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EXISTENCE AND EXPONENTIAL DECAY OF A LOGARITHMIC WAVE EQUATION WITH DISTRIBUTED DELAY

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Abstract. In this article, we deal with a logarithmic wave equation with distributed delay. Firstly, we establish the well-posedness by utilizing the semigroup theory. Later, we obtain the global existence of solutions by using the well-depth method. Moreover, under appropriate assumptions on the weight of the distributed delay and that of strong damping, we get the exponential decay results.

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1. INTRODUCTION

In this paper, we deal with the following logarithmic wave equation with distributed delay:

$$\begin{cases} u_{tt}-\Delta u-\mu_1\Delta u_t\left(x,t\right)-\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)\Delta u_t\left(x,t-s\right)ds & \\ =u\left|u\right|^{p-2}\ln\left|u\right|^k, & \text{in } \Omega\times(0,\infty), \\ u\left(x,t\right)=0, & \text{on } \partial\Omega\times(0,\infty), \\ u_t\left(x,-t\right)=f_0\left(x,-t\right), & 0<t\leq\tau_2, \\ u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and regular domain of R^n. k, μ_1 are positive constants, the integral term denotes the distributed delay for $\tau_1<\tau_2$ and $\mu_2:[\tau_1,\tau_2]\to R$ is a bounded function and u_0, u_1, f_0 are the functions of the initial data to be specified later.

The study of the asymptotic behavior of wave equation is an old and large areas because it is important in the applications, for this reasons wave equation has been taking different forms and names according to the phenomena it describes and also the material using in the experiment. In this paper we will study the nonlinear wave equation (1.1) with time delay. Also, in the same equation, we add the logarithmic source term which seems in inflation cosmology, nuclear physics, geophysics and optics, for instance (see [3–5,7,12]). To give a problem describe a phenomena enough perfect, we must take a consideration the delay which appears in practical phenomena like physical, biological, economic and some of themes. Furthermore, delay term influenced on the stability and instability of the studied system and many papers has been discussed the both situations, for example [8–10, 16].

During the past decades, many authors considered extensively on existence, nonexistence, stability and blow-up of solutions for the strongly damped wave equations with source term as follows:

$$u_{tt}-\Delta u-\omega\Delta u_t+\mu u_t=f(u). \tag{1.2}$$

Firstly, Sattinger [28] studied the existence of local as well as global solutions for the equation (1.2) with $\omega=\mu=0$ by introducing the concepts of stable and unstable sets. In [13], Ikehata looked into the decay and blow-up of solutions for the equation (1.2) with linear damping that is $\omega=0$ and $\mu>0$. In [11], Gazzola and Squassina obtained the global existence and blow-up results for the equation (1.2) with both weak and strong damping ($\omega>0$). In the case of logarithmic source term $f(u)=u\ln|u|^k$, Ma and Fang [20] obtained the existence and blow-up results for the equation (1.2) with $\omega=1$, $\mu=0$ and $k=2$. In [18], Lian and Xu established the global existence, energy decay and blow-up results in the case $\omega\geq 0$ and $\mu>-\omega\lambda_1$, here λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions.

It is well known that, in the n-dimensional case, the equation:

$$u_{tt}-\Delta u+a_0u_t(x,t)+au_t(x,t-\tau)=0, \tag{1.3}$$

is exponentially stable without delay ($a=0,\ a_0>0$), (see [19, 35]). With delay term ($a>0$), Nicaise and Pignotti [23] considered the equation (1.3) and established, under the condition that the weight of the feedback is larger than the weight of the delay ($a<a_0$), that the energy of (1.3) is exponentially stable. If the delay term in the equation (1.3) is replaced by the distributed delay:

$$\int_{\tau_1}^{\tau_2}a(s)u_t(x,t-s)\,ds,$$

exponential stability of solutions have been proved in Ref. [22] under the assumption:

$$\int_{\tau_1}^{\tau_2}a(s)\,ds<a_0.$$

Without logarithmic nonlinearity ($u|u|^{p-2}\ln|u|^k$), the equation (1.1) takes the following form:

$$u_{tt}-\Delta u-\mu_1\Delta u_t(x,t)-\int_{\tau_1}^{\tau_2}\mu_2(s)\Delta u_t(x,t-s)\,ds=0, \tag{1.4}$$

Messaoudi et al. [21], established the well-posedness and obtained an exponential decay results under a suitable assumption on the weight of the damping and the weight of the delay of the equation (1.4). Furthermore, in [21], they extended the results obtained to the case of strong delay ($\mu_2\Delta u_t(x,t-\tau)$) in same work. Some other researchers considered delayed hyperbolic-type equations (see [2,14,15,25,27,29–34]).

Motivated by previous works and in the presence of strong damping ($-\mu_1\Delta u_t(x,t)$), distributed delay ($\int_{\tau_1}^{\tau_2}\mu_2(s)\Delta u_t(x,t-s)\,ds$) and logarithmic ($u|u|^{p-2}\ln|u|^k$) nonlinearity, we consider the well-posedness, global existence and exponential decay for the logarithmic wave equation (1.1) with strong damping and distributed delay.

The contents of this work is organized as follows: In Section 2, firstly, we obtain the well-posedness by utilizing the semigroup theory. Then, in Section 3, we establish the global existence results by the well-depth method. Finally, in Section 4, we get the exponential decay of solutions.

2. Well-posedness

As usual, the notation $\|.\|_p$ denotes L^p norm and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$ (see [1,26], for details). Similar to the [23], we introduce the new variable

$$z(x,\rho,s,t)=u_t(x,t-\rho s),\ \text{in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Hence, we get

$$sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0 \text{ in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Therefore, the problem (1.1) becomes the form:

$$\begin{cases} u_{tt}-\Delta u-\mu_1\Delta u_t(x,t)-\int_{\tau_1}^{\tau_2}\mu_2(s)\Delta z(x,1,s,t)\,ds & \\ \qquad =u|u|^{p-2}\ln|u|^k, & \text{in } \Omega\times(0,+\infty)\\ sz_t(x,\rho,s,t)+z_\rho(x,\rho,s,t)=0, & \text{in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,+\infty)\\ u(x,t)=0, & \text{on } \partial\Omega\times(0,\infty)\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & \text{in } \Omega\\ z(x,\rho,s,0)=f_0(x,-\rho s), & \text{in } \Omega\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{2.1}$$

We establish the local existence utilizing the semigroup theory [17,24]. Let $\upsilon=u_t$ and denote by

$$\Phi=(u,\upsilon,z)^T,\ \Phi(0)=\Phi_0=(u_0,u_1,f_0(\cdot,\rho s))^T \text{ and } J(\Phi)=\left(0,u|u|^{p-2}\ln|u|^k,0\right)^T.$$

Hence, (2.1) can be written as an initial-value problem:

$$\begin{cases} \partial_t\Phi+\mathcal{A}\Phi &=J(\Phi),\\ \quad\Phi(0) &=\Phi_0, \end{cases} \tag{2.2}$$

where the linear operator $\mathcal{A}:D(\mathcal{A})\longrightarrow\mathcal{H}$ is defined by

$$\mathcal{A}\Phi=\begin{pmatrix} -\upsilon\\ -\Delta u-\mu_1\Delta\upsilon-\int_{\tau_1}^{\tau_2}\mu_2(s)\Delta z(x,1,s,t)\,ds\\ \frac{1}{s}z_\rho \end{pmatrix},$$

where $D(\mathcal{A})$ and $\mathcal{H}$ are introduced below.

The setting space of Φ is the Hilbert space

$$\mathcal{H}=H_0^1(\Omega)\times L^2(\Omega)\times L^2\left((0,1)\times(\tau_1,\tau_2);H_0^1(\Omega)\right),$$

equipped with the inner product

$$\left\langle \Phi,\widetilde{\Phi}\right\rangle_{\mathcal{H}}=\int_{\Omega}(\nabla u\nabla\widetilde{u}+\upsilon\widetilde{\upsilon})\,dx+\int_{\Omega}\int_{\tau_1}^{\tau_2}s\left|\mu_2(s)\right|\int_0^1\nabla z\nabla\widetilde{z}\,d\rho\, ds\,dx,$$

for all $\Phi=(u,\upsilon,z)^T$ and $\widetilde{\Phi}=(\widetilde{u},\widetilde{\upsilon},\widetilde{z})^T$ in $\mathcal{H}$. The domain of $\mathcal{A}$ is

$$D(\mathcal{A})=\left\{\begin{array}{c}\Phi\in\mathcal{H}:\left(u+\mu_1\upsilon+\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,z\right)\right)\in H^2(\Omega),\\ z,z_\rho\in L^2\left((0,1)\times(\tau_1,\tau_2);H_0^1(\Omega)\right),\ z(\cdot,0,\cdot)=\upsilon\end{array}\right\}.$$

Now, we obtain the local existence and uniqueness of solutions as follows:

Theorem 1. *Suppose that* $\mu_1>\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds$ *and*

$$\left\{\begin{array}{ll} p>2 & \text{if } n=1,2,\\ 2<p<\frac{2(n-1)}{n-2} & \text{if } n\geq 3.\end{array}\right. \tag{2.3}$$

Then, for any $\Phi_0\in\mathcal{H}$*, problem* (2.2) *has a unique weak solution*

$$\Phi\in C([0,T);\mathcal{H}).$$

Furthermore, if $\Phi_0\in D(\mathcal{A})$*, the solution of* (2.2) *satisfies*

$$\Phi\in C^1([0,T),\mathcal{H})\cap C([0,T),D(\mathcal{A})).$$

Proof. We will utilize the Hille-Yoside theorem [6, 17]. For this aim, we indicate that $\mathcal{A}$ is monotone. Thus, for all $\Phi\in D(\mathcal{A})$, we obtain

$$\begin{aligned}\langle\mathcal{A}\Phi,\Phi\rangle_{\mathcal{H}}&=-\int_{\Omega}\nabla\upsilon\nabla u\,dx\\&\quad-\int_{\Omega}\upsilon\left[\Delta u+\mu_1\Delta\upsilon+\int_{\tau_1}^{\tau_2}\mu_2(s)\,\Delta z(x,1,s,t)\,ds\right]dx\\&\quad+\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\int_0^1\nabla z\nabla z_\rho\,d\rho\,ds\,dx\\&=\mu_1\int_{\Omega}|\nabla\upsilon|^2dx+\int_{\Omega}\nabla\upsilon\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\nabla z(x,1,s,t)\,ds\right)dx\\&\quad+\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\left|\nabla z(x,1,s,t)\right|^2ds\,dx\\&\quad-\frac{1}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\left|\nabla\upsilon\right|^2ds\,dx.\end{aligned}\tag{2.4}$$

By using the Young's inequality, the estimate (2.4) becomes the form:

$$\begin{aligned}\langle\mathcal{A}\Phi,\Phi\rangle_{\mathcal{H}}&\geq\mu_1\int_{\Omega}|\nabla\upsilon|^2dx-\frac{|\mu_2|}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\nabla\upsilon|^2ds\,dx-\frac{|\mu_2|}{2}\int_{\Omega}\int_{\tau_1}^{\tau_2}|\nabla\upsilon|^2ds\,dx\\&=\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2|(s)\,ds\right)\int_{\Omega}|\nabla\upsilon|^2dx\geq 0.\end{aligned}$$

Therefore, $\mathcal{A}$ is a monotone operator.

To indicate that $\mathcal{A}$ is maximal, we establish that for each:

$$F = (f, g, h)^T \in \mathcal{H},$$

there exists $V = (u, \upsilon, z)^T \in D(\mathcal{A})$ such that $(I + \mathcal{A})V = F$. That is,

$$\begin{cases} u - \upsilon & = f \\ \upsilon - \Delta u - \mu_1 \Delta \upsilon - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \Delta z(x,1,s,t)\, ds & = g, \\ sz + z_\rho & = sh. \end{cases} \tag{2.5}$$

As $z(.,0,.) = u - f = \upsilon$, by the third equation of (2.5), we infer that

$$z(.,\rho,.) = (u - f) e^{-\rho s} + s e^{-\rho s} \int_0^\rho h(.,\gamma,s) e^{\gamma s} d\gamma. \tag{2.6}$$

By substituting (2.6) in the second equation of (2.5), we have

$$u - k\Delta u = G,$$

where

$$k = 1 + \mu_1 + \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)|\, ds > 0,$$

$$G = g + f - \left(\mu_1 + \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} ds \right) \Delta f \tag{2.7}$$

$$+ s \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} \int_0^1 \Delta h(\gamma, x) e^{\gamma s} d\gamma ds \in H^{-1}(\Omega). \tag{2.8}$$

We define, over $H_0^1(\Omega)$, the bilinear and linear forms:

$$B(u,w) = \int_\Omega uw dx + k \int_\Omega \nabla u \nabla w dx, \; L(w) = \langle G, w \rangle_{H^{-1} \times H_0^1}.$$

We see that B is coercive and continuous, and L is continuous on $H_0^1(\Omega)$.

Then, Lax-Milgram theorem specifies that the equation

$$B(u,w) = L(w), \forall w \in H_0^1(\Omega), \tag{2.9}$$

has a unique solution $u \in H_0^1(\Omega)$. Thus, $\upsilon = u - f \in H_0^1(\Omega)$.

As a result, by (2.6), we obtain

$$z, z_\rho \in L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)) \text{ and } z(.,0,.) = \upsilon.$$

Replacing Δf by $\Delta(u - \upsilon)$ and $se^{-s} \int_0^1 \Delta h(\gamma, x) e^{\gamma s} d\gamma$ by $\Delta z(x,1,s) - e^{-s} \Delta \upsilon$ in the right-hand side of (2.9) and utilizing the Green's formula, we get

$$\int_\Omega u\upsilon + \int_\Omega \nabla \left(u + \mu_1 \upsilon + \int_{\tau_1}^{\tau_2} \mu_2(s) e^{-s} z(x,1,s)\, ds \right) \nabla w = \int_\Omega (f+g) w, \forall w \in H_0^1(\Omega).$$

From the standard elliptic regularity theory [6], we have

$$u + \mu_1 \upsilon + \int_{\tau_1}^{\tau_2} \mu_2(s) e^{-s} z(.,1,s)\, ds \in H^2(\Omega).$$

Hence,

$$\Delta\left(u+\mu_1\upsilon+\int_{\tau_1}^{\tau_2}\mu_2(s)e^{-s}z(.,1,s)\,ds\right)=g+f-u\in L^2(\Omega).$$

Thus,

$$V=(u,\upsilon,z)^T\in D(\mathcal{A}).$$

As a result, $I+\mathcal{A}$ is surjective and then $\mathcal{A}$ is maximal.

Finally, we denote that $J:\mathcal{H}\longrightarrow\mathcal{H}$ is locally Lipschitz. Hence, if we set:

$$f(s)=|s|^{p-2}s\ln|s|^k,\ |s|\neq 0 \text{ and } f(s)=0,\ |s|=0,$$

then

$$f'(s)=k[1+(p-1)\ln|s|]\,|s|^{p-2},\ |s|\neq 0 \text{ and } f'(s)=0,\ |s|=0.$$

Thus,

$$\begin{aligned}\left\|J(\Phi)-J\left(\widetilde{\Phi}\right)\right\|_{\mathcal{H}}^2&=\left\|\left(0,|u|^{p-2}u\ln|u|^k-|\widetilde{u}|^{p-2}\widetilde{u}\ln|\widetilde{u}|^k,0\right)\right\|_{\mathcal{H}}^2\\&=\left\||u|^{p-2}u\ln|u|^k-|\widetilde{u}|^{p-2}\widetilde{u}\ln|\widetilde{u}|^k\right\|_{L^2}^2\\&=\|f(u)-f(\widetilde{u})\|_{L^2}^2.\end{aligned}$$

By the mean value theorem, we obtain, for some $\theta\in[0,1]$,

$$\begin{aligned}|f(u)-f(\widetilde{u})|&=\left|f'(\theta u+(1-\theta)\widetilde{u})(u-\widetilde{u})\right|\\&=k\left|[1+(p-1)\ln|\theta u+(1-\theta)\widetilde{u}|]\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}(u-\widetilde{u})\right|\\&\leq k[1+(p-1)\,|\ln|\theta u+(1-\theta)\widetilde{u}||]\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}|u-\widetilde{u}|\\&=k(p-1)\,|\ln|\theta u+(1-\theta)\widetilde{u}||\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}|u-\widetilde{u}|\\&\quad+k\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}|u-\widetilde{u}|.\end{aligned}$$

To control the logarithmic term

$$\ln|\theta u+(1-\theta)\widetilde{u}|\,|\theta u+(1-\theta)\widetilde{u}|^{p-2},$$

seems in the last inequality we remind that, for any $\varepsilon>0$,

$$\lim_{|s|\to+\infty}\frac{\ln|s|}{|s|^{\varepsilon}}=0.$$

Then, there exists $B>0$ such that

$$\frac{\ln|s|}{|s|^{\varepsilon}}<1,\ \forall|s|>B.$$

Hence, whenever $|\theta u+(1-\theta)\widetilde{u}|>B$, we get

$$\ln|\theta u+(1-\theta)\widetilde{u}|\leq|\theta u+(1-\theta)\widetilde{u}|^{\varepsilon},$$

and

$$\ln|\theta u+(1-\theta)\widetilde{u}|\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}\leq|\theta u+(1-\theta)\widetilde{u}|^{p-2+\varepsilon}.$$

Since $p>2$, then for some $A>0$ and $|\theta u+(1-\theta)\widetilde{u}|\leq B$, we obtain

$$\ln|\theta u+(1-\theta)\widetilde{u}|\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}\leq A.$$

Thus, we get

$$\ln|\theta u+(1-\theta)\widetilde{u}|\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}\leq A+|\theta u+(1-\theta)\widetilde{u}|^{p-2+\varepsilon}.$$

Then, we obtain the following estimation for

$$\begin{aligned}|f(u)-f(\widetilde{u})|&\leq k(p-1)\,|\theta u+(1-\theta)\widetilde{u}|^{p-2+\varepsilon}\,|u-\widetilde{u}|\\&\qquad+k\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}\,|u-\widetilde{u}|+kA(p-1)\,|u-\widetilde{u}|\\&\leq k(p-1)\,(|u|+|\widetilde{u}|)^{p-2+\varepsilon}\,|u-\widetilde{u}|\\&\qquad+k\,(|u|+|\widetilde{u}|)^{p-2}\,|u-\widetilde{u}|+kA(p-1)\,|u-\widetilde{u}|.\end{aligned}\tag{2.10}$$

As u, $\widetilde{u}\in H_0^1(\Omega)$, by utilizing Hölder's inequality, (2.3) and the Sobolev embedding

$$H_0^1(\Omega)\hookrightarrow L^r(\Omega),\ 1\leq r\leq\frac{2n}{n-2},$$

to have

$$\begin{aligned}\int_\Omega\left[(|u|+|\widetilde{u}|)^{p-2}\,|u-\widetilde{u}|\right]^2dx&=\int_\Omega(|u|+|\widetilde{u}|)^{2(p-2)}\,|u-\widetilde{u}|^2\,dx\\&\leq C\left(\int_\Omega(|u|+|\widetilde{u}|)^{2(p-1)}\,dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega|u-\widetilde{u}|^{2(p-1)}\,dx\right)^{\frac{1}{p-1}}\\&\leq C\left[\|u\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}+\|\widetilde{u}\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}\right]^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|_{L^{2(p-1)}(\Omega)}^2\\&\leq C\left[\|u\|_{H_0^1(\Omega)}^{2(p-1)}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{2(p-1)}\right]^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_\Omega\left[(|u|+|\widetilde{u}|)^{p-2+\varepsilon}\,|u-\widetilde{u}|\right]^2dx&=\int_\Omega(|u|+|\widetilde{u}|)^{2(p-2+\varepsilon)}\,|u-\widetilde{u}|^2\,dx\\&\leq\left(\int_\Omega(|u|+|\widetilde{u}|)^{\frac{2(p-2+\varepsilon)(p-1)}{p-2}}\,dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega|u-\widetilde{u}|^{2(p-1)}\,dx\right)^{\frac{1}{p-1}}\\&\leq\left(\int_\Omega(|u|+|\widetilde{u}|)^{2(p-1)+\frac{2\varepsilon(p-1)}{p-2}}\,dx\right)^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|_{L^{2(p-1)}(\Omega)}^2.\end{aligned}$$

Since $p<\frac{2(n-1)}{n-2}$, by choosing $\varepsilon>0$ small enough, such that

$$p^*=2(p-1)+\frac{2\varepsilon(p-1)}{p-2}\leq\frac{2n}{n-2}.$$

Thus, we get

$$\int_{\Omega}(|u|+|\widetilde{u}|)^{2(p-2+\varepsilon)}|u-\widetilde{u}|^2\,dx \le C\left[\|u\|_{L^{p^*}(\Omega)}^{p^*}+\|\widetilde{u}\|_{L^{p^*}(\Omega)}^{p^*}\right]^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|_{L^{2(p-1)}(\Omega)}^{2}$$
$$\le C\left[\|u\|_{H_0^1(\Omega)}^{p^*}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{p^*}\right]^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|_{H_0^1(\Omega)}^{2}.$$

Thus, by combining the last three estimations, we get

$$\begin{aligned}\left\|J(\Phi)-J\left(\widetilde{\Phi}\right)\right\|_{\mathcal{H}}^2 &\le \left[k^2(p-1)^2A^2\right]\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2\\ &\quad +C\left[\left(\|u\|_{H_0^1(\Omega)}^{2(p-1)}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{2(p-1)}\right)^{\frac{p-2}{p-1}}\right.\\ &\quad \left.+\left(\|u\|_{H_0^1(\Omega)}^{p^*}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{p^*}\right)^{\frac{p-2}{p-1}}\right]\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2\\ &\le C\left(\|u\|_{H_0^1(\Omega)},\|\widetilde{u}\|_{H_0^1(\Omega)}\right)\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2.\end{aligned}$$

Hence, J is locally Lipschitz. Then, as a consequence of the Theorem 1.2 page 184, Pazy [24] (see also a remark in the beginning of page 118, Komornik [17]), we complete the proof. □

3. Global existence

In this section, we establish that the solution of (2.1) is uniformly bounded and global in time. The energy functional for the problem (2.1) is,

$$\begin{aligned}E(t)&=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{k}{p^2}\|u\|_p^p-\frac{1}{p}\int_{\Omega}|u|^p\ln|u|^k\,dx\\ &\quad+\frac{1}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)||\nabla z(x,\rho,s,t)|^2\,dsd\rho dx.\end{aligned}\tag{3.1}$$

The following lemma indicates that the energy functional is nonincreasing:

Lemma 1. *The energy $E(t)$ satisfies, along the solution (u,z) of (2.1), the estimate*

$$E'(t)\le-\left[\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right]\int_{\Omega}|\nabla u_t|^2\,dx<0.\tag{3.2}$$

Proof. Multiply the first equation in (2.1) by u_t and integrate over Ω and the second equation in (2.1) by $-|\mu_2|\Delta z$ and integrate over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, sum up, we obtain

$$\begin{aligned}&\frac{d}{dt}\begin{bmatrix}\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{k}{p^2}\|u\|_p^p-\frac{1}{p}\int_{\Omega}|u|^p\ln|u|^k\,dx\\ +\frac{1}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)||\nabla z(x,\rho,s,t)|^2\,dsd\rho dx\end{bmatrix}\\ &=-\mu_1\int_{\Omega}|\nabla u_t|^2\,dx-\int_{\Omega}\nabla u_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|\nabla z(x,1,s,t)\,dsdx\end{aligned}$$

$$-|\mu_2(s)|\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\nabla z\nabla z_\rho(x,\rho,s,t)\,dsd\rho dx. \tag{3.3}$$

Now, we handle the last two terms of the right-hand side for (3.3) as:

$$\begin{aligned}
-|\mu_2(s)|&\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\nabla z\nabla z_\rho(x,\rho,s,t)\,dsd\rho dx\\
&=-\frac{|\mu_2(s)|}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\frac{\partial}{\partial\rho}\left[|\nabla z(x,\rho,s,t)|^2\right]dsd\rho dx\\
&=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_\Omega|\nabla u_t|^2\,dx\\
&\quad-\frac{|\mu_2(s)|}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\nabla z(x,1,s,t)|^2\,dsdx
\end{aligned}$$

and

$$\begin{aligned}
-&\int_\Omega\nabla u_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,\nabla z(x,1,s,t)\,dsdx\\
&\le\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega|\nabla u_t|^2\,dx+|\mu_2(s)|\int_\Omega\int_{\tau_1}^{\tau_2}|\nabla z(x,1,s,t)|^2\,dsdx\right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\frac{dE(t)}{dt}\le&-\mu_1\int_\Omega|\nabla u_t|^2\,dx+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_\Omega|\nabla u_t|^2\,dx\\
&-\frac{|\mu_2(s)|}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\nabla z(x,1,s,t)|^2\,dsdx\\
&+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega|\nabla u_t|^2\,dx+|\mu_2(s)|\int_\Omega\int_{\tau_1}^{\tau_2}|\nabla z(x,1,s,t)|^2\,dsdx\right).
\end{aligned}$$

Therefore, we get

$$E'(t)\le-\left[\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right]\int_\Omega|\nabla u_t|^2\,dx\le 0.$$

□

Firstly, we set

$$\begin{aligned}
I(t)&=\|\nabla u\|^2-\int_\Omega|u|^p\ln|u|^k\,dx,\\
J(t)&=\frac{1}{2}\int_\Omega|\nabla u|^2\,dx+\frac{k}{p^2}\|u\|_p^p-\frac{1}{p}\int_\Omega|u|^p\ln|u|^k\,dx\\
&\quad+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\,|\nabla z(x,\rho,s,t)|^2\,dsd\rho dx.
\end{aligned} \tag{3.4}$$

Thus, we obtain

$$E(t) = J(t) + \frac{1}{2}\|u_t\|^2.$$

Lemma 2. *Assume that the initial data u_0, $u_1 \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying*

$$I(0) > 0 \text{ and } \beta = kC_{p+l}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2+l}{2}} < 1. \tag{3.5}$$

Then, $I(t) > 0$, for any $t \in [0,T]$.

Proof. Since $I(0) > 0$, we infer from continuity that there exists $T^* \leq T$ such that $I(t) \geq 0$ for all $t \in [0,T^*]$. This implies that, for all $t \in [0,T^*]$,

$$\begin{aligned} J(t) = & \frac{p-2}{2p}\|\nabla u\|^2 + \frac{k}{p^2}\|u\|_p^p + \frac{1}{p}I(t) \\ & + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| |\nabla z(x,\rho,s,t)|^2 \, ds d\rho dx. \end{aligned}$$

Thus, we get

$$J(t) \geq \frac{p-2}{2p}\|\nabla u\|^2.$$

Therefore,

$$\|\nabla u\|^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0). \tag{3.6}$$

On the other hand, by using the fact that $\ln|u| < |u|^l$, we obtain

$$\int_\Omega |u|^p \ln|u| \, dx \leq \int_\Omega |u|^{p+l} dx, \tag{3.7}$$

where l is chosen to be $0 < l < \frac{2}{n-2}$, such that

$$p + l < \frac{2n-2}{n-2} + l < \frac{2n}{n-2}.$$

Hence, the embedding $H_0^1(\Omega) \hookrightarrow L^{p+l}(\Omega)$, satisfies

$$\begin{aligned} \int_\Omega |u|^p \ln|u| \, dx \leq & C_{p+l}\|\nabla u\|^{p+l} \\ & = C_{p+l}\|\nabla u\|^2 \|\nabla u\|^{p-2+l} = C_{p+l}\|\nabla u\|^2 \left(\|\nabla u\|^2\right)^{\frac{p-2+l}{2}} \\ & \leq C_{p+l}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2+l}{2}} \|\nabla u\|^2, \end{aligned} \tag{3.8}$$

where C_{p+l} is the embedding constant.

As a result, from (3.4) and (3.5), we have

$$I(t) > \|\nabla u\|^2 - \beta\|\nabla u\|^2 > 0, \forall t \in [0,T^*]. \tag{3.9}$$

Therefore, T^* can be extended to T. □

Theorem 2. *Assume that the initial data u_0, u_1 satisfy the conditions of Lemma 2, then the solution of* (2.1) *is uniformly bounded and global in time.*

Proof. It suffices to indicate that $\|\nabla u\|^2+\|u_t\|^2$ is bounded independently of t. We have,

$$\begin{aligned}E(0)\geq E(t)&=\frac{1}{2}\|u_t\|^2+J(t)\geq\frac{1}{2}\|u_t\|^2+\frac{k}{p^2}\|u\|_p^p+\frac{1}{p}I(t)\\&+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)||\nabla z(x,\rho,s,t)|^2dsd\rho dx\\&\geq\frac{1}{2}\|u_t\|^2+\frac{1}{p}(1-\beta)\|\nabla u\|^2.\end{aligned}$$

Therefore,

$$\|\nabla u\|^2+\|u_t\|^2\leq CE(0),$$

where C is a positive constant depending only on k, p and C_{p+1}. □

4. Exponential decay

In this section, we establish the exponential decay results. Firstly, we give the lemmas as follows:

Lemma 3. *[21] The functional*

$$F_1(t)=\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}se^{-s\rho}|\mu_2(s)||\nabla z(x,\rho,s,t)|^2dsd\rho dx,$$

satisfies, along the solution of (2.1),

$$\begin{aligned}F_1'(t)\leq&\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\int_\Omega|\nabla u_t|^2dx-\int_\Omega\int_{\tau_1}^{\tau_2}e^{-s}|\mu_2(s)||\nabla z(x,1,s,t)|^2dsdx\\&-\int_\Omega\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\int_0^1e^{-s\rho}|\nabla z(x,\rho,s,t)|^2d\rho dsdx.\end{aligned}\tag{4.1}$$

Lemma 4. *The functional*

$$F_2(t)=NE(t)+\varepsilon\int_\Omega uu_tdx+\frac{\varepsilon\mu_1}{2}\int_\Omega|\nabla u|^2dx,$$

satisfies, along the solution of (2.1),

$$\begin{aligned}F_2'(t)\leq&-\left\{N\left[\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\right]-\varepsilon\right\}\int_\Omega|\nabla u_t|^2dx\\&-\varepsilon(1-\beta-\delta)\|\nabla u\|^2+\varepsilon\frac{c_*}{4\delta}\int_\Omega\int_{\tau_1}^{\tau_2}|\nabla z(x,1,s,t)|^2dsdx,\end{aligned}\tag{4.2}$$

where N, α and ε are positive constants.

Proof. A direct differentiation, from the equations in (2.1), satisfies

$$F_2'(t) \leq -N\left[\mu_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\right] \int_\Omega |\nabla u_t|^2\, dx$$
$$+\varepsilon\left(\int_\Omega |u_t|^2\, dx - \int_\Omega |\nabla u|^2\, dx + \int_\Omega |u|^p \ln|u|^k\, dx\right)$$
$$-\varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, \nabla u \nabla z(x,1,s,t)\, ds dx. \tag{4.3}$$

From the Young's inequality and the boundness property of $\mu_2(s)$, we get, for any $\delta > 0$ and some $c_* > 0$,

$$-\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, \nabla u \nabla z(x,1,s,t)\, ds dx$$
$$\leq \delta \|\nabla u\|^2 + \frac{c_*}{4\delta} \int_\Omega \int_{\tau_1}^{\tau_2} |\nabla z(x,1,s,t)|^2\, ds dx. \tag{4.4}$$

Combining (3.7), (4.3) and (4.4), the result follows: □

Theorem 3. *Assume that (3.5) holds. Then, there exist two positive constants c_3 and c_4 such that*

$$E(t) \leq c_3 e^{-c_4 t}.$$

Proof. By setting

$$F_3(t) = F_1(t) + F_2(t).$$

It is easy to see, for ε small enough, that

$$F_3(t) \sim E(t). \tag{4.5}$$

From the (4.1) and (4.2), we get

$$F_3'(t) \leq -\left\{N\left[\mu_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\right] - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) - \varepsilon\right\} \|\nabla u_t\|^2$$
$$-\varepsilon(1-\beta-\delta)\|\nabla u\|^2 - \left(e^{-s}|\mu_2(s)| - \varepsilon\frac{c_*}{4\delta}\right) \int_\Omega \int_{\tau_1}^{\tau_2} |\nabla z(x,1,s,t)|^2\, ds dx$$
$$-\int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)| \int_0^1 e^{-s\rho} |\nabla z(x,\rho,s,t)|^2\, d\rho ds dx. \tag{4.6}$$

Since $\beta < 1$, by choosing δ small enough, such that $\alpha = 1-\beta-\delta > 0$.

For some $\omega > 0$, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ satisfies

$$\|u\|_p^p \leq C\|\nabla u\|_2^p \leq C\left(\|\nabla u\|^2\right)^{\frac{p-2}{2}} \|\nabla u\|^2 \leq C(E(0))^{\frac{p-2}{2}} \|\nabla u\|^2$$
$$\leq \omega \|\nabla u\|^2,$$

or

$$-\frac{\varepsilon\alpha\omega^{-1}}{2} \|u\|_p^p \geq -\frac{\varepsilon\alpha}{2} \|\nabla u\|_2^2.$$

Therefore, (4.6) becomes the form

$$
\begin{aligned}
F_3'(t) \le & -\left\{N\left[\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right]-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)-\varepsilon\right\}\|\nabla u_t\|^2 \\
& -\frac{\varepsilon\alpha}{2}\|\nabla u\|^2-\frac{\varepsilon\alpha\omega^{-1}}{2}\|u\|_p^p-\left(e^{-s}|\mu_2(s)|-\varepsilon\frac{c_*}{4\delta}\right)\int_\Omega\int_{\tau_1}^{\tau_2}|\nabla z(x,1,s,t)|^2\,dsdx \\
& -\int_\Omega\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\int_0^1 e^{-s\rho}|\nabla z(x,\rho,s,t)|^2\,d\rho dsdx. \qquad (4.7)
\end{aligned}
$$

Whence δ is fixed, by choosing N to be large enough, such that

$$
N\left[\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right]-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)-\varepsilon>0 \text{ and } e^{-s}|\mu_2(s)|-\varepsilon\frac{c_*}{4\delta}>0.
$$

Hence, (4.7) takes the form, for some $C>0$,

$$
\begin{aligned}
F_3'(t) &\le -C\left[\|\nabla u_t\|^2+\|\nabla u\|^2+\|u\|_p^p\right. \\
&\qquad \left.+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)||\nabla z(x,\rho,s,t)|^2\,dsd\rho dx\right] \\
&\le -CE(t).
\end{aligned}
$$

From the equivalence relation (4.5) and taking a simple integration over $(0,t)$, the result is established. □

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