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BRUNN-MINKOWSKI INEQUALITY FOR L_p-MIXED INTERSECTION BODIES

CHANG-JIAN ZHAO AND MIHÁLY BENCZE

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Abstract. In this paper, we establish L_p-Brunn-Minkowski inequality for dual Quermassintegral of L_p-mixed intersection bodies. As application, we give the well-known Brunn-Minkowski inequality for mixed intersection bodies.

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1. INTRODUCTION

The intersection operator and the class of intersection bodies were defined by Lutwak [9]. The closure of the class of intersection bodies was studied by Goody, Lutwak, and Weil [5]. The intersection operator and the class of intersection bodies played a critical role in Zhang [12] and Gardner [2] on the solution of the famous Busemann-Petty problem (See also Gardner, Koldobsky, Schlumprecht [4]).

As Lutwak [9] shows (and as is further elaborated in Gardner's book [3]), there is a kind of duality between projection and intersection bodies. Consider the following illustrative example: It is well known that the projections (onto lower dimensional subspaces) of projection bodies are themselves projection bodies. Lutwak conjectured the "dualiy": When intersection bodies are intersected with lower dimensional subspaces, the results are intersection bodies (within the lower dimensional subspaces). This was proven by Fallert, Goodey and Weil [1].

In [7] (see also [10] and [8]), Lutwak introduced mixed projection bodies and proved the following Brunn-Minkowski inequality for mixed projection bodies:

Theorem 1. *If $K, L \in \mathcal{K}^n$ and $0 \le i < n$, then*

$$W_i(\mathbf{P}(K+L))^{1/(n-i)(n-1)} \ge W_i(\mathbf{P}K)^{1/(n-i)(n-1)} + W_i(\mathbf{P}L)^{1/(n-i)(n-1)}, \quad (1.1)$$

with equality if and only if K and L are homothetic.

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Where, $\mathcal{K}^n$ denotes the set of convex bodies in $\mathbb{R}^n$.

$$W_i(K) = V(\underbrace{K,\ldots,K}_{n-i},\underbrace{B,\ldots,B}_{i})$$

denotes the classical Quermassintegral of convex body K. $\mathbf{P}K$ denotes the projection body of convex body K.

In 2008, the Brunn-Minkowski inequality for mixed intersection bodies was established as follows [13].

Theorem 2. *If $K,L\in\varphi^n$, $0\le i<n$, then*

$$\tilde{W}_i(\mathbf{I}(K\tilde{+}L))^{1/(n-i)(n-1)} \le \tilde{W}_i(\mathbf{I}K)^{1/(n-i)(n-1)} + \tilde{W}_i(\mathbf{I}L)^{1/(n-i)(n-1)}, \tag{1.2}$$

with equality if and only if K and L are dilates.

Where, φ^n denotes the set of star bodies in $\mathbb{R}^n$. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot): S^{n-1}\to\mathbb{R}$, defined for $u\in S^{n-1}$, by

$$\rho(K,u) = Max\{\lambda\ge 0:\lambda u\in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Moreover, $\mathbf{I}K$ denotes the intersection body of star body K and the sum $\tilde{+}$ denotes the radial Minkowski sum and $\tilde{W}_i(K) = \tilde{V}(\underbrace{K,\ldots,K}_{n-i},\underbrace{B,\ldots,B}_{i})$ denotes the classical dual Quermassintegral of star body K.

In 2006, Haberl and Ludwig [6] introduced L_p-intersection bodies($p<1$). For $K\in\mathcal{P}_0^n$, where $\mathcal{P}_0^n$ denotes the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors. The star body $\mathbf{I}_p^+K$ is defined for $u\in S^{n-1}$ by

$$\rho(\mathbf{I}_p^+K,u)^p = \int_{K\cap u^+}|u\cdot x|^{-p}dx, \tag{1.3}$$

where $u^+=\{x\in\mathbb{R}^n: u\cdot x\ge 0\}$, and define $\mathbf{I}_p^-K=\mathbf{I}_p^+(-K)$. For $p<1$, the centrally symmetric star body $\mathbf{I}_pK = \mathbf{I}_p{}^+K + \mathbf{I}_p{}^-K$ is called as the L_p *intersection body* of K. So for $u\in S^{n-1}$,

$$\rho^p(\mathbf{I}_pK,u) = \int_K |u\cdot x|^{-p}dx. \tag{1.4}$$

The purpose of this paper is to establish Brunn-Minkowski inequality for L_p-mixed intersection bodies as follows

Theorem 3. *If $K,L\in\varphi^n$, and $0\le i<n$, then for $p<1$*

$$\tilde{W}_i(\mathbf{I}_p(K\tilde{+}L))^{1/(n-i)(n-1)} \le \tilde{W}_i(\mathbf{I}_pK)^{1/(n-i)(n-1)} + \tilde{W}_i(\mathbf{I}_pL)^{1/(n-i)(n-1)}, \tag{1.5}$$

with equality if and only if K and L are dilates.

Where, $\mathbf{I}_pK$ denotes the above L_p-intersection body of star body K which was defined by Haberl and Ludwig [6].

Remark 1. Let $p \to 1^-$ in (1.5), (1.5) changes to (1.2).

To prove Theorem 3, the paper first introduce a new notion L_p*-dual mixed volumes*, then generalize Haberl and Ludwig's L_p-intersection bodies to L_p-mixed intersection bodies ($p < 1$). Moreover, we use a new way which is different from the way of [13].

2. Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathbb{C}^n$ denote the set of non-empty convex figures(compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathbb{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of convex body K. The support function of $K \in \mathcal{K}^n$, $h(K,\cdot)$, defined on $\mathbb{R}^n$ by $h(K,\cdot) = Max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \varphi^n$, then $\tilde{\delta}(K,L) = |\rho_K - \rho_L|_\infty$.

2.1. L_p-*dual mixed volumes*

We define vector addition $\tilde{+}$ on $\mathbb{R}^n$, which we shall call the radial addition, as follows. For any $x_1, \ldots, x_r \in \mathbb{R}^n$, $x_1 \tilde{+} \cdots \tilde{+} x_r$ is defined to be the usual vector sum of $x_1, \ldots, x_r$ if they all lie in a 1-dimensional subspace of $\mathbb{R}^n$, and as the zero vector otherwise.

If $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}.$$

The following property will be used later. If $K, L \in \varphi^n$ and $\lambda, \mu \geq 0$

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot). \tag{2.1}$$

For $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski liner combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i [11],

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n} \tag{2.2}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (2.1.2) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is written as

$\tilde{V}(K_{i_1},\ldots,K_{i_n})$ and is called the *dual mixed volume* of $K_{i_1},\ldots,K_{i_n}$. If $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=L$, the dual mixed volumes is written as $\tilde{V}_i(K,L)$. The dual mixed volumes $\tilde{V}_i(K,B)$ is written as $\tilde{W}_i(K)$.

If $K_i\in\varphi^n(i=1,2,\ldots,n-1)$, then the dual mixed volume of $K_i\cap E_u(i=1,2,\ldots,n-1)$ will be denoted by $\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)$. If $K_1=\ldots=K_{n-1-i}=K$ and $K_{n-i}=\ldots=K_{n-1}=L$, then $\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)$ is written $\tilde{v}_i(K\cap E_u,L\cap E_u)$. If $L=B$, then $\tilde{v}_i(K\cap E_u,B\cap E_u)$ is written $\tilde{w}_i(K\cap E_u)$.

L_p-dual mixed volumes was defined as follows [14].

$$\tilde{V}_p(K_1,\ldots,K_n)=\omega_n\left(\frac{1}{n\omega_n}\int_{S^{n-1}}\rho^p(K_1,u)\cdots\rho^p(K_n,u)dS(u)\right)^{1/p},\ p\neq 0,\tag{2.3}$$

where $K_1,\ldots,K_n\in\varphi^n$.

If $K_1=\ldots=K_{n-1-i}=K$ and $K_{n-i}=\ldots=K_{n-1}=L$, will write $\tilde{V}_p(\underbrace{K,\ldots,K}_{n-1-i},\underbrace{L,\ldots,L}_{i})$ as $\tilde{V}_{p,i}(K,L)$. If $K_1=\ldots=K_n=K$, will write $\tilde{V}_p(\underbrace{K,\ldots,K}_{n})$ as $\tilde{V}_p(K)$. If $L=B$, then write $\tilde{V}_p(\underbrace{K,\ldots,K}_{n-i},\underbrace{B,\ldots,B}_{i})$ as $\tilde{V}_{p,i}(K)$ and is called L_p-dual Quermassintegral as follows.

$$\tilde{V}_{p,i}(K)=\omega_n\left(\frac{1}{n\omega_n}\int_{S^{n-1}}\rho^{p(n-i)}(K,u)dS(u)\right)^{1/p},\quad p\neq 0.\tag{2.4}$$

Remark 2. Apparently, let $p=1$, then L_p-dual mixed volumes $\tilde{V}_p$ and L_p-dual Quermassintegral $\tilde{V}_{p,i}$ change to the classical dual mixed volumes $\tilde{V}$ and dual Quermassintegral $\tilde{W}_i$, respectively.

2.2. *L_p-mixed intersection bodies*

Since [6]

$$v(K\cap u^+)=\lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\int_K|u\cdot x|^{-1+\varepsilon}dx.\tag{2.5}$$

and

$$\rho(\mathbf{I}K,u)=\lim_{p\to 1^-}\frac{1-p}{2}\rho^p(\mathbf{I}_pK,u),\tag{2.6}$$

that is, the intersection body of K is obtained as a limit of L_p intersection bodies of K. Also note that a change to polar coordinates in (2.6) shows that up to a normalization factor $\rho^p(\mathbf{I}_pK,u)$ equals the Cosine transform of $\rho(K,u)^{n-p}$.

Here, we introduce the L_p*-mixed intersection bodies* of $K_1,\ldots,K_{n-1}$. It is written as $\mathbf{I}_p(K_1,\ldots,K_{n-1})(p<1)$, whose radial function is defined by

$$\rho^p(\mathbf{I}_p(K_1,\ldots,K_{n-1}),u)=\frac{2}{1-p}\tilde{v}_p^*(K_1\cap E_u,\ldots,K_{n-1}\cap E_u),\tag{2.7}$$

where, $\tilde{v}_p^*(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$ denotes the p-dual mixed volumes of $K_1 \cap E_u, \ldots, K_{n-1} \cap E_u$ in $(n-1)$-dimensional space. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $\tilde{v}_p^*(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$ is written as $\tilde{v}_{p,i}^*(K \cap E_u, L \cap E_u)$. If $L = B$, then $\tilde{v}_{p,i}^*(K \cap E_u, L \cap E_u)$ is written as $\tilde{v}_{p,i}^*(K \cap E_u)$.

Remark 3. From the definition, which introduces a new star body, namely the L_p-mixed intersection body of $n-1$ given bodies.

From the definition, $V_p(K_1, \ldots, K_n)$ is continuous function for any $K_i \in \varphi^n$, $i = 1, 2, \ldots, n$, then

$$\begin{aligned}
&\lim_{p \to 1^-} \frac{1-p}{2} \rho^p(\mathbf{I}_p(K_1, \ldots, K_{n-1}), u) \\
&\qquad = \lim_{p \to 1^-} \omega_n \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho^p(K_1, u) \cdots \rho^p(K_{n-1}, u) dS(u) \right)^{1/p} \\
&\qquad = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_{n-1}, u) dS(u).
\end{aligned}$$

On the other hand, by using definition of mixed intersection bodies(see [3] and [14]), we have

$$\begin{aligned}
\rho(\mathbf{I}(K_1, \ldots, K_{n-1}), u) &= \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u) \\
&= \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_{n-1}, u) dS(u).
\end{aligned}$$

Hence

$$\lim_{p \to 1^-} \frac{1-p}{2} \rho^p(\mathbf{I}_p(K_1, \ldots, K_{n-1}), u) = \rho(\mathbf{I}(K_1, \ldots, K_{n-1}), u).$$

For the L_p-mixed intersection bodies, $\mathbf{I}_p(K_1, \ldots, K_{n-1})$, if $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{I}_p(K_1, \ldots, K_{n-1})$ is written as $\mathbf{I}_p(K, L)_i$. If $L = B$, then $\mathbf{I}_p(K, L)_i$ is written as $\mathbf{I}_p K_i$ is called the ith L_p-intersection body of K. For $\mathbf{I}_p K_0$ simply write $\mathbf{I}_p K$, this is just the L_p-intersection bodies of star body K.

The following properties will be used later: If $K, L, M, K_1, \ldots, K_{n-1} \in \varphi^n$, and $\lambda, \mu, \lambda_1, \ldots, \lambda_{n-1} > 0$, then

$$\mathbf{I}_p(\lambda K \tilde{+} \mu L, M) = \lambda \mathbf{I}_p(K, M) \tilde{+} \mu \mathbf{I}_p(L, M), \tag{2.8}$$

where $M = (K_1, \ldots, K_{n-2})$.

3. Main results

3.1. *Some Lemmas*

The following results will be required to prove our main Theorems.

Lemma 1. *If $K, L \in \varphi^n$, $0 \le i < n, 0 \le j < n-1, i, j \in \mathbb{N}$ and $p < 1$, then*

$$\tilde{W}_i(\mathbf{I}_p(K,L)_j) = \frac{1}{n}\left(\frac{2}{1-p}\right)^{\frac{n-i}{p}} \int_{S^{n-1}} \tilde{v}^*_{p,j}(K \cap E_u, L \cap E_u)^{\frac{(n-i)}{p}} dS(u). \quad (3.1)$$

From (2.4) and (2.7), identity (3.1) in Lemma 1 easy follows.

Lemma 2. *If $K_1, \dots, K_n \in \varphi^n$, $1 < r \le n$, $0 \le j < n-1, j \in \mathbb{N}$ and $p \ne 0$, then*

$$\tilde{V}_p(K_1, \dots, K_n)^r \le \prod_{j=1}^{r} \tilde{V}_p(\underbrace{K_j, \dots, K_j}_{r}, K_{r+1}, \dots, K_n), \quad (3.2)$$

with equality if and only if $K_1, \dots, K_n$ are all dilations [14].

From (3.1), (3.2) and in view of Hölder inequality for integral, we obtain

Lemma 3. *If $K, L \in \varphi^n$, $0 \le i < n$, $0 < j < n-1$, and $p < 1$, then*

$$\tilde{W}_i(\mathbf{I}_p(K,L))^{n-1} \le \tilde{W}_i(\mathbf{I}_p K)^{n-j-1} \cdot \tilde{W}_i(\mathbf{I}_p L)^j, \quad (3.3)$$

with equality if and only if K and K are dilations.

3.2. *Brunn-Minkowski inequality for L_p-mixed intersection bodies*

The Brunn-Minkowski inequality for L_p-intersection bodies, which will be established is: If $K, L \in \varphi^n$, $p < 1$ then

$$V(\mathbf{I}_p(K \tilde{+} L))^{1/n(n-1)} \le V(\mathbf{I}_p K)^{1/n(n-1)} + V(\mathbf{I}_p L)^{1/n(n-1)}, \quad (3.4)$$

with equality if and only if K and L are dilates.

This is just the special case $i = 0$ of:

Theorem 4. *If $K, L \in \varphi^n$, and $0 \le i < n$, then*

$$\tilde{W}_i(\mathbf{I}_p(K \tilde{+} L))^{1/(n-i)(n-1)} \le \tilde{W}_i(\mathbf{I}_p K)^{1/(n-i)(n-1)} + \tilde{W}_i(\mathbf{I}_p L)^{1/(n-i)(n-1)}, \quad (3.5)$$

with equality if and only if K and L are dilates.

Proof. Let $M = (L_1, \dots, L_{n-2})$, from (2.1), (2.4), (2.8) and in view of Minkowski inequality for integral, we obtain that

$$\begin{aligned}
\tilde{W}_i(\mathbf{I}_p(K \tilde{+} L, M))^{1/(n-i)} &= n^{-1/(n-i)} \|\rho(\mathbf{I}_p(K \tilde{+} L, M), u)\|_{n-i} \\
&= n^{-1/(n-i)} \|\rho(\mathbf{I}_p(K, M) \tilde{+} \mathbf{I}_p(L, M), u)\|_{n-i} \\
&= n^{-1/(n-i)} \|\rho(\mathbf{I}_p(K, M), u) + \rho(\mathbf{I}_p(L, M), u)\|_{n-i} \\
&\le n^{-1/(n-i)} \left(\|\rho(\mathbf{I}_p(K, M), u)\|_{n-i} + \|\rho(\mathbf{I}_p(L, M), u)\|_{n-i}\right) \\
&= \tilde{W}_i(\mathbf{I}_p(K, M))^{1/(n-i)} + \tilde{W}_i(\mathbf{I}_p(L, M))^{1/(n-i)}. \quad (3.6)
\end{aligned}$$

On the other hand, taking $L_1 = \cdots = L_{n-2} = K \tilde{+} L$ to (3.6) and apply Lemma 2 and Lemma 3, and get

$$\begin{aligned}
&\tilde{W}_i(\mathbf{I}_p(K\tilde{+}L))^{1/(n-i)} \leq \\
&\qquad \tilde{W}_i\mathbf{I}_p(K,K\tilde{+}L)_{n-2})^{1/(n-i)} + \tilde{W}_i(\mathbf{I}_p(L,K\tilde{+}L)_{n-2})^{1/(n-i)} \\
&\qquad \leq \tilde{W}_i(\mathbf{I}_pK)^{1/(n-1)(n-i)}\tilde{W}_i(\mathbf{I}_p(K\tilde{+}L))^{(n-2)/(n-1)(n-i)} \\
&\qquad\qquad + \tilde{W}_i(\mathbf{I}_pL)^{1/(n-1)(n-i)}\tilde{W}_i(\mathbf{I}_p(K\tilde{+}L))^{(n-2)/(n-1)(n-i)}, \qquad (3.7)
\end{aligned}$$

with equality if and only if K, L and $M = K\tilde{+}L$ are dilates, combine this with the equality condition of (3.6), it follows that the condition holds if and only if K and L are dilates.

Dividing both sides of (3.7) by $\tilde{W}_i(\mathbf{I}_p(K\tilde{+}L))^{(n-2)/(n-1)(n-i)}$, we get the inequality (3.5).

The proof is complete. □

Remark 4. Let $i = 0$ and $p \to 1^-$ in (2.6), we get the well-known Brunn-Minkowski inequality for mixed intersection bodies as follows:

$$\tilde{V}(\mathbf{I}(K\tilde{+}L))^{1/n(n-1)} \leq \tilde{V}(\mathbf{I}K)^{1/n(n-1)} + \tilde{V}(\mathbf{I}L)^{1/n(n-1)}$$

with equality if and only if K and L are dilates.

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Authors' addresses

Chang-Jian Zhao
Department of Mathematics, China Jiliang University, Hangzhou 310018, P.R.China
E-mail address: chjzhao@163.com.com chjzhao@aliyun.com

Mihály Bencze
Str. Hărmanului 6, 505600 Săcele-Něgyfalu, Jud, Braşov, Romania, Romania
E-mail address: benczemihaly@yahoo.com benczemihaly@gmail.com