Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1477–1493
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4159

NEW FIXED POINT RESULTS OF SOME ENRICHED CONTRACTIONS IN CAT(0) SPACES

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Received 04 March, 2022

Abstract. Enriched contraction, a class that contains the Picard -Banach contractions and some nonexpansive mappings, are generalized from Banach space framework into a nonlinear context, namely, geodesic metric spaces of nonpositive curvature. We establish general implications that extend the well-known results for enriched mappings formed on Banach spaces. We also look at the results for fixed points involving the contractions i.e., the limit shadowing property and well-posedness as well as some instances to back up our findings.

2010 *Mathematics Subject Classification:* 47H10; 54H25; 54E50

Keywords: CAT(0) spaces, enriched contraction, fixed point, iterative method, strong convergence

1. INTRODUCTION

In the analysis of the solutions to nonlinear functional systems, fixed point theory provides useful tools. The desired solution of a functional equation is written as the fixed point of an appropriate operator i.e.,

$$a = Sa,$$

where S is a self map defined on a nonempty set E.

A variety of contractions have been proposed by a number of mathematicians and employed them in a variety of frameworks to produce fixed points, common fixed points, and coincidence points for a map. Enriched contraction is one such contraction, which was very recently introduced by Berinde and Pacurar [2]. First, let's review the notion of enriched contraction.

"Let S be a self map on a normed linear space E. S is said to be an enriched contraction if there exist $b^* \in [0,\infty)$ and $\theta \in [0, b^*+1)$ such that

$$\|b^*(a-b) + Sa - Sb\| \leq \theta \|a-b\|$$

for all $a,b \in E$. In this situation the map S is also known as a (b^*,θ)-enriched contraction." Picard-Banach contractions along with several nonexpansive maps, are included in the enriched contractions category.

Many of the contractions used to solve fixed point problems have gradually been generalized from linear spaces to differentiable manifolds. We continue along these lines by introducing symmetric contractions in the context of CAT(0) spaces. In addition, Mondal et al. [9] published a work presenting two different kinds of enriched contractions in Banach spaces.

The goal of this paper is to introduce enriched contractions into nonpositive curvature metric spaces. Fixed point results of enriched contractions are next investigated, and strong convergence theorems for the Kransnoselskii iteration which is adapted to approximate fixed points, are established. Examples are also given to demonstrate the generality of our new findings.

2. Preliminaries

Considering a self map S on a convex subset C of a linear space E, then for any $\lambda \in (0,1)$, the so-called averaged mapping (a term coined in [1]) S_λ given by

$$S_\lambda a = \lambda a + (1-\lambda)Sa, \qquad \forall a \in C.$$

Lemma 1. *[4] Let (E, d) be a CAT(0) space. Then*

$$d((1-\lambda)a \oplus \lambda b, c) \le (1-\lambda)d(a,c) + \lambda d(b,c)$$

for all $a,b,c \in E$ and $\lambda \in [0,1]$.

Proposition 1. *[4] Let (E, d) be a CAT(0) space. For each $\lambda \in [0,1]$ there is a unique point $c \in [a,b]$ such that*

$$d(a,c) = \lambda d(a,b), \qquad d(b,c) = (1-\lambda)d(a,b),$$

for all $a,b \in E$.

The following lemma is a partial extension from Banach space to CAT(0) space setting of a conclusion presented in Corollary to Theorem 5 in [5] (it can be also found in [10]).

Lemma 2. *Assume that S is a self map on CAT(0) space E. Define the map $S_\lambda a : E \to E$ by*

$$S_\lambda a = \lambda a \oplus (1-\lambda)Sa, \qquad a \in E$$

Then, for any $\lambda \in [0,1)$,

$$Fix(S) = Fix(S_\lambda).$$

Proof. $S_\lambda = S$ when $\lambda = 0$, and the claim is simple. Consider that $\lambda \in (0,1)$ and that $a \in Fix(S)$. This suggests $a = Sa$ and as a result

$$d(a, S_\lambda a) = d(a, \lambda a \oplus (1-\lambda)Sa) \le \lambda d(a,a) + (1-\lambda)d(a,Sa) = 0.$$

i.e., $d(a,S_\lambda a)=0 \Rightarrow a \in Fix(S_\lambda)$.

Conversely, assume $a \in Fix(S_\lambda)$. This implies that $d(a,S_\lambda a)=0$, so,

$$d(a,\lambda a \oplus (1-\lambda)Sa)=0.$$

By Proposition 1,

$$\begin{aligned} d(a,Sa) &= \lambda d(a,Sa) \\ d(a,Sa)(1-\lambda) &= 0 \end{aligned}$$

given that $\lambda \neq 1$, this means that $d(a,Sa)=0$. □

3. ENRICHED CONTRACTIONS IN CAT(0) SPACE

Definition 1. Let (E,d) be a CAT(0) space. If there exist $c^* \in [0,1)$ and $\lambda \in [0,1)$, then a self mapping S defined on E is an enriched contraction such that

$$d(\lambda a \oplus (1-\lambda)Sa, \lambda b \oplus (1-\lambda)Sb) \leq c^* d(a,b), \forall a,b \in E.$$

(1) S is also known as a (λ, c^*)-enriched contraction because it specifies the parameters c^* and λ. "A $(0,c^*)$-enriched contraction is a usual Banach contraction which can be easily seen."

(2) Let $E=[0,1]\times[0,1]$. Define the radial distance d_r between $a,b \in E$ to be the usual distance if they are on the same ray emanating from origin; otherwise take

$$d_r(a,b)=d(a,0)+d(0,b).$$

(Here d denotes the usual Euclidean distance and 0 denotes the origin.) Note that d_r is a metric on E and is known as a radial metric on E ([8]). Let S be a self map on E defined as $Sa=-a$, for all $a \in E$. Then S is an enriched contraction, not a contraction.

However, if S is a contraction, then there would exist $c^* \in [0,1)$ by the definition of contraction, such that

$$d(Sa,Sb)=d(-a,-b)=\|b-a\|=d(a,b)\leq c^*.d(a,b), \forall a,b \in [0,1]\times[0,1].$$

This leads to the contradiction $1 \leq c^* < 1$ for any $a \neq b$. Take $a,b \in E$ and $\lambda = \frac{1}{2}$. If a and b are on the same ray emanating from the origin, the enriched contraction condition is identical to

$$d\left(\frac{1}{2}a+\frac{1}{2}Sa, \frac{1}{2}b+\frac{1}{2}Sb\right) \leq c^* d(a,b).$$

Now we show that the above inequality holds for all $a,b \in E$.

$$\begin{aligned} d\left(\frac{1}{2}a+\frac{1}{2}Sa, \frac{1}{2}b+\frac{1}{2}Sb\right) &= \frac{1}{2}\|a+Ta-b-Tb\| = \frac{1}{2}\|a-b+Ta-Tb\| \\ &= \frac{1}{2}\|a-b-a+b\| = 0 \leq c^* d(a,b), \end{aligned}$$

which is true. Now, if a and b are not on the same ray emanating from the origin, then enriched contraction condition becomes:

$$d_r\left(\frac{1}{2}a+\frac{1}{2}Sa, \frac{1}{2}b+\frac{1}{2}Sb\right) \leq c^* d_r(a,b),$$

which can equivalently be written :

$$d\left(\frac{1}{2}a+\frac{1}{2}Sa,0\right)+d\left(0,\frac{1}{2}b+\frac{1}{2}Sb\right) \leq c^* d(a,0)+d(0,b)$$

$$\frac{1}{2}(|a+Sa|+|b+Sb|) \leq c^*|a|+|b|$$

$$\Rightarrow \qquad \frac{1}{2}(|a+(-a)|+|b+(-b)|)=0 \leq \ c^*|a|+|b|,$$

which shows that S is an enriched contraction.

Theorem 1. *A self map S defined on CAT(0) space E is a (λ, c^*)-enriched contraction. Then,*

(a): *$Fix(S)=\{p\}$, for some $p \in E$.*

(b): *The sequence $\{a_n\}$ obtained from the Krasnoselskii iterative method [3]*

$$a_{n+1} = \lambda a_n \oplus (1-\lambda)Ta_n, n \geq 0, \tag{3.1}$$

converges to p, for any $a_0 \in E$.

(c): *Also, the following holds*

$$d(a_{n+i-1},p) \leq \frac{(c^*)^i}{1-(c^*)} \cdot d(a_n,a_{n-1}) \qquad n=1,2,\ldots;i=1,2,\ldots$$

Proof. The mapping $S_\lambda : E \to E$ described by $S_\lambda a = (1-\lambda)a \oplus \lambda Sa$ satisfies enriched contractive condition, so

$$d(S_\lambda a, S_\lambda b) \leq c^* \cdot d(a,b), \tag{3.2}$$

for all $a,b \in E$, i.e., S_λ is a contraction. We also notice that the Picard iteration linked with S_λ is the Krasnoselskii iterative process $\{a_n\}$ linked with S and described by (3.2), i.e.,

$$a_{n+1} = S_\lambda a_n, n \geq 0. \tag{3.3}$$

Now, we take $a=a_n$ and $b=a_{n-1}$ in (3.2) to get

$$d(a_{n+1},a_n) \leq c^* \cdot d(a_n,a_{n-1}), n \geq 1, \tag{3.4}$$

which inductively implies

$$d(a_{n+1},a_n) \leq (c^*)^n \cdot d(a_1,a_0), n \geq 1, \tag{3.5}$$

We can derive from $c^* \in [0,1)$ that

$$\lim_{n\to\infty} d(a_{n+1},a_n) = 0, \tag{3.6}$$

i.e., $\{a_n\}$ is asymptotically regular. for all n and $k>0$, we have the following using the triangle inequality and (3.2),

$$\begin{aligned} d(a_{n+k+1},a_{n+1}) &\le c^*\cdot d(a_{n+k},a_n) \\ &\le c^*[d(a_{n+k},a_{n+k+1})+d(a_{n+k+1},a_{n+1})+d(a_{n+1},a_n)] \\ &\le \frac{c^*}{1-c^*}[d(a_{n+k},a_{n+k+1})+d(a_{n+1},a_n)]. \end{aligned}$$

As a result of (3.6), $\lim_{n\to\infty} d(a_{n+k+1},a_{n+1})=0$, uniformly w.r.t. k, demonstrating that $\{a_n\}$ is Cauchy sequence. Hence, $\{a_n\}$ is convergent and let us represent

$$p=\lim_{n\to\infty} a_n. \tag{3.7}$$

By taking $n\to\infty$ in (3.3) and employing the continuity of S_λ (which follows by the fact that S_λ is contraction), we obtain immediately

$$p=S_\lambda p,$$

i.e., $p\in Fix(S_\lambda)$.

Next, we show that p is the only fixed point of S_λ. Suppose that $q\neq p$ is yet another fixed point of S_λ. After that, by (3.2),

$$0<d(p,q)=d(S_\lambda p,S_\lambda q)\le c\cdot d(p,q)<d(p,q),$$

This is a contradiction. As a result, $Fix(S_\lambda)=\{p\}$ and, as a result of Lemma 2, $p\in Fix(S)$, which demonstrates (a).

(3.7) proves the conclusion (b).

To prove (c), first by (3.4) and (3.5), The following estimates are obtained commonly,

$$d(a_{n+m},a_n)\le (c^*)^n\cdot\frac{(1-(c^*)^m)}{1-c^*}\cdot d(a_1,a_0), \qquad n\ge 1, m\ge 1. \tag{3.8}$$

$$d(a_{n+m},a_n)\le \frac{c^*}{1-c^*}\cdot d(a_n,a_{n-1}), \qquad n\ge 1, m\ge 1. \tag{3.9}$$

By taking $m\to\infty$ in (3.8) and (3.9), we obtain

$$d(a_n,p)\le\frac{(c^*)^n}{1-c^*}\cdot d(a_1,a_0), n\ge 1 \tag{3.10}$$

and

$$d(a_n,p)\le\frac{c^*}{1-c^*}\cdot d(a_n,a_{n-1}), n\ge 1, \tag{3.11}$$

respectively. Finally, (3.10) and (3.11) can be combined to obtain the unifying error estimate in conclusion (c). □

The following example support the above theorem:

Example 1. Let $E=[0,1]$ and consider the usual metric d on E. Define a function $S:E\to E$ by

$$Sa=1-a$$

for all $a\in[0,1]$. Here, S is an enriched contraction. We note that Picard iteration $\{x_n\}$ defined by $a_{n+1}=Sa_n$ does not converge, for any initial value a_0 except the unique fixed point $\frac{1}{2}$ of S while Krasnoselskij iteration $\{x_n\}$ defined by (3.1) converges for any $\lambda\in(0,1)$ and any initial value $a_0\in[0,1]$. We also observe that Krasnoselskij iteration converges faster for λ closer to but different from the unique fixed point $\frac{1}{2}$. The fact is illustrated by some numerical experiments performed for the starting value $a_0=1$ and for four different values of the control parameter $\lambda,\lambda\in\{0.1,0.25,0.3,0.49\}$ which are presented in Table 1. Also graphical representation is also given in Figure 1

TABLE 1. **Results of the numerical experiments for** $Ta=1-a$

$\lambda=0.1$	$\lambda=0.25$	$\lambda=0.3$	$\lambda=0.49$
1	1	1	1
0.1	0.25	0.3	0.49
0.82	0.625	0.58	0.5002
0.244	0.4375	0.468	0.499996
0.7048	0.53125	0.5128	0.50000008
0.33616	0.484375	0.49488	0.499999998
0.631072	0.5078125	0.502048	0.5
0.3951424	0.49609375	0.4991808	0.5
0.58388608	0.501953125	0.50032768	0.5
0.432891136	0.499023438	0.499868928	0.5
0.553687091	0.500488281	0.500052429	0.5
0.457050327	0.499755859	0.499979028	0.5
0.534359738	0.50012207	0.500008389	0.5
0.472512209	0.499938965	0.499996645	0.5
0.521990233	0.500030518	0.500001342	0.5
0.482407814	0.499984741	0.499999463	0.5
0.514073749	0.500007629	0.500000215	0.5
0.488741001	0.499996185	0.499999914	0.5
0.509007199	0.500001907	0.500000034	0.5
0.492794241	0.499999046	0.499999986	0.5
0.505764608	0.500000477	0.500000005	0.5
0.495388314	0.499999762	0.499999998	0.5
0.503689349	0.500000119	0.500000001	0.5
0.497048521	0.49999994	0.5	0.5

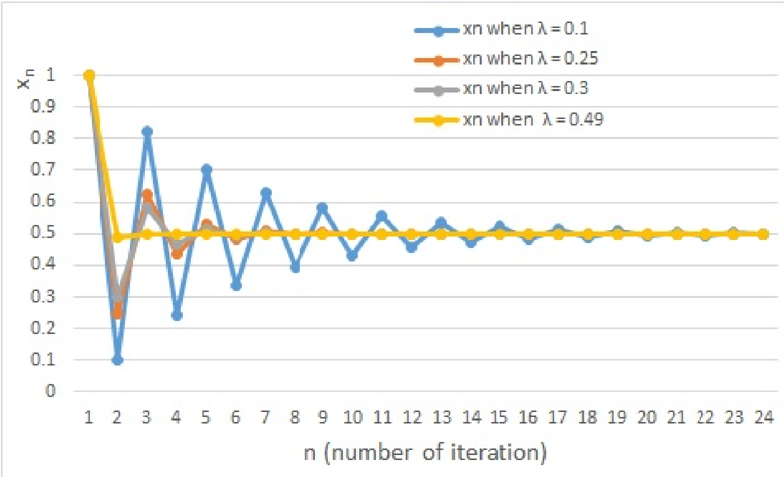

FIGURE 1. Convergence of x_n to the unique fixed point $\frac{1}{2}$ for different values of λ.

Remark 1. The contraction map principle in the context of a CAT(0) space is obtained by Theorem 1 if S is a $(0,c^*)$-contraction. The Krasnoselskii type iterative method simplifies to the Picard iteration in this case.

"The local variant of Banach contraction mapping principle (see, [6]), which involves an open ball B in a complete metric space (E,d) and a nonself contraction map of B into E with the key property of not displacing the center of the ball too far, is important in actual applications." The following theorem is an analogue of this conclusion in the case CAT(0) spaces.

Theorem 2. *Consider (E,d) is a complete CAT(0) space, $a_0 \in E, r > 0, B = B(a_0,r) := \{a \in E : d(a,a_0) < r\}$ and also, $S : B \to E$ is a (λ,c^*)-enriched contraction and*

$$d(Sa_0,a_0) < \frac{1-c^*}{1-\lambda} \cdot r,$$

then S has a fixed point.

Proof. We can select $\varepsilon < r$ in such a way that

$$d(Sa_0,a_0) < \frac{1-c^*}{1-\lambda} \cdot \varepsilon < \frac{1-c^*}{1-\lambda} \cdot r. \tag{3.12}$$

Since S is a (λ,c^*)-enriched contraction, there exists $c^* \in [0,1)$ such that

$$d(S_\lambda a, S_\lambda b) \le c^* d(a,b), \qquad \text{for all } a,b \in B,$$

for any $\lambda \in (0,1)$, where we denote as before $S_\lambda a = (1-\lambda)a \oplus \lambda Sa$.

By Proposition 1, we have

$$d(S_\lambda a_0, a_0) = d((1-\lambda)a_0 \oplus \lambda S a_0, a_0) = (1-\lambda)d(a_0, Sa_0)$$

and therefore (3.12) implies that $d(S_\lambda a_0, a_0) \leq (1-c^*)\varepsilon$.

We now prove that the closed ball $\overline{B_\varepsilon} := \{a \in E : d(a, a_0) \leq \varepsilon\}$ is invariant with respect to S_λ. Indeed, for any $a \in \overline{B_\varepsilon}$, we have

$$d(S_\lambda a, a_0) \leq d(S_\lambda a, S_\lambda a_0) + d(S_\lambda a_0, a_0) \leq c^* d(a, a_0) + (1-c^*)\varepsilon \leq \varepsilon.$$

Since $\overline{B_\varepsilon}$ is complete, the conclusion follows by Theorem 1. □

The following example demonstrates that while some mappings S are not contractions, a certain iterate of them is.

Example 2. Let $E = \mathbb{R}$ and $S : E \to E$ be given by

$$Sa = \begin{cases} 0, & \text{if} \quad a \in (-\infty, 3] \\ \frac{-1}{2} & \text{if} \quad a \in (3, \infty). \end{cases}$$

Then S is not a contraction as it is discontinuous at $a = 3$. But S^2 is a contraction as $S^2 a = 0$, if $a \in (-\infty, \infty)$.

We cannot use Theorem 1 in these instances, but the following result could be beneficial in applications.

Theorem 3. *Let (E, d) be a complete CAT(0) space and let V be a self mapping defined on E with the property that there exists a positive integer N such that V^N is a (λ, c^*)-enriched contraction. Then,*

- **(a):** $Fix(V) = \{p\}$
- **(b):** *The sequence $\{a_n\}$ obtained from the iterative process*

$$a_{n+1} = \lambda a_n \oplus (1-\lambda) S^N a_n, n \geq 0,$$

converges to p, for any $a_0 \in E$.

Proof. We apply Theorem 1 (a) for the mapping $S = V^N$ and obtain that $Fix(V^N) = \{p\}$. We also have

$$V^N(V(p)) = V^{N+1}(p) = V(V^N(p)) = V(p),$$

which shows that $V(p)$ is a fixed point of V^N. However, V^N has a unique fixed point, p, hence $V(p) = p$ and so $p \in Fix(V)$. The remaining part of the proof follows by Theorem 1 □

4. ENRICHED $\mathcal{A}$-CONTRACTION AND ENRICHED $\mathcal{A}'$-CONTRACTION IN CAT(0) SPACES

Mondal et al. [9] proposed two new forms of enriched contractions, and found fixed points of mappings meeting such contractions using the fixed point property of the average operator of a map . We will introduce enriched $\mathcal{A}$-contraction and enriched $\mathcal{A}'$-contraction in the context of CAT(0) space in this section, as follows.

Definition 2. "Let (E,d) be a CAT(0) space. Let $\mathcal{A}$ be the collection of all mappings $f:\mathbb{R}^3_+ \to \mathbb{R}$ satisfying the following conditions:

($\mathcal{A}_1$): f is continuous;
($\mathcal{A}_2$): if $r \leq f(s,r,s)$ or $r \leq f(r,s,s)$, then there exists $k \in [0,1)$ such that $r \leq ks$;
($\mathcal{A}_3$): for $\lambda > 0$ and for all $r,s,t \in \mathbb{R}_+, \lambda f(r,s,t) \leq f(\lambda r, \lambda s, \lambda t)$.

Let $S : E \to E$ be a mapping such that there exist $f \in \mathcal{A}$ with

$$d(S_\lambda a, S_\lambda b) \leq \lambda f((b^*+1)d(a,b), d(a,Sa), d(b,Sb)) \tag{4.1}$$

for all $a,b \in E$ with $a \neq b$ and $b^* \in [0,\infty)$. Then S is said to be an enriched $\mathcal{A}$-contraction."

Example 3. Let $E = [0,1]\times[0,1]$. Define the radial distance d_r between $a,b \in E$ to be the usual distance if they are on the straight ray initiating from origin; otherwise take

$$d_r(a,b) = d(a,0) + d(0,b).$$

(Here d denotes the usual Euclidean distance and 0 denotes the origin.) Let S be a self map on E defined as $Sa = -2a$, for all $a \in E$. S is not an enriched contraction but is an enriched $\mathcal{A}$-contraction.

Indeed, if S would be an enriched contraction, then there would exist $b^* \in [0,+\infty)$ and $\theta \in [0, b^*+1)$ such that $\forall\, a,b \in E$,

$$\begin{aligned} d(\lambda a \oplus (1-\lambda)Sa, \lambda b \oplus (1-\lambda)Sb) &\leq \theta d(a,b) \\ \|b^*(a-b) + Sa - Sb\| &\leq \theta \|(a-b)\|. \end{aligned}$$

Take $\lambda = 0$ and then $\theta \in [0,1)$,

$$\begin{aligned} d(Sa,Sb) &\leq \theta d(a,b) \\ \|Sa - Sb\| &\leq \theta \|(a-b)\| \\ \|-2a+2b\| &\leq \theta \|(a-b)\| \\ 2\|a-b\| &\leq \theta \|(a-b)\|, \end{aligned}$$

which yields to the contradiction $2 \leq 1$.

S is an enriched $\mathcal{A}$-contraction for $b^* = \frac{5}{4}$, $\lambda = \frac{1}{2}$ and $f(r,s,t) = \frac{s+t}{3}$. Assume $a,b \in E$ and a,b are on the straight ray initiating from origin, then the contractive condition becomes;

$$d\left(\frac{1}{2}a+\frac{1}{2}Sa,\frac{1}{2}b+\frac{1}{2}Sb\right) \leq \frac{1}{2}f\left((b^*+1)d(a,b),d(a,Sa),d(b,Sb)\right),$$

which can be written as

$$\begin{aligned}\frac{1}{2}\|a-2a+b-2b\| &\leq \frac{1}{2}f\left((b^*+1)\|a-b\|,\|a-Sa\|,\|b-Sb\|\right)\\ \frac{1}{2}\|-a-b\| &\leq \frac{1}{2}\times\frac{1}{3}\left[\|a-Sa\|+\|b-Sb\|\right]\\ \frac{1}{2}\|a+b\| &\leq \frac{1}{6}\left[\|a+2a\|+\|b+2b\|\right]\\ \frac{1}{2}\|a+b\| &\leq \frac{3}{6}\left[\|a\|+\|b\|\right]=\frac{1}{2}\left[\|a\|+\|b\|\right],\end{aligned}$$

which is true.

Now, if a and b are not on the straight ray initiating from the origin, then the contractive condition becomes;

$$d_r\left(\frac{1}{2}a+\frac{1}{2}Sa,\frac{1}{2}b+\frac{1}{2}Sb\right) \leq \frac{1}{2}f\left((b^*+1)d_r(a,b),d_r(a,Sa),d_r(b,Sb)\right).$$

It gives that

$$\begin{aligned}d\left(\frac{1}{2}a+\frac{1}{2}Sa,0\right)d\left(0,\frac{1}{2}b+\frac{1}{2}Sb\right) &\leq \frac{1}{2}f\left((b^*+1)d(a,0)+d(0,b),d(a,0)\right.\\ &\qquad \left.+d(0,Sa),d(b,0)+d(0,Sb)\right)\\ \left\|\frac{1}{2}a+\frac{1}{2}Ta\right\|+\left\|\frac{1}{2}b+\frac{1}{2}Sb\right\| &\leq \frac{1}{6}[d(a,0)+d(0,Ta)+d(b,0)+d(0,Sb)]\\ \frac{1}{2}\|a-2a\|+\frac{1}{2}\|b-2b\| &\leq \frac{1}{6}[\|a\|+\|2a\|+\|b\|+\|2b\|]\\ \frac{1}{2}[\|a\|+\|b\|] &\leq \frac{3}{6}[\|a\|+\|b\|]=\frac{1}{2}[\|a\|+\|b\|],\end{aligned}$$

that is true. This shows that S is an enriched $\mathcal{A}$-contraction.

Theorem 4. *Let (E,d) be a complete CAT(0) space and S be an enriched $\mathcal{A}$-contraction. Then S has a unique fixed point in E and there exists $\lambda\in(0,1]$ such that the sequence $\{a_n\}$ defined by $a_{n+1}=\lambda a_n\oplus(1-\lambda)Sa_n$, $n\geq 0$ converges to that fixed point , for any $a_0\in E$.*

Proof. Let $b^*>0$. Set $\lambda=\frac{1}{b^*+1}>0$. Then

$$\begin{aligned}d(S_\lambda a,S_\lambda b) &\leq \lambda f\left(\frac{1}{\lambda}d(a,b),d(a,Sa),d(b,Sb)\right)\\ &\leq f(d(a,b),\lambda d(a,Sa),\lambda d(b,Sb))\\ d(S_\lambda a,S_\lambda b) &\leq f(d(a,b),d(a,S_\lambda a),d(b,S_\lambda b)) \qquad (4.2)\end{aligned}$$

Let $a_0 \in E$ be arbitrary. Define $a_n = S_\lambda^n a_0$ for all $n \geq 1$. Let us put $a = a_n$ and $y = a_{n-1}$ in (4.2). Then

$$d(a_{n+1}, a_n) \leq f(d(a_n, a_{n-1}), d(a_n, a_{n+1}), d(a_{n-1}, a_n))$$

which implies that

$$d(a_{n+1}, a_n) \leq k.d(a_n, a_{n-1})$$

for some $k \in [0,1]$. From this we get

$$d(a_{n+1}, a_n) \leq k^n.d(a_1, a_0).$$

Now for all $m, n \geq 1$, we have

$$\begin{aligned} d(a_{n+m}, a_n) &\leq d(a_{n+m}, a_{n+m-1}) + d(a_{n+m-1}, a_{n+m-2}) + \ldots + d(a_{n+1}, a_n) \\ &\leq (k^{n+m-1} + k^{n+m-2} + \ldots + k^n) d(a_1, a_0) \\ &= k^n \left(\frac{1-k^m}{1-k} \right) d(a_1, a_0) \end{aligned}$$

This implies that $d(a_{n+m}, a_n) \to 0$ as $m, n \to \infty$. Hence $\{a_n\}$ is a Cauchy sequence in E and hence there exists an element $p \in E$ such that $a_n \to p$ as $n \to \infty$.

Now

$$\begin{aligned} d(S_\lambda p, a_{n+1}) &= d(S_\lambda p, S_\lambda a_n) \\ &\leq f(d(p, a_n), d(p, Sp), d(a_n, a_{n+1})) \\ &= f(d(p, a_n), d(p, S_\lambda p), d(a_n, a_{n+1})) \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$d(S_\lambda p, p) \leq f(d(p,p), d(p, S_\lambda p), d(p,p)) \leq k \cdot d(p,p) = 0.$$

Hence, $S_\lambda p = p$ and consequently $Sp = p$. Therefore, p is a fixed point of S.

Let p^* be another fixed point of S and consequently, a fixed point of S_λ. Then

$$\begin{aligned} d(p, p^*) = d(S_\lambda p, S_\lambda p^*) &\leq f(d(p, p^*), d(p, Sp), d(p^*, Sp^*)) \\ &= f(d(p, p^*), d(p,p), d(p^*, p^*)) = f(d(p, p^*), 0, 0). \end{aligned}$$

This gives that $d(p, p^*) \leq k.0 = 0$ for some $k \in [0,1)$. Hence $p = p^*$.

If $b = 0$, then (4.1) reduces to

$$d(S_\lambda a, S_\lambda b) \leq \lambda f(d(a,b), d(a, Sa), d(b, Sb))$$

for all $a, b \in E$ with $a \neq b$ which can easily be solved. □

Definition 3. Let (E,d) be a CAT(0) space. Let $\mathcal{A}'$ be the collection of all mappings $f : \mathbb{R}_+^3 \to \mathbb{R}$ satisfying the following conditions:

- **($\mathcal{A}_1'$):** f is continuous;
- **($\mathcal{A}_2'$):** if $r \leq f(r,s,s)$ or $r \leq f(s,s,r)$, then there exists $k \in [0,1)$ such that $r \leq ks$;
- **($\mathcal{A}_3'$):** for $\lambda > 0$ and for all $r,s,t \in \mathbb{R}_+, \lambda f(r,s,t) \leq f(\lambda r, \lambda s, \lambda t)$;

($\mathcal{A}_4'$): if $t \leq t_1$, then $f(r,s,t) \leq f(r,s,t_1)$;
($\mathcal{A}_5'$): if $r \leq f(s,0,r+s)$, then $r \leq ks$ for some $k \in [0,1)$;
($\mathcal{A}_6'$): if $r \leq f(r,r,r)$ then $r = 0$.

Let $T : E \to E$ be a mapping such that there exist $f \in \mathcal{A}'$ with

$$d(S_\lambda a, S_\lambda b) \leq \lambda f((b^*+1)d(a,b),(b^*+1)d(a,Sb),(b^*+1)d(b,Sa)) \tag{4.3}$$

for all $a,b \in E$ with $a \neq b$ and $b^* \in [0,\infty)$. Then S is said to be an enriched $\mathcal{A}'$-contraction.

Example 4. Let $E = \mathbb{R}$ be endowed with the usual norm and $S : E \to E$ be defined by $Sa = 2 - a$, for all $a \in E$. S is not an enriched contraction but is an enriched $\mathcal{A}'$-contraction.

Indeed, if S would be an enriched contraction, then there would exist $b^* \in [0,+\infty)$ and $\theta \in [0,b^*+1)$ such that $\forall\, a,b \in \mathbb{R}$,

$$\|b^*(a-b) + Sa - Sb\| \leq \theta \|(a-b)\|$$

Take $b^* = 0$ and then $\theta \in [0,1)$,

$$\|a-b\| \leq \theta \|a-b\|,$$

which, for $a = 0$ and $b = 1$ yields the contradiction $1 \leq \theta < 1$. S is an enriched $\mathcal{A}'$-contraction for $b^* = 1$ and $f(r,s,t) = \frac{s+t}{6}$ (see [9]).

Theorem 5. *Let (E,d) be a complete CAT(0) space and S be an enriched $\mathcal{A}'$-contraction. Then S has a unique fixed point in E and there exists $\lambda \in (0,1]$ such that the sequence $\{a_n\}$ defined by $a_{n+1} = \lambda a_n \oplus (1-\lambda)Sa_n$, $n \geq 0$ converges to that fixed point , for any $a_0 \in E$.*

Proof. Let $b^* > 0$. Set $\lambda = \frac{1}{b^*+1}$ so that $0 < \lambda < 1$. Now note that

$$\begin{aligned} d(S_\lambda a, S_\lambda b) &\leq \lambda f(\frac{1}{\lambda}d(a,b), \frac{1}{\lambda}d(a,Sb), \frac{1}{\lambda}d(b,Sa)) \\ &\leq f(d(a,b),d(a,Sb),d(b,Sa)) \\ d(S_\lambda a, S_\lambda b) &\leq f(d(a,b),d(a,S_\lambda b),d(b,S_\lambda a)) \end{aligned} \tag{4.4}$$

for all $a,b \in E$ with $a \neq b$.

Let $a_0 \in E$ be arbitrary. Define $a_n = S_\lambda^n a_0$ for all $n \geq 1$. Let us put $a = a_n$ and $b = a_{n-1}$ in (4.4). Then

$$\begin{aligned} d(a_{n+1},a_n) &\leq f(d(a_n,a_{n-1}),d(a_n,a_n),d(a_{n-1},a_{n+1})) \\ &\leq f(d(a_n,a_{n-1}),d(a_n,a_n),d(a_{n-1},a_n),d(a_n,a_{n+1})) \end{aligned}$$

which implies that

$$d(a_{n+1},a_n) \leq k.d(a_n,a_{n-1})$$

for some $k \in [0,1]$. From this we get

$$d(x_{n+1},x_n) \leq k^n.d(x_1,x_0).$$

Now for all $m,n \geq 1$, we have

$$\begin{aligned} d(a_{n+m},a_n) &\leq d(a_{n+m},a_{n+m-1}) + d(a_{n+m-1},a_{n+m-2}) + \ldots + d(a_{n+1},a_n) \\ &\leq (k^{n+m-1} + k^{n+m-2} + \ldots + k^n) d(a_1,a_0) \\ &= k^n \left(\frac{1-k^m}{1-k} \right) d(a_1,a_0) \end{aligned}$$

So, $d(a_{n+m},a_n) \to 0$ as $m,n \to \infty$. Hence $\{a_n\}$ is a Cauchy sequence in E and hence there exists an element $p \in E$ such that $a_n \to p$ as $n \to \infty$.

Now

$$d(S_\lambda p, a_{n+1}) = d(S_\lambda p, S_\lambda a_n) \leq f(d(p,a_n), d(p,a_{n+1}), d(a_n,Sp))$$

Taking limit as $n \to \infty$, we get

$$d(S_\lambda p, p) \leq f(d(p,p), d(p,p), d(p,S_\lambda p)) \leq k \cdot d(p,p) = 0.$$

Hence, $S_\lambda p = p$ and consequently $Sp = p$. Therefore, p is a fixed point of S.

Let p^* be another fixed point of S and hence, a fixed point of S_λ. Then

$$\begin{aligned} d(p,p^*) &= d(S_\lambda p, S_\lambda p^*) \\ &\leq f(d(p,p^*), d(p,Sp^*), d(p^*,Sp)) \\ &= f(d(p,p^*), d(p,p^*), d(p,p^*)) \end{aligned}$$

This gives that $d(p,p^*) = 0$. Hence $p = p^*$.

If $b = 0$, then (4.3) reduces to

$$d(S_\lambda a, S_\lambda b) \leq \lambda f(d(a,b), d(a,Sa), d(b,Sb))$$

for all $a,b \in E$ with $a \neq b$ which can easily be solved. □

Example 5. Let $E = \mathbb{R}$ be a CAT(0) space equipped with usual norm. Let S be a self map on E defined as $Sa = 4 - a$ for all $a \in E$.

We choose $f \in \mathcal{A}$, defined by $f(r,s,t) = \frac{1}{4} max\{s,t\}$ for $r,s,t \in \mathbb{R}_+$ and $\lambda = \frac{1}{2}, b = 1$. Then

$$d(S_\lambda a, S_\lambda b) = |S_\lambda a - S_\lambda b| = 0$$

and

$$\begin{aligned} \lambda f((b^*+1)d(a,b), d(a,Sa), d(b,Sb)) &= \frac{1}{2} \cdot \frac{1}{4} max\{d(a,Sa),(b,Sb)\} \\ &= \frac{1}{8} max\{|2a-4|, |2b-4|\} \end{aligned}$$

Therefore,

$$d(S_\lambda a, S_\lambda b) \leq \lambda f((b^*+1)d(a,b), d(a,Sa), d(b,Sb))$$

for all $a,b \in E$. Hence, S is an enriched $\mathcal{A}$-contraction on E. So, by Theorem 4, S has a unique fixed point. It is to be noticed that $a = 2$ is the unique fixed point of S.

Example 6. Let $E=\mathbb{R}$ be a CAT(0) space equipped with usual norm. Let S be a self map on E defined as $Sa=1-a$ for all $a\in E$.

Let us consider the mapping $f\in\mathcal{A}'$, defined by $f(r,s,t)=\frac{s+t}{2}$ for $r,s,t\in\mathbb{R}_+$ and $\lambda=\frac{1}{2}, b=1$. Then

$$d(S_\lambda a,S_\lambda b)=|S_\lambda a-S_\lambda b|=0$$

and

$$\begin{aligned}\lambda f((b^*+1)d(a,b),(b+1)d(a,Sb),(b+1)d(b,Sa))&=\frac{2(d(a,Sb)+d(b,Sa))}{4}\\&=\frac{|a-Sb|+|b-Sa|}{2}\\&=\frac{|a-1+b|+|a+b-1|}{2}\end{aligned}$$

Hence,

$$d(S_\lambda a,S_\lambda b)\le\lambda f((b^*+1)d(a,b),(b^*+1)d(a,Sb),(b^*+1)d(b,Sa))$$

for all $a,b\in E$. Therefore, S is an enriched $\mathcal{A}'$-contraction on E. So, by Theorem 5, S has a unique fixed point. It is to be noticed that $a=\frac{1}{2}$ is the unique fixed point of S.

Now, we study well-posedness and limit shadowing property of fixed point problem for both types of contractions.

Definition 4. Let S be a self map defined on a CAT(0) space (E,d). Then the fixed point problem concerning S is known as well-posed if the followings hold:

(i): S has a unique fixed point $p\in E$;

(ii): for any sequence $\{a_n\}$ in E with $\lim\limits_{n\to\infty}d(a_n,Sa_n)=0$, we have

$$\lim_{n\to\infty}d(a_n,p)=0.$$

Theorem 6. *Let (E,d) be a complete CAT(0) space and S be an enriched $\mathcal{A}$-contraction (resp. enriched $\mathcal{A}'$-contraction). Then the fixed point problem is well posed.*

Proof. Theorem 1(resp. in Theorem 2) ensures that S possesses a unique fixed point p, say.

$$\begin{aligned}d(a_n,S_\lambda a_n)&=d(a_n,\lambda a_n+(1-\lambda)Sa_n)\\&\le(1-\lambda)d(a_n,Sa_n).\end{aligned}$$

for all $a\in E$. Let $\{a_n\}$ be a sequence in E such that $\lim_{n\to\infty}d(a_n,Sa_n)=0$. Then $\lim_{n\to\infty}d(a_n,S_\lambda a_n)=0$.

Then

$$\begin{aligned}d(a_n,p)&\le d(a_n,S_\lambda a_n)+d(S_\lambda a_n,p)\\&=d(a_n,S_\lambda a_n)+d(S_\lambda a_n,S_\lambda p).\end{aligned}$$

Now

$$d(S_\lambda a_n, S_\lambda p) \leq f(d(a_n,p), d(a_n,S_\lambda a_n), d(p,Sp)).$$

Therefore, we have

$$d(a_n,p) \leq d(a_n,S_\lambda a_n) + f(d(a_n,p), d(a_n,S_\lambda a_n), d(p,Sp)).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(a_n,p) \leq f(\lim_{n\to\infty} d(a_n,p),0,0).$$

So, there exist $k \in [0,1)$ such that

$$\lim_{n\to\infty} d(a_n,p) \leq k \cdot 0 \qquad \Rightarrow \qquad \lim_{n\to\infty} d(a_n,p) = 0.$$

Hence the result follows. □

Definition 5. Let S be a self map defined on a CAT(0) space (E,d). Then the fixed point problem involving S is said to posses limit shadowing property in E if for any sequence $\{a_n\}$ in E such that $\lim_{n\to\infty} d(a_n,Sa_n) = 0$, we have $p \in E$ with $\lim_{n\to\infty} d(S^n p, a_n) = 0$.

Theorem 7. *Let (E,d) be a complete CAT(0) space and S be an enriched $\mathcal{A}$-contraction (resp. enriched $\mathcal{A}'$-contraction). Then the fixed point problem involving S possesses limit shadowing property.*

Proof. We have already shown in Theorem 4 (resp. in Theorem 5) that $\lim_{n\to\infty} d(a_n,p) = 0$, where p is the unique fixed point of S. Then for any $n \in \mathbb{N}, S^n p = p$ and therefore

$$\lim_{n\to\infty} d(a_n, S^n p) = 0.$$

i.e.,

$$\lim_{n\to\infty} d(S^n p, a_n) = 0.$$

Hence the fixed point problems has limit shadowing property in both the cases. □

5. Conclusion

(1) The existence and approximation of fixed points for the class of enriched contractions in the context of CAT(0) space are introduced and investigated in this article.
(2) We show that in a complete CAT(0) space, any enriched contraction has a unique fixed point that can be approximated using a Kransnoselskii type iterative procedure
(3) For enriched contractions in CAT(0) spaces, we derive a local fixed point conclusion (Theorem 2) along with an asymptotic fixed point consequence (Theorem 3). These results expand the results in [2] from Banach spaces to complete CAT(0) spaces in a significant way.

(4) Some convergence results are obtained for enriched $\mathcal{A}$-contraction and enriched $\mathcal{A}'$-contraction in the context of CAT(0) space and numerical examples are also provided to illustrate the findings of our paper. These outcomes are generalization of the results in [9] from Banach spaces to complete CAT(0) spaces.

6. Future Work

(1) Other important conclusions on the solution of the fixed point problem in CAT(0) spaces or in Banach spaces, metric spaces and generalized metric spaces could be derived using the approach considered in this study along with the ideas given in [7].
(2) In the case of CAT(k) spaces for $k > 0$, an approach identical to that described in this paper can be applied.

Acknowledgements

The authors thank Editor-in-Chief/Area Editors and Referee(s) for their insightful comments and suggestions, which were very much useful to improve the paper significantly.

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