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CONSTRUCTION OF A NEW CLASS OF FUNCTIONS WITH THEIR SOME PROPERTIES AND CERTAIN INEQUALITIES: *n*-FRACTIONAL POLYNOMIAL CONVEX FUNCTIONS

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Abstract. In this paper, we introduce a new function class called n-fractional polynomial convex function and their some algebric properties. We obtain some refinements of the right-hand side of Hermite-Hadamard inequality for the class of functions whose derivatives in absolutely value at certain powers are n-fractional polynomial convex. Also, we compare the results obtained with both Hölder, Hölder-İşcan inequalities and power-mean, improved-power-mean integral inequalities and show that the result obtained with Hölder-İşcan and improved power-mean inequalities give better approach than the others. Some applications to special means of real numbers are also given.

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1. PRELIMINARIES

Due to its robustness, convex functions and convex sets have been generalized and extended in many mathematics branches, in particular, many inequalities can be found in the literature via convexity theory. To the best of our knowledge, the Hermite–Hadamard inequality is a well-known, paramount and extensively useful inequality in the applied literature of mathematical inequalities. Let $\varphi\colon I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \varphi(x)dx \leq \frac{\varphi(u)+\varphi(v)}{2} \tag{1.1}$$

for all $u,v \in I$ with $u < v$. Both inequalities hold in the reversed direction if the function φ is concave. This double inequality is well known as the Hermite-Hadamard inequality [5]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping φ. Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. See articles [1,3,4,6,8,9,11] and the references therein.

Definition 1 ([12, Definition 4]). Let $h\colon J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $\varphi\colon I \to \mathbb{R}$ is an h-convex function, or that φ belongs to the class $SX(h,I)$, if φ is non-negative and for all $u,v \in I$, $\alpha \in (0,1)$ we have

$$\varphi(\alpha u + (1-\alpha)v) \leq h(\alpha)\varphi(u) + h(1-\alpha)\varphi(v).$$

If this inequality is reversed, then φ is said to be h-concave, i.e. $\varphi \in SV(h,I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [2] for studies on h-convexity.

In [7], İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 1 (Hölder-İşcan integral inequality [7, Theorem 2.1]). *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If φ and ψ are real functions defined on interval $[u,v]$ and if $|\varphi|^p$, $|\psi|^q$ are integrable functions on $[u,v]$ then*

$$\int_u^v |\varphi(x)\psi(x)|\,dx \leq \frac{1}{v-u}\left\{\left(\int_u^v (v-x)\,|\varphi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_u^v (v-x)\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}} + \left(\int_u^v (x-u)\,|\varphi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_u^v (x-u)\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}}\right\}$$

An refinement of power-mean integral inequality as a different version of the Hölder-İşcan integral inequality can be given as follows:

Theorem 2 (Improved power-mean integral inequality [10]). *Let $q \geq 1$. If φ and ψ are real functions defined on interval $[u,v]$ and if $|\varphi|$, $|\varphi|\,|\psi|^q$ are integrable functions on $[u,v]$ then*

$$\begin{aligned}
&\int_u^v |\varphi(x)\psi(x)|\,dx \\
&\leq \frac{1}{v-u}\left\{\left(\int_u^v (v-x)\,|\varphi(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_u^v (v-x)\,|\varphi(x)|\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_u^v (x-u)\,|\varphi(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_u^v (x-u)\,|\varphi(x)|\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

The main purpose of this paper is to introduce the concept of n-fractional polynomial convex functions and establish some results connected with the right-hand side of new inequalities similar to the Hermite-hadamard inequality for these classes of functions. Some applications to special means of positive real numbers are also given.

2. The definition of n-fractional polynomial convex functions

In this section, we introduce a new concept, which is called n-fractional polynomial convexity and we give by setting some algebraic properties for the n-fractional polynomial convex functions, as follows:

Definition 2. Let $n \in \mathbb{N}$. A non-negative function $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$ is called n-fractional polynomial convex function if the inequality

$$\varphi(tu+(1-t)v) \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i}\varphi(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi(v) \tag{2.1}$$

holds for every $u,v \in I$ and $t \in [0,1]$.

We will denote by $FPC(I)$ the class of all fractional n-fractional polynomial convex functions on interval I.

We note that, every fractional n-fractional polynomial convex function is a h-convex function with the function $h(t) = \frac{1}{n}\sum_{i=1}^{n} t^{1/i}$. Therefore, if $\varphi, \psi \in FPC(I)$, then

(i) $\varphi+\psi \in FPC(I)$ *and for* $c \in \mathbb{R}$ *(*$c \geq 0$*)* $c\varphi \in FPC(I)$. (see [12, Proposition 9]).

(ii) if φ and g be a similarly ordered functions on I , then $\varphi\psi \in FPC(I)$. (See [12, Proposition 10]).

Also, if $\varphi\colon I \to J$ is a convex and $\psi \in FPC(J)$ and nondecreasing, then $\psi \circ \varphi \in FPC(I)$ (see [12, Theorem 15]).

Remark 1. We especially note that; if we take $n = 1$ in the inequality (2.1), then the 1-polynomial convexity reduces to the clasical convexity.

With the help of the following remark, we can give all non-negative convex functions as an example of an n-fractional polynomial convex functions.

Remark 2. Every nonnegative convex function is also a n-fractional polynomial convex function. Indeed, since

$$t \leq t^{1/2} \leq t^{1/3} \leq \ldots \leq t^{1/n}$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. We can write

$$t \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i} \text{ and } 1-t \leq \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. and thus, if φ is an nonnegative convex function on an interval $I \subseteq \mathbb{R}$, then we have

$$\varphi(tu+(1-t)v) \leq t\varphi(u)+(1-t)\varphi(v) \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i}\varphi(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi(v)$$

for all $u,v \in I$, $t \in [0,1]$ and $n \in \mathbb{N}$.

Theorem 3. *Let* $\varphi\colon [u,v] \to \mathbb{R}$ *be an n-fractional polynomial convex function, then* φ *is bounded on* $[u,v]$.

Proof. Let $M = \varphi(u) + \varphi(v)$ and $x \in [u,v]$ be an arbitrary point. There exists an $t \in [0,1]$ such that $x = tu + (1-t)v$. Also, since

$$\frac{1}{n}\sum_{i=1}^{n} t^{1/i} \le 1 \text{ and } \frac{1}{n}\sum_{i=1}^{n} (1-t)^{1/i} \le 1$$

for all $t \in [0,1]$, we can write

$$\varphi(x) \le \frac{1}{n}\sum_{i=1}^{n} t^{1/i}\varphi(u) + \frac{1}{n}\sum_{i=1}^{n} (1-t)^{1/i}\varphi(v) \le M.$$

It is also bounded from below as we see by writing an arbitrary point $x \in [u,v]$ in the form $(u+v)/2+t$ or $(u+v)/2-t, t \in [0,(v-u)/2]$ .We can accept $x = (u+v)/2+t$ since it will not loss the generality. Then

$$\varphi\left(\frac{u+v}{2}\right) \le \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}\left[\varphi(x)+\varphi\left(\frac{u+v}{2}-t\right)\right] \le \left[\varphi(x)+\varphi\left(\frac{u+v}{2}-t\right)\right]$$

or

$$\varphi(x) \ge \varphi\left(\frac{u+v}{2}\right) - \varphi\left(\frac{u+v}{2}-t\right).$$

Using M as the upper bound, $-\varphi\left(\frac{u+v}{2}-t\right) \ge -M$, so

$$\varphi(x) \ge \varphi\left(\frac{u+v}{2}\right) - M = m.$$

Thus, the proof is completed. □

Theorem 4. *Let* $\varphi_\alpha\colon [u,v] \to \mathbb{R}$ *be an arbitrary family of n-fractional polynomial convex functions and let* $\varphi(m) = \sup_\alpha \varphi_\alpha(m)$. *If* $J = \{m \in [u,v] : \varphi(m) < \infty\}$ *is nonempty, then* J *is an interval and* φ *is a n-fractional polynomial convex function on* J.

Proof. Let $t \in [0,1]$ and $m, r \in J$ be arbitrary. Then

$$\begin{aligned}
\varphi(tm+(1-t)r) &= \sup_\alpha \varphi_\alpha(tm+(1-t)r) \\
&\le \sup_\alpha \left[\frac{1}{n}\sum_{i=1}^{n} t^{1/i}\varphi_\alpha(m) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi_\alpha(r)\right] \\
&\le \frac{1}{n}\sum_{i=1}^{n} t^{1/i}\sup_\alpha \varphi_\alpha(m) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\sup_\alpha \varphi_\alpha(r) \\
&= \frac{1}{n}\sum_{i=1}^{n} t^{1/i}\varphi(m) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi(r) < \infty.
\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that φ is a n-fractional polynomial convex function. This completes the proof of theorem. $\square$

3. HERMITE-HADAMARD INEQUALITY FOR N-FRACTIONAL POLYNOMIAL CONVEX FUNCTIONS

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for n-fractional polynomial convex functions. In this section, we will denote by $L[u,v]$ the space of (Lebesgue) integrable functions on $[u,v]$.

Theorem 5. *Let $\varphi : [u,v] \to \mathbb{R}$ be a n-fractional polynomial convex function. If $u < v$ and $\varphi \in L[u,v]$, then the following Hermite-Hadaamrd type inequalities hold:*

$$\frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}\varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \varphi(x)dx \leq \left(\frac{\varphi(u)+\varphi(v)}{n}\right)\sum_{k=1}^{n}\frac{k}{k+1}. \tag{3.1}$$

Proof. From the propery of the n-fractional polynomial convex function of φ, we get

$$\begin{aligned}
\varphi\left(\frac{u+v}{2}\right) &= \varphi\left(\frac{(tu+(1-t)v)+[(1-t)u+tv]}{2}\right) \\
&= \varphi\left(\frac{1}{2}(tu+(1-t)v)+\frac{1}{2}[(1-t)u+tv]\right) \\
&\leq \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\varphi(tu+(1-t)v)+\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\varphi((1-t)u+tv) \\
&= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\left[\varphi(tu+(1-t)v)+\varphi((1-t)u+tv)\right].
\end{aligned}$$

By taking integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$\begin{aligned}
\varphi\left(\frac{u+v}{2}\right) &\leq \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\left[\int_0^1 \varphi(tu+(1-t)v)\,dt+\int_0^1 \varphi((1-t)u+tv)\,dt\right] \\
&= \frac{2}{(v-u)n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\int_u^v \varphi(x)dx.
\end{aligned}$$

By using the property of the n-fractional polynomial convex function φ, if the variable is changed as $x = tu+(1-t)v$, then

$$\begin{aligned}
\frac{1}{v-u}\int_u^v \varphi(x)du &= \int_0^1 \varphi(tu+(1-t)v)\,dt \\
&\leq \int_0^1\left[\frac{1}{n}\sum_{k=1}^{n}t^{1/k}\varphi(u)+\frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\varphi(v)\right]dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varphi(u)}{n}\int_0^1 \sum_{k=1}^{n} t^{1/k}dt + \frac{\varphi(v)}{n}\int_0^1 \sum_{k=1}^{n} (1-t)^{1/k}dt \\
&= \frac{\varphi(u)}{n+1}\sum_{i=1}^{n}\int_0^1 t^{1/k}dt + \frac{\varphi(v)}{n}\sum_{k=1}^{n}\int_0^1 (1-t)^{1/k}dt \\
&= \frac{\varphi(u)}{n}\sum_{k=1}^{n}\frac{k}{k+1} + \frac{\varphi(v)}{n}\sum_{k=1}^{n}\frac{k}{k+1} \\
&= \left[\frac{\varphi(u)+\varphi(v)}{n}\right]\sum_{k=1}^{n}\frac{k}{k+1}
\end{aligned}
$$

where

$$
\int_0^1 t^{1/k}dt = \int_0^1 (1-t)^{1/k} = \frac{k}{k+1}.
$$

This completes the proof of theorem. □

Remark 3. In case of $n = 1$, the inequality (3.1) coincides with the the inequality (1.1)

4. Trapezoid type inequalities for n-fractional polynomial convex functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is n-fractional polynomial convex function. Dragomir and Agarwal [3] used the following lemma:

Lemma 1. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $u,v \in I^\circ$ with $u < v$. If $\varphi' \in L[u,v]$.The following identity holds:*

$$
\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx = \frac{v-u}{2}\int_0^1 (1-2t)\varphi'(tu+(1-t)v)\,dt.
$$

Theorem 6. *Let $\varphi\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $u,v \in I^\circ$ with $u < v$ and assume that $\varphi' \in L[u,v]$. If $|\varphi'|$ is n-fractional polynomial convex function on interval $[u,v]$, then the following inequality holds*

$$
\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \qquad (4.1)\\
&\le \frac{v-u}{n}\sum_{k=1}^{n}\left[\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right] A\left(\left|\varphi'(u)\right|, \left|\varphi'(v)\right|\right),
\end{aligned}
$$

where $A(.,.)$ is the arithmetic mean.

Proof. Using Lemma 1 and the inequality

$$\left|\varphi'(tu+(1-t)v)\right| \le \frac{1}{n}\sum_{k=1}^{n} t^{1/k}\left|\varphi'(u)\right| + \frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|,$$

we get

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \\
&\le \left|\frac{v-u}{2}\int_0^1 (1-2t)\varphi'(tu+(1-t)v)\,dt\right| \\
&\le \frac{v-u}{2}\int_0^1 |1-2t|\left(\frac{1}{n}\sum_{k=1}^{n} t^{1/k}\left|\varphi'(u)\right| + \frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|\right)dt \\
&\le \frac{v-u}{2n}\left(\left|\varphi'(u)\right|\int_0^1 |1-2t|\sum_{k=1}^{n} t^{1/k}dt + \left|\varphi'(v)\right|\int_0^1 |1-2t|\sum_{k=1}^{n}(1-t)^{1/k}dt\right) \\
&= \frac{v-u}{2n}\left(\left|\varphi'(u)\right|\sum_{k=1}^{n}\int_0^1 |1-2t|\,t^{1/k}dt + \left|\varphi'(v)\right|\sum_{k=1}^{n}\int_0^1 |1-2t|\,(1-t)^{1/k}dt\right) \\
&= \frac{v-u}{2n}\left(\left|\varphi'(u)\right|\sum_{k=1}^{n}\left[\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right] + \left|\varphi'(v)\right|\sum_{k=1}^{n}\left[\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right]\right) \\
&= \frac{v-u}{n}\sum_{k=1}^{n}\left[\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right] A\left(\left|\varphi'(u)\right|,\left|\varphi'(v)\right|\right)
\end{aligned}$$

where

$$\int_0^1 |1-2t|\,t^{1/k}dt = \int_0^1 |1-2t|\,(1-t)^{1/k} = \frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}.$$

This completes the proof of theorem. □

Corollary 1. *If we take $n=1$ in the inequality (4.1), we get the following inequality:*

$$\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \le \frac{v-u}{4}A\left(\left|\varphi'(u)\right|,\left|\varphi'(v)\right|\right).$$

This inequality coincides with the inequality in [3].

Theorem 7. *Let $\varphi: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $u,v\in I^\circ$ with $u<v$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and assume that $\varphi'\in L[u,v]$. If $|\varphi'|^q$ is an n-fractional polynomial convex function on interval $[u,v]$, then the following inequality holds*

$$\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \tag{4.2}$$

$$\leq \frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^{q},\left|\varphi'(v)\right|^{q}\right),$$

where $A(.,.)$ is the arithmetic mean.

Proof. Using Lemma 1, Hölder's integral inequality and the following inequality

$$\left|\varphi'\left(tu+(1-t)v\right)\right|^{q} \leq \frac{1}{n}\sum_{k=1}^{n} t^{1/k}\left|\varphi'(u)\right|^{q} + \frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|^{q}$$

which is the n-fractional polynomial convex function of $|\varphi'|^{q}$, we get

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_{u}^{v}\varphi(x)dx\right| \\
&\leq \frac{v-u}{2}\left(\int_{0}^{1}|1-2t|^{p}\,dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\varphi'\left(tu+(1-t)v\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\leq \frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^{q}}{n}\sum_{k=1}^{n}\int_{0}^{1}t^{1/k}dt+\frac{|\varphi'(v)|^{q}}{n}\sum_{k=1}^{n}\int_{0}^{1}(1-t)^{1/k}dt\right)^{\frac{1}{q}} \\
&= \frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^{q},\left|\varphi'(v)\right|^{q}\right),
\end{aligned}$$

where

$$\int_{0}^{1}|1-2t|^{p}\,dt=\frac{1}{p+1}.$$

This completes the proof of theorem. □

Corollary 2. *If we take $n=1$ in the inequality (4.2), we get the following inequality:*

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_{u}^{v}\varphi(x)dx\right| \leq \frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^{q},\left|\varphi'(v)\right|^{q}\right).$$

This inequality coincides with the inequality in [3].

Theorem 8. *Let $\varphi\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $u,v\in I^{\circ}$ with $u<v$, $q\geq 1$and assume that $\varphi'\in L[u,v]$. If $|\varphi'|^{q}$ is an n-fractional polynomial convex function on the interval $[u,v]$, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_{u}^{v}\varphi(x)dx\right| \qquad (4.3)\\
&\leq \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^{q},\left|\varphi'(v)\right|^{q}\right),
\end{aligned}$$

where $A(.,.)$ is the arithmetic mean.

Proof. Assume first that $q>1$. From Lemma 1, power mean integral inequality and the property of the n-fractional polynomial convex function of $|\varphi'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v \varphi(x)dx\right| \\
&\leq \frac{v-u}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left[\frac{1}{n}\sum_{k=1}^n t^{1/k}\left|\varphi'(u)\right|^q\right.\right. \\
&\qquad\qquad \left.\left. +\frac{1}{n}\sum_{k=1}^n (1-t)^{1/k}\left|\varphi'(v)\right|^q dt\right]\right)^{\frac{1}{q}} \\
&= \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^n\int_0^1 |1-2t|\,t^{1/k}dt\right. \\
&\qquad\qquad \left. +\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^n\int_0^1 |1-2t|\,(1-t)^{1/k}dt\right]^{\frac{1}{q}} \\
&= \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{k=1}^n \frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^q,\left|\varphi'(v)\right|^q\right).
\end{aligned}$$

For $q=1$ we use the estimates from the proof of Theorem 6, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 3. *Under the assumption of Theorem 8 with $q=1$, we get the conclusion of Theorem 6.*

Corollary 4. *If we take $n=1$ in the inequality (4.3), we get the following inequality:*

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v \varphi(x)dx\right| \leq \frac{v-u}{4}A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^q,\left|\varphi'(v)\right|^q\right).$$

If we take $q=1$ in the above inequality, then obtained inequality coincides with the inequality in [3].

Now, we will prove the Theorem 7 by using Hölder-İşcan integral inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 7.

Theorem 9. *Let $\varphi\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $u,v\in I^\circ$ with $u<v$, $q>1, \frac{1}{p}+\frac{1}{q}=1$ and assume that $\varphi'\in L[u,v]$. If $|\varphi'|^q$ is an n-fractional* *polynomial convex function on interval* $[u,v]$*, then the following inequality holds*

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v \varphi(x)dx\right| \tag{4.4}$$
$$\leq \frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_1(n)+\frac{|\varphi'(v)|^q}{n}K_2(n)\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_2(n)+\frac{|\varphi'(v)|^q}{n}K_1(n)\right)^{\frac{1}{q}},$$

where

$$K_1(n)=\sum_{k=1}^{n}\frac{k^2}{(k+1)(2k+1)}, \qquad K_2(n)=\sum_{k=1}^{n}\frac{k}{2k+1}.$$

Proof. Using Lemma 1, Hölder-İşcan integral inequality and the following inequality

$$\left|\varphi'(tu+(1-t)v)\right|^q \leq \frac{1}{n}\sum_{k=1}^{n}t^{1/k}\left|\varphi'(u)\right|^q+\frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|^q$$

which is the n-fractional polynomial convex function of $|\varphi'|^q$, we get

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v \varphi(x)dx\right|$$
$$\leq \frac{v-u}{2}\left(\int_0^1(1-t)|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\int_0^1 t|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)t^{1/k}dt\right.$$
$$\left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)(1-t)^{1/k}dt\right)^{\frac{1}{q}}$$
$$\leq \frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1 t.t^{1/k}dt\right.$$
$$\left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1 t(1-t)^{1/k}dt\right)^{\frac{1}{q}}$$
$$=\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_1(n)+\frac{|\varphi'(v)|^q}{n}K_2(n)\right)^{\frac{1}{q}}$$

$$+\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_2(n)+\frac{|\varphi'(v)|^q}{n}K_1(n)\right)^{\frac{1}{q}}$$

where

$$\int_0^1(1-t)|1-2t|^p\,dt=\int_0^1 t\,|1-2t|^p\,dt=\frac{1}{2(p+1)},$$

$$\int_0^1(1-t)t^{1/k}dt=\int_0^1 t(1-t)^{1/k}dt=\frac{k^2}{(k+1)(2k+1)},$$

$$\int_0^1(1-t)(1-t)^{1/k}dt=\int_0^1 t.t^{1/k}dt=\frac{k}{2k+1}.$$

This completes the proof of theorem. □

Corollary 5. *If we take $n=1$ in the inequality (4.4), we get the following inequality:*

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v\varphi(x)dx\right|$$
$$\leq\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\varphi'(u)|^q+2|\varphi'(v)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2|\varphi'(u)|^q+|\varphi'(v)|^q}{6}\right)^{\frac{1}{q}}\right].$$

Remark 4. The inequality (4.4) gives better results than the inequality (4.2). Indeed, using the inequality $x^r+y^r\leq 2^{1-r}(x+y)^r$, $x,y\in[0,\infty)$, $0<r\leq 1$, by sample calculation we get

$$\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_1(n)+\frac{|\varphi'(v)|^q}{n}K_2(n)\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_2(n)+\frac{|\varphi'(v)|^q}{n}K_1(n)\right)^{\frac{1}{q}}$$
$$\leq\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}2^{1-1/q}\left(\frac{K_1(n)+K_2(n)}{n}\right)^{\frac{1}{q}}\left(|\varphi'(u)|^q+|\varphi'(v)|^q\right)^{\frac{1}{q}}$$
$$=\frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|\varphi'(u)|^q,|\varphi'(v)|^q\right)$$

which is the required.

Theorem 10. *Let $\varphi: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $u,v\in I^\circ$ with $u<v$, $q\geq 1$ and assume that $\varphi'\in L[u,v]$. If $|\varphi'|^q$ is an n-fractional polynomial convex function on the interval $[u,v]$, then the following inequality holds for $t\in[0,1]$.*

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v\varphi(x)dx\right| \tag{4.5}$$

$$\leq \frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}M_1(k)+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}M_2(k)\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}M_2(k)+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}M_1(k)\right)^{\frac{1}{q}},$$

where

$$M_1(k)=\frac{k^2\left[\left(\frac{1}{2}\right)^{1+1/k}(5k+1)+1-k\right]}{(k+1)(2k+1)(3k+1)},\qquad M_2(k)=\frac{k\left[\left(\frac{1}{2}\right)^{1+1/k}k+1+k\right]}{(2k+1)(3k+1)}.$$

Proof. Assume first that $q>1$. From Lemma 1, improved power-mean integral inequality and the property of the n-fractional polynomial convex function of $|\varphi'|^q$, we obtain

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v\varphi(x)dx\right|$$
$$\leq \frac{v-u}{2}\left(\int_0^1(1-t)|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)|1-2t|\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\int_0^1 t|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|1-2t|\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{v-u}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)|1-2t|t^{1/k}dt\right.$$
$$\left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)|1-2t|(1-t)^{1/k}dt\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1 t|1-2t|t^{1/k}dt\right.$$
$$\left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1 t|1-2t|(1-t)^{1/k}dt\right)^{\frac{1}{q}}$$
$$=\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}M_1(k)+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}M_2(k)\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}M_2(k)+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}M_1(k)\right)^{\frac{1}{q}}$$

where

$$\int_0^1 (1-t)\,|1-2t|\,dt = \int_0^1 t\,|1-2t|\,dt = \frac{1}{4},$$

$$\begin{aligned}
M_1(k) &= \int_0^1 (1-t)\,|1-2t|\,t^{1/k} = \int_0^1 t\,|1-2t|\,(1-t)^{1/k} dt \\
&= \frac{k^2\left[\left(\frac{1}{2}\right)^{1+1/k}(5k+1)+1-k\right]}{(k+1)\,(2k+1)(3k+1)}, \\
M_2(k) &= \int_0^1 t\,|1-2t|\,t^{1/k} dt = \int_0^1 (1-t)\,|1-2t|\,(1-t)^{1/k} dt \\
&= \frac{k\left[\left(\frac{1}{2}\right)^{1+1/k} k+1+k\right]}{(2k+1)(3k+1)}.
\end{aligned}$$

For $q=1$ we use the estimates from the proof of Theorem 6, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 6. *If we take $n=1$ in the inequality (4.5), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \\
&\leq \frac{v-u}{8}\left[\left(\frac{|\varphi'(u)|^q}{4}+\frac{3\,|\varphi'(v)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\,|\varphi'(u)|^q}{4}+\frac{|\varphi'(v)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 5. The inequality (4.5) gives better result than the inequality (4.3). Indeed, If we use the inequality $x^r+y^r \leq 2^{1-r}(x+y)^r$, $x,y\in[0,\infty)$, $0<r\leq 1$, we get

$$\begin{aligned}
&\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{s=1}^n M_1(k)+\frac{|\varphi'(v)|^q}{n}\sum_{s=1}^n K_2(s)\right)^{\frac{1}{q}} \\
&+\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{s=1}^n M_2(k)+\frac{|\varphi'(v)|^q}{n}\sum_{s=1}^n K_1(s)\right)^{\frac{1}{q}} \\
&\leq \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^n [M_1(k)+M_2(k)]\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|\varphi'(u)|^q,|\varphi'(v)|^q\right),
\end{aligned}$$

where

$$M_1(k)+M_2(k) = \frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}$$

which completes the proof of remark.

5. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers u, v with $v > u$:

1. The arithmetic mean
$$A := A(u,v) = \frac{u+v}{2}.$$
2. The geometric mean
$$G := G(u,v) = \sqrt{uv}, \quad u,v \geq 0.$$
3. The harmonic mean
$$H := H(u,v) = \frac{2uv}{u+v}, \quad u,v > 0.$$
4. The logarithmic mean
$$L := L(u,v) = \begin{cases} \frac{v-u}{\ln v - \ln u}, & u \neq v; \\ u, & u = v; \end{cases} \quad u,v > 0.$$
5. The p-logaritmic mean
$$L_p := L_p(u,v) = \begin{cases} \left(\frac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right)^{\frac{1}{p}}, & u \neq v, p \in \mathbb{R}\setminus\{-1,0\}; \\ u, & u = v; \end{cases} \quad u,v > 0.$$
6. The identric mean
$$I := I(u,v) = \frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}}, \quad u,v > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:
$$H \leq G \leq L \leq I \leq A.$$
It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 1. *Let $u, v \in [0,\infty)$ with $u < v$ and $n \in (-\infty,0) \cup [1,\infty) \setminus \{-1\}$. Then, the following inequalities are obtained:*
$$\frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}} A^n(u,v) \leq L_n^n(u,v) \leq A(u^n,v^n)\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}.$$

Proof. The assertion follows from the inequalities (3.1) for the function
$$\varphi(x) = x^n, \;\; x \in [0,\infty).$$
□

Proposition 2. *Let* $u,v\in(0,\infty)$ *with* $u<v$ *. Then, the following inequalities are obtained:*

$$\frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}\leq L^{-1}(u,v)\leq\frac{2}{n}H^{-1}(u,v)\sum_{k=1}^{n}\frac{k}{k+1}.$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$\varphi(x)=x^{-1},\ \ x\in(0,\infty).$$

□

Proposition 3. *Let* $u,v\in(0,1]$ *with* $u<v$*. Then, the following inequalities are obtained:*

$$\frac{2\ln G(u,v)}{n}\sum_{k=1}^{n}\frac{k}{k+1}\leq\ln I\leq\frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}\ln A(u,v).$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$\varphi(x)=-\ln x,\ \ x\in(0,1].$$

□

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