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REPDIGITS AS SUMS OF THREE HALF-COMPANION PELL NUMBERS

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Abstract. In this paper, we find all repdigits which can be expressed as the sum of three Half-companion Pell numbers. To prove our main result, we use the combined approach of lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker Davenport reduction method.

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1. INTRODUCTION

Diophantine equations involving recurrence sequences have been studied for a long time. One of the most interesting of these equations is the equations involving repdigits.

A repdigit, short for "repeated digit", T is a natural number composed of repeated instances of the same digit in its decimal expansion. That is, T is of the form

$$x \cdot \left(\frac{10^t - 1}{9}\right)$$

for some positive integers x, t with $t \geq 1$ and $1 \leq x \leq 9$.

Some of the most recent papers related to the Diophantine equations and repdigits with well known recurrence sequences are [2–5, 7, 8, 10, 11]. In this note, we use Half-companion Pell sequence in our main result.

The companion Pell numbers or Pell–Lucas numbers are defined by the recurrence relation

$$Q_n = \begin{cases} 2 & \text{if } n = 0, \\ 2 & \text{if } n = 1, \\ 2Q_{n-1} + Q_{n-2} & \text{otherwise.} \end{cases}$$

The first few terms of the sequence are (sequence A002203 in the OEIS): $2, 2, 6, 14$, 34, $82, 198, 478, \ldots$.

Binet's formula for Companion Pell numbers numbers is

$$Q_n = \varphi^n + \psi^n$$

where $\varphi = \left(1+\sqrt{2}\right)$ (the silver ratio) and $\psi = \left(1-\sqrt{2}\right)$ are roots of the characteristic equation $x^2-2x-1=0$. From this formula, one can easiliy prove that

$$\varphi^{n-2} \leq Q_n \leq 2\varphi^{n-1}.$$

by the induction on n.

Half-companion Pell numbers H_n that play an important role in obtaning Lucas-Balancing numbers C_n as $C_n = H_{2n}$ are defined by $H_n = Q_n/2$. So, it is obvious that

$$H_n = \frac{\varphi^n + \psi^n}{2}.$$

Further, for every positive integer $n \geq 2$,

$$\varphi^{n-1} \leq 2H_n \leq \varphi^{n+1}. \tag{1.1}$$

In this study, our main result is the following:

Theorem 1.1. *All nonnegative integer solutions (n,m,k,x,t) with $1 \leq x \leq 9$ satisfying the Diophantine equation*

$$N = H_n + H_m + H_k = x \cdot \left(\frac{10^t - 1}{9}\right) \tag{1.2}$$

as follows:

$$\begin{aligned}(n,m,k,x,t) \in \{&(0,1,1,3,1),(0,1,2,5,1),(0,1,3,9,1),(0,2,1,5,1),(0,2,2,7,1),\\ &(0,2,3,1,2),\ (0,3,1,9,1),(0,3,2,1,2),(1,1,1,3,1),(1,1,2,5,1),\\ &(1,1,3,9,1),(1,2,1,5,1),\ (1,2,2,7,1),(1,2,3,1,2),(1,3,1,9,1),\\ &(1,3,2,1,2),(2,1,1,5,1),(2,1,2,7,1),\ (2,1,3,1,2),(2,2,1,7,1),\\ &(2,2,2,9,1),(2,3,1,1,2),(3,1,1,9,1),(3,1,2,1,2),\ (3,2,1,1,2),\\ &(3,3,5,5,2),(3,5,3,5,2),(4,5,5,9,2),(5,3,3,5,2),(5,4,5,9,2),\\ &(5,5,4,9,2)\}\end{aligned}$$

2. PRELIMINARIES

Before proceeding with the proof of our main result, let us give some necessary information for proof. We give the definition of the logarithmic height of an algebraic number and its some properties.

Definition 2.1. Let z be an algebraic number of degree $d(z)$ with minimal polynomial

$$f(x) = a_0 x^{d(z)} + a_1 x^{d(z)-1} + \cdots + a_{d(z)} = a_0 \cdot \prod_{i=1}^{d(z)} (x - z_i)$$

where a_i's are relatively prime integers with $a_0 > 0$ and z_i's are conjugates of z. Then

$$h(z) = \frac{1}{d(z)} \left(\log a_0 + \sum_{i=1}^{d(z)} \log(\max\{|z_i|, 1\}) \right)$$

is called *the logarithmic height of* z. The following proposition gives some properties of logarithmic height that can be found in [12].

Proposition 2.1. *Let $z, z_1, z_2, \ldots, z_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

(1) $h(z_1 \cdots z_t) \le \sum_{i=1}^{t} h(z_i)$
(2) $h(z_1 + \cdots + z_t) \le \log t + \sum_{i=1}^{t} h(z_i)$
(3) $h(z^m) = |m| h(z)$.

We will use the following theorem (see [9] or Theorem 9.4 in [1]) for proving our results.

Theorem 2.1. *Let $z_1, z_2, \ldots, z_s$ be nonzero elements of algebraic number field $\mathbb{F}$ of degree D over $\mathbb{Q}$ and let $b_1, b_2, \ldots, b_s \in \mathbb{Z}$. Set*

$$B := \max\{|b_1|, \ldots, |b_s|\}$$

and

$$\Lambda := z_1^{b_1} \ldots z_s^{b_s} - 1.$$

If Λ is nonzero, then

$$\log|\Lambda| > -3 \cdot 30^{s+4} \cdot (s+1)^{5.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log(sB)) \cdot A_1 \cdots A_s$$

where

$$A_i \ge \max\{D \cdot h(z_i), |\log z_i|, 0.16\}$$

for all $1 \le i \le s$. If $\mathbb{F} = \mathbb{R}$, then

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

Another main tool for our proof is a variant of Baker and Davenport reduction method due to [6].

Let $\psi_1, \psi_2, \varepsilon \in \mathbb{R}$ be given and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Gamma = \varepsilon + x_1 \psi_1 + x_2 \psi_2. \tag{2.1}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Gamma| < c \cdot \exp(-\delta \cdot Y), \tag{2.2}$$

$$Y \leq X \leq X_0. \tag{2.3}$$

Set $\psi = -\psi_1/\psi_2$. Let the continued fraction expansion of ψ be given by $[a_0, a_1, a_2, \dots]$ and let the k^{th} convergent of ψ be p_k/q_k for $k = 0, 1, 2, \dots$. Without loss of the generality, we may assume that $|\psi_1| < |\psi_2|$ and that $x_1 > 0$. We have the following two result from [6].

Lemma 2.1. *Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

If (2.2) *and* (2.3) *hold for* x_1, x_2 *and* $\varepsilon = 0$ *in* (2.1)*, then*

$$Y < \frac{1}{\delta} \log\left(\frac{c\,(A+2)\,X_0}{|\psi_2|}\right).$$

If $\varepsilon \neq 0$*, then setting* $\phi = \varepsilon/\psi_2$ *we have that*

$$\frac{\Gamma}{\psi_2} = \phi - x_1\psi + x_2.$$

Lemma 2.2. *Let* p/q *be a convergent of* ψ *with* $q > X_0$*. Suppose that*

$$\|q\phi\| > \frac{2X_0}{q}$$

where $\|\cdot\|$ *denotes the distance from the nearest integer. Then, the solutions of* (2.2) *and* (2.3) *satisfy*

$$Y < \frac{1}{\delta} \log\left(\frac{q^2 c}{|\psi_2|\,X_0}\right).$$

3. The Proof of Theorem 1.1

Let us assume that $n \geq m \geq k$, $t > 0$ and $x \in \{1, 2, \dots, 9\}$. A quick search in Mathematica reveals that the solutions of equation (1.2) for $0 \leq n \leq 500$ are as stated in Theorem 1.1. Exactly, the solutions of equation (1.2) are

$$N \in \{3, 5, 7, 9, 11, 55, 99\}.$$

We will assume $n > 500$ for the remainder of the work. Further, since Half-companion Pell numbers are all odd, (1.2) has no solution for even x. Thus, with $A = \{1, 3, 5, 7, 9\}$, $x \in A$.

The equations (1.1) and (1.2) imply that

$$H_{501} \leq H_n \leq H_n + H_m + H_k = x \cdot \left(\frac{10^t - 1}{9}\right) \leq 10^t - 1.$$

It follows that

$$191 \leq \frac{\log(1 + H_{501})}{\log(10)} \leq t. \tag{3.1}$$

Now,

$$10^{t-1} \le x \cdot \left(\frac{10^t - 1}{9}\right) = H_n + H_m + H_k \le 3H_n$$
$$\le \frac{3}{2}\left(\varphi^n + |\psi|^n\right) < \frac{3}{2}\left(\varphi^n + 1\right) < \frac{3}{2}\varphi^n < \varphi^{n+0.47}$$

which means that

$$2.61\,(t-1) < (t-1)\,\frac{\log 10}{\log \varphi} < n + 0.47.$$

It can be easily seen from the last inequality that

$$t < 2.61t - 3.08 < n$$

and from (3.1) that

$$191 < t < n.$$

Let us rewrite the equation (1.2) as

$$\frac{\varphi^n + \psi^n}{2} + \frac{\varphi^m + \psi^m}{2} + \frac{\varphi^k + \psi^k}{2} = x \cdot \left(\frac{10^t - 1}{9}\right). \tag{3.2}$$

The equality (3.2) can be simply converted to

$$\frac{\varphi^n + \psi^n}{2} + \frac{\varphi^m + \psi^m}{2} + \frac{\varphi^k + \psi^k}{2} - \frac{x \times 10^t}{9} = -\frac{x}{9}. \tag{3.3}$$

In order to obtain a bound on n, we examine equation (3.3) in three different cases.

Case 1. *We have that*

$$\frac{\varphi^n}{2}\left(1 + \varphi^{m-n} + \varphi^{k-n}\right) - \frac{x \times 10^t}{9} = -\frac{x}{9} - \frac{\psi^n}{2} - \frac{\psi^m}{2} - \frac{\psi^k}{2}.$$

This yields

$$\left|\frac{\varphi^n}{2}\left(1 + \varphi^{m-n} + \varphi^{k-n}\right) - \frac{x \times 10^t}{9}\right| \le \frac{x}{9} + \frac{|\psi|^n}{2} + \frac{|\psi|^m}{2} + \frac{|\psi|^k}{2} < \frac{1}{2}\,(2+3)\,.$$

From this inequality, we get

$$\left|\frac{\varphi^n}{2}\left(1 + \varphi^{m-n} + \varphi^{k-n}\right) - \frac{x \times 10^t}{9}\right| < \frac{\varphi^2}{2}. \tag{3.4}$$

By multiplying both sides of the inequality (3.4) *by*

$$\frac{2\varphi^{-n}}{1 + \varphi^{m-n} + \varphi^{k-n}}$$

we obtain

$$\left|1 - \varphi^{-k} 10^t \left(\frac{2x}{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)}\right)\right| < \frac{\varphi^{2-n}}{1 + \varphi^{m-n} + \varphi^{k-n}} < \varphi^{2-n}. \tag{3.5}$$

Now let us check whether the obtained inequality (3.5) conforms to the hypothesis in Theorem 2.1. Set

$$\Delta_1 := 1 - \varphi^{-k} 10^t \left(\frac{2x}{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)} \right).$$

Suppose that $\Delta_1 = 0$. Then, we have

$$\varphi^n + \varphi^m + \varphi^k = \frac{10^t \times 2x}{9}$$

and hence, $\varphi^n + \varphi^m + \varphi^k \in \mathbb{Q}$, which is not possible for any $n, m, k > 0$. Therefore $\Delta_1 \neq 0$. Thus, we can apply Theorem 2.1 to the inequality (3.5). Take,

$$z_1 = \varphi, \quad z_2 = 10, \quad z_3 = \frac{2x}{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)}, \quad b_1 = -k, \quad b_2 = t, \quad b_3 = 1$$

where $z_1, z_2, z_3 \in \mathbb{Q}\sqrt{2}$ and $b_1, b_2, b_3 \in \mathbb{Z}$. Since $z_i \in \mathbb{Q}\sqrt{2}$ for $i \in \{1,2,3\}$ $D = 2$. So, we can take

$$\begin{aligned}
0.9 = A_1 &\geq 2 \cdot h(\varphi) = 2 \cdot \frac{1}{2} \log(\varphi) = \log(\varphi) \sim 0.88 \\
4.7 = A_2 &\geq 2 \cdot h(10) < 2 \cdot \log(10) \sim 4.6.
\end{aligned}$$

Let us compute A_3. We have

$$z_3 = \frac{2x}{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)} < 2$$

and

$$z_3^{-1} = \frac{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)}{2x} < \frac{27}{2} \varphi^{n-k}.$$

So, $|\log(z_3)| < 3 + (n-k)\log\varphi$. Also,

$$\begin{aligned}
h(z_3) &\leq h(2x) + h(9) + h\left(\varphi^{n-k} + \varphi^{m-k} + 1\right) \\
&\leq h(18) + h(9) + h\left(\varphi^{n-k} + \varphi^{m-k}\right) + \log 2 \\
&\leq h(18) + h(9) + \log 2 + h\left(\varphi^{m-k}\left(\varphi^{n-m} + 1\right)\right) \\
&\leq \log 18 + \log 9 + 2\log 2 + h\left(\varphi^{m-k}\right) + h\left(\varphi^{n-m}\right) \\
&\leq \log 18 + \log 9 + 2\log 2 + (m-k)\, h(\varphi) + (n-m)\, h(\varphi) \\
&= \log 18 + \log 9 + 2\log 2 + \frac{1}{2}(n-k)\log\varphi.
\end{aligned}$$

Therefore, $2h(z_3) \leq 13 + (n-k)\log\varphi$. This yields

$$13 + (n-k)\log\varphi = A_3 \geq \max\{2h(z_3), |\log(z_3)|, 0.16\}.$$

Finally, $B = \max\{k,t,1\} < n$. *Set* $K_1 = 1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)\cdot A_1\cdot A_2$. *By applying Theorem 2.1 to* Δ_1, *we get*

$$\log|\Delta_1| > -K_1\cdot(1+\log(n))\cdot(13+(n-k)\log\varphi)$$

and by using the inequality (3.5), *we obtain*

$$n\log\varphi < 2\log\varphi + K_1\cdot(1+\log(n))\cdot(13+(n-k)\log\varphi)\,. \tag{3.6}$$

Case 2. *Let us rewrite the equation* (3.3) *as*

$$\frac{\varphi^n}{2}\left(1+\varphi^{m-n}\right) - \frac{x\times 10^t}{9} = -\frac{x}{9} - \frac{\psi^n}{2} - \frac{\psi^m}{2} - \frac{\varphi^k}{2} - \frac{\psi^k}{2}. \tag{3.7}$$

This equation yields

$$\begin{aligned}\left|\frac{\varphi^n}{2}\left(1+\varphi^{m-n}\right) - \frac{x\times 10^t}{9}\right| &\le \frac{x}{9} + \frac{|\psi|^n}{2} + \frac{|\psi|^m}{2} + \frac{\varphi^k}{2} + \frac{|\psi|^k}{2}\\ &\le 1 + \frac{1}{2} + \frac{1}{2} + \frac{\varphi^k}{2} + \frac{1}{2}\\ &\le \frac{1}{2}(2+4)\,\varphi^k\end{aligned}$$

and from the last inequality we have that

$$\left|\frac{\varphi^n}{2}\left(1+\varphi^{m-n}\right) - \frac{x\times 10^t}{9}\right| < \frac{\varphi^{k+3}}{2}. \tag{3.8}$$

By multiplying both sides of the inequality (3.8) *by*

$$\frac{2\varphi^{-n}}{1+\varphi^{m-n}}$$

we get

$$\left|1 - \varphi^{-m}10^t\left(\frac{2x}{9\left(1+\varphi^{n-m}\right)}\right)\right| < \frac{\varphi^{k-n+3}}{1+\varphi^{m-n}}.$$

Finally, we can write

$$\left|1 - \varphi^{-m}10^t\left(\frac{2x}{9\left(1+\varphi^{n-m}\right)}\right)\right| < \varphi^{k-n+3}. \tag{3.9}$$

Set

$$\Delta_2 := 1 - \varphi^{-m}10^t\left(\frac{2x}{9\left(1+\varphi^{n-m}\right)}\right).$$

Similar to the proof given in the previous case, it can be seen easily that $\Delta_2 \neq 0$. *Using the notations in Theorem 2.1, we deduce that*

$$z_1 = \varphi, \qquad z_2 = 10, \qquad z_3 = \frac{2x}{9\left(1+\varphi^{n-m}\right)}, \qquad b_1 = -m, \qquad b_2 = t, \qquad b_3 = 1$$

where $z_1, z_2, z_3 \in \mathbb{Q}\sqrt{2}$ *and* $b_1, b_2, b_3 \in \mathbb{Z}$. *Since* $z_i \in \mathbb{Q}\sqrt{2}$ *for* $i \in \{1,2,3\}$ $D = 2$ *and* $B = \max\{m,t,1\} < n$. *So, we can take* $A_1 = 0.9$ *and* $A_2 = 4.7$ *as in the previous case. Let us compute* A_3. *We have*

$$z_3 = \frac{2x}{9\left(1+\varphi^{n-m}\right)} < 2$$

and

$$z_3^{-1} = \frac{9\left(1+\varphi^{n-m}\right)}{2x} < \frac{18}{2}\varphi^{n-m},$$

so $|\log(z_3)| < 3 + (n-m)\log\varphi$. *Also,*

$$\begin{aligned} h(z_3) &\leq h(2x) + h(9) + h\left(1+\varphi^{n-m}\right) \\ &\leq h(18) + h(9) + \log 2 + (n-m)\, h(\varphi) \\ &= h(18) + h(9) + \log 2 + \frac{1}{2}(n-m)\log\varphi. \end{aligned}$$

Therefore, $2h(z_3) \leq 12 + (n-m)\log\varphi$, *and so*

$$12 + (n-m)\log\varphi = A_3 \geq \max\{2h(z_3), |\log(z_3)|, 0.16\}.$$

Applying Theorem 2.1 to Δ_2 *gives us*

$$\log|\Delta_2| > -K_1 \cdot (1+\log(n)) \cdot (12 + (n-m)\log\varphi)$$

with K_1 *as given in the previous case. By using the inequality (3.9), we get*

$$(n-k)\log\varphi < 3\log\varphi + K_1 \cdot (1+\log(n)) \cdot (12 + (n-m)\log\varphi). \tag{3.10}$$

Case 3. *Let us consider equation (3.3) for the last time as follows:*

$$\frac{\varphi^n}{2} - \frac{x \times 10^t}{9} = -\frac{x}{9} - \frac{\psi^n}{2} - \frac{\varphi^m}{2} - \frac{\psi^m}{2} - \frac{\varphi^k}{2} - \frac{\psi^k}{2}. \tag{3.11}$$

This yields

$$\begin{aligned} \left|\frac{\varphi^n}{2} - \frac{x \times 10^t}{9}\right| &\leq \frac{x}{9} + \frac{|\psi|^n}{2} + \frac{\varphi^m}{2} + \frac{|\psi|^m}{2} + \frac{\varphi^k}{2} + \frac{|\psi|^k}{2} \\ &\leq 1 + \frac{1}{2} + \frac{\varphi^m}{2} + \frac{1}{2} + \frac{\varphi^k}{2} + \frac{1}{2} \leq \frac{1}{2}(2+5)\varphi^m \end{aligned}$$

and so

$$\left|\frac{\varphi^n}{2} - \frac{x \times 10^t}{9}\right| < \frac{\varphi^{m+3}}{2}. \tag{3.12}$$

Multiplying through (3.12) by $2\varphi^{-n}$, *we obtain*

$$\left|1 - \varphi^{-n} 10^t \left(\frac{2x}{9}\right)\right| < \varphi^{m-n+3}. \tag{3.13}$$

Set

$$\Delta_3 := 1 - \varphi^{-n} 10^t \left(\frac{2x}{9}\right).$$

Again it is clear that $\Delta_3 \neq 0$. *We apply Theorem 2.1 with the data*

$$z_1 = \varphi, \qquad z_2 = 10, \qquad z_3 = \frac{2x}{9}, \qquad b_1 = -n, \qquad b_2 = t, \qquad b_3 = 1$$

where $z_1, z_2, z_3 \in \mathbb{Q}\sqrt{2}$ *and* $b_1, b_2, b_3 \in \mathbb{Z}$. *Since* $z_i \in \mathbb{Q}\sqrt{2}$ *for* $i \in \{1,2,3\}$ $D = 2$. *We have the same* A_1 *and* A_2 *as in the previous cases. Let us compute* A_3.

$$z_3 = \frac{2x}{9} \leq 2$$

and

$$z_3^{-1} = \frac{9}{2x} \leq \frac{9}{2},$$

so $|\log z_3| < 1.51$. *Also,*

$$h(z_3) \leq h(2x) + h(9) \leq h(18) + h(9)$$

which implies that

$$2h(z_3) < 10.2.$$

Therefore, we obtain

$$10.2 = A_3 \geq \max\{2h(z_3), |\log z_3|, 0.16\}.$$

By applying Theorem 2.1 to (3.13), we get

$$(n-m)\log\varphi < 3\log\varphi + 10.2\cdot K_1\cdot(1+\log n) < 10.4\cdot K_1\cdot(1+\log n).$$

Combining this result with the inequality (3.10), we get

$$\begin{aligned}(n-k)\log\varphi &< 3\log\varphi + K_1\cdot(1+\log(n))\cdot(12+10.4\cdot K_1\cdot(1+\log(n)))\\ &< 10.6\cdot K_1^2\cdot(1+\log(n))^2.\end{aligned}$$

The last result and the inequality (3.6) show that

$$\begin{aligned}n\log\varphi &< 2\log\varphi + K_1\cdot(1+\log(n))\cdot\left(13+10.6\cdot K_1^2\cdot(1+\log(n))^2\right)\\ &< 10.8\cdot K_1^3\cdot(1+\log(n))^3.\end{aligned}$$

Simplifying the last inequality we obtain

$$n \leq 9.68\times 10^{44}.$$

Now let us try to lower this bound. Set

$$\Gamma_1 = -n\log\varphi + t\log 10 + \log\left(\frac{2x}{9}\right).$$

If we insert the Γ_1 into the equation (3.11), we get

$$\frac{\varphi^n}{2}\left(1-e^{\Gamma_1}\right) = -\frac{x}{9} - \frac{\psi^n}{2} - H_m - H_k < 0$$

and so $\Gamma_1 > 0$. Thus, we obtain

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 < \varphi^{m-n+3}$$

from the inequality (3.13). It follows that

$$|\Gamma_1| < \varphi^{3.1} \exp\left(-0.88\,(n-m)\right)$$

with $n-m \leq n \leq 9.68 \times 10^{44}$. If we divide Γ_1 by $\log 10$, we get

$$\frac{\Gamma_1}{\log 10} = \frac{\log(2x/9)}{\log 10} - n\frac{\log\varphi}{\log 10} + t.$$

Hence, we can take

$$c = \varphi^{3.1}, \qquad \delta = 0.88, \qquad X_0 = 9.68 \times 10^{44}, \qquad \phi = \frac{\log(2x/9)}{\log 10}$$

$$\psi = \frac{\log\varphi}{\log 10}, \qquad \psi_1 = -\log\varphi, \qquad \psi_2 = \log 10, \qquad \varepsilon = \log(2x/9).$$

For all values of $x \in \{1, 2, \ldots, 9\}$, $q = q_{99}$ satisfies the conditions $q > X_0$ and $\|q\phi\| > \frac{2X_0}{q}$ given in the Lemma 2.2. So, applying Lemma 2.2, we obtain

$$n - m < 128.$$

Set

$$\Gamma_2 = -m\log\varphi + t\log 10 + \log\left(\frac{2x}{9\left(\varphi^{n-m}+1\right)}\right).$$

From (3.7), we have that

$$\frac{\varphi^n}{2}\left(1+\varphi^{m-n}\right)\left(1-e^{\Gamma_2}\right) = -\frac{x}{9} - \frac{\psi^n}{2} - \frac{\psi^m}{2} - H_k < 0$$

which implies $\Gamma_2 > 0$. From the inequality (3.9), we obtain

$$0 < \Gamma_2 < e^{\Gamma_2} - 1 < \varphi^{k-n+3}.$$

Hence,

$$|\Gamma_2| < \varphi^{3.1} \exp\left(-0.88\,(n-k)\right)$$

where $n-k \leq n \leq 9.68 \times 10^{44}$. $q = q_{103}$ satisfies the hypothesis of Lemma 2.2 and applying the Lemma 2.2, we obtain $n-k \leq 141$. Finally, set

$$\Gamma_3 = -k\log\varphi + t\log 10 + \log\left(\frac{2x}{9\left(\varphi^{n-k}+\varphi^{m-k}+1\right)}\right).$$

It is easy to see from previous trials that $\Gamma_3 > 0$. So, the inequality (3.5) implies that

$$0 < \Gamma_3 < e^{\Gamma_3} - 1 < \varphi^{2-n} < \varphi^{2.1} \cdot \exp\left(-0.88 \cdot n\right)$$

which yields

$$|\Gamma_3| < \varphi^{2.1} \cdot \exp\left(-0.88 \cdot n\right).$$

We see that hypothesis of Lemma 2.2 holds for $q = q_{109}$. Thus, applying Lemma 2.2 we obtain $n < 155$ which contradicts our assumption that $n > 500$.

This completes our proof.

4. Conclusion

In this paper we have given all solutions of equation (1.2). A similar work can be done for the quadruple sum.

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Conflict of Interest

The author declare no conflicts of interest.

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