we get

$$\delta(x)[\delta(x),x]\in P \quad \text{for all } x\in R. \tag{2.16}$$

Substituting $x+ky$ for x in equation (2.16), where $1\le k\le 2n$, and implementing Lemma 1 we arrive at

$$\delta(x)[\delta(x),y]+n\delta(x)[\Delta(x,x,\dots,y),x]+n\Delta(x,x,\dots,y)[\delta(x),x]\in P. \tag{2.17}$$

Replacing y by yx in the last equation, one can see that

$$\delta(x)[\delta(x),yx]+n\delta(x)[y\delta(x)+\Delta(x,x,\dots,y)x,x]+ \\ n\big(y\delta(x)+\Delta(x,x,\dots,y)x\big)[\delta(x),x]\in P,$$

and thus

$$\delta(x)y[\delta(x),x]+\delta(x)[\delta(x),y]x+n\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)+ \\ n\delta(x)[\Delta(x,x,\dots,y),x]x+ny\delta(x)[\delta(x),x]+n\Delta(x,x,\dots,y)x[\delta(x),x]\in P. \tag{2.18}$$

Using (2.17) together with (2.18), we get

$$(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)-n\Delta(x,x,\dots,y)[\delta(x),x]x+ \\ n\Delta(x,x,\dots,y)x[\delta(x),x]\in P,$$

hence

$$(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)-n\Delta(x,x,\dots,y)[[\delta(x),x],x]\in P.$$

Accordingly,

$$(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)\in P \quad \text{for all } x,y\in R. \tag{2.19}$$

Substituting xy for y in (2.19)

$$(n+1)\delta(x)xy[\delta(x),x]+n\delta(x)x[y,x]\delta(x)\in P \quad \text{for all } x,y\in R. \tag{2.20}$$

Left multiplying (2.19) by x, we obtain

$$(n+1)x\delta(x)y[\delta(x),x]+nx\delta(x)[y,x]\delta(x)\in P \quad \text{for all } x,y\in R. \tag{2.21}$$

Combining (2.20) and (2.21), we get

$$(n+1)[\delta(x),x]y[\delta(x),x]+n[\delta(x),x][y,x]\delta(x)\in P \quad \text{for all } x,y\in R. \tag{2.22}$$

Substituting yz for y in (2.6), we have

$$(2n+1)[yz,x][\delta(x),x]+n\big[[yz,x],x\big]\delta(x)\in P,$$

and thus

$$(2n+1)y[z,x][\delta(x),x]+(2n+1)[y,x]z[\delta(x),x]+ny\big[[z,x],x\big]\delta(x)+$$
$$n[y,x][z,x]\delta(x)+n[y,x][z,x]\delta(x)+n\big[[y,x],x\big]z\delta(x)\in P.$$

Using (2.6) one can see that

$$(2n+1)[y,x]z[\delta(x),x]+2n[y,x][z,x]\delta(x)+n\big[[y,x],x\big]z\delta(x)\in P.$$

Replacing y by $\delta(x)$ in the above relation we get

$$(2n+1)[\delta(x),x]z[\delta(x),x]+2n[\delta(x),x][z,x]\delta(x)\in P \quad \text{for all } x,z\in R. \tag{2.23}$$

Combining (2.22) and (2.23) we obtain

$$[\delta(x),x]z[\delta(x),x]\in P \quad \text{for all } x,z\in R.$$

Since P is a semi-prime ideal, we get $[\delta(x),x]\in P$ for all $x\in R$. □

In ([4], Theorem 2.5) it is proved that if $n\geq 2$ is a fixed positive integer and R is a $n!$-torsion free semiprime ring provided with a permuting n-derivation $\Delta\colon R^n\longrightarrow R$ such that the trace δ of Δ is centralizing on R, then δ is commuting on R. In the following corollary which is an immediate result from Theorem 1 we extends the result of Park to the case of any semiprime ideals rather than the zero ideal.

Corollary 1. *Let $n\geq 2$ be a fixed positive integer, P be a semiprime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If there exists a permuting n-derivation $\Delta\colon R^n\longrightarrow R$ such that the trace δ of Δ is P-centralizing then δ is P-commuting.*

The following corollary extends the main result in ([4], Theorem 2.6), which is an analogue of Posner's Theorem for permuting n-derivations.

Corollary 2. *Let $n\geq 2$ be a fixed positive integer, P be a prime ideal of a ring R such that R/P is $n!$-torsion free. If there exists a permuting n-derivation $\Delta\colon R^n\longrightarrow R$ such that the trace δ of Δ is P-centralizing, then $\Delta(R^n)\subseteq P$ or R/P is commutative.*

Proof. Suppose that R/P is noncommutative, then it follows from Corollary 1 that δ is P-commuting. Hence Lemma 2 gives $\Delta(R^n)\subseteq P$. This guarantees the conclusion of the corollary. □

The fundamental result of [4] is immediate from Corollary 2 as follows

Corollary 3. *Let $n\geq 2$ be a fixed positive integer, R be a $n!-$torsion free prime ring. If there exists a permuting n-derivation $\Delta\colon R^n\longrightarrow R$ such that the trace δ of Δ is centralizing on R, then $\Delta=0$ or R is commutative.*

We continue with the next theorem for symmetric n-derivations which present a more general situation than P-centralizing and the identity $\big[[\delta(x),x],x\big]\in P$ for all $x\in R$. In fact we suggest to extend the result of [[9], Theorem 2] proving that if $n\geq 2$ is a fixed positive integer and R is a $(n+1)!$-torsion free semiprime ring admitting a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ satisfies $[[\delta(x),x],x] \in Z(R)$ for all $x \in R$, then δ is commuting on R.

Theorem 2. *Let $n \geq 2$ be a fixed positive integer, P be a semiprime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If R admits a permuting n-derivation Δ such that the trace δ of Δ satisfies $\overline{[[\delta(x),x],x]} \in Z(R/P)$ for all $x \in R$, then δ is P-commuting.*

Proof. Assume that

$$\Big[[[\delta(x),x],x],r\Big] \in P \quad \text{for all} \;\; r,x \in R. \tag{2.24}$$

Consider a positive integer $k, 1 \leq k \leq n+1$. Replacing x by $x+ky$ in (2.24), we obtain

$$kQ_1(x,y)+k^2Q_2(x,y)+\cdots+k^{n+1}Q_{n+1}(x,y) \in P \quad \text{for all} \;\; x,y \in R \tag{2.25}$$

where $Q_i(x,y)$ denotes the sum of the terms in which y appears i times. Using (2.25) together with Lemma 1 we have

$$\Big[[[\delta(x),x],y],r\Big]+\Big[[[\delta(x),y],x],r\Big]+n\Big[[[\Delta(x,x,\ldots,y),x],x],r\Big] \in P. \tag{2.26}$$

Replacing y by xy in (2.26), we get

$$\Big[(n+2)\big[[\delta(x),x],x\big]y+x\Big(\big[[\delta(x),x],y\big]+\big[[\delta(x),y],x\big]+ n\big[[\Delta(x,x,\ldots,y),x],x\big]\Big)+n\delta(x)\big[[y,x],x\big]+(2n+1)[\delta(x),x][y,x],r\Big] \in P. \tag{2.27}$$

Combining (2.26) with (2.27), we find that

$$(3n+3)\big[[\delta(x),x],x\big]\,[y,x]+(3n+1)[\delta(x),x]\,\big[[y,x],x\big]+ n\delta(x)\Big[\big[[y,x],x\big],x\Big] \in P. \tag{2.28}$$

Substituting $\delta(x)$ for y in (2.28) to get

$$(6n+4)\big[[\delta(x),x],x\big]\,[\delta(x),x] \in P.$$

Commuting with x gives

$$(6n+4)\big[[\delta(x),x],x\big]^2 \in P. \tag{2.29}$$

Replacing y by $[\delta(x),x]$ in (2.28), we arrive at

$$(3n+3)\big[[\delta(x),x],x\big]^2 \in P. \tag{2.30}$$

Now combining (2.29) and (2.30) to get

$$2\big[[\delta(x),x],x\big]R\big[[\delta(x),x],x\big] \in P.$$

Since R is $(n+1)!$-torsion free and also the center of a semi-prime ring is free from nilpotent elements, we have $\left[[\delta(x),x],x\right] \in P$. Using the result of Theorem 1 we conclude that δ is P-commuting. □

Combining Theorem 2 with Lemma 2, we can prove the following corollary which generalize the main result of ([9], Corollary 1).

Corollary 4. *Let $n \geq 2$ be a fixed positive integer, P be a prime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If R admits a permuting n-derivation Δ such that the trace δ of Δ satisfies $\overline{\left[[\delta(x),x],x\right]} \in Z(R/P)$ for all $x \in R$, then $\Delta(R^n) \subseteq P$ or R/P is commutative.*

The following example proves that the `semiprimeness` hypothesis in Theorems 1 and 2 is necessary.

Example 1. Let us consider $R = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & z_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} | x_1, x_2, x_3, y_1, y_2, z_1 \in \mathbb{Z} \right\}$ provided with the derivation defined by $d_A(X) = [A,X]$ with $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

It is straightforward to check that for $X = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ we have $[d_A(X), X] = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0_R$ proving that d_A is not commuting. Moreover, $[[d_A(X), X], Y] = 0$ for all $X, Y \in R$ and thus d_A is centralizing on R. Hence d_A satisfies the condition of Theorem 1 and 2 but d_A in not commuting.

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