Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1569–1579

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4135

NJ-SEMICOMMUTATIVE RINGS

SANJIV SUBBA AND TIKARAM SUBEDI

Received 22 February, 2022

Abstract. We call a ring R NJ-semicommutative if $wh \in N(R)$ implies $wRh \subseteq J(R)$ for any $w,h \in R$. The class of NJ-semicommutative rings is large enough that it contains semicommutative rings, left (right) quasi-duo rings, J-clean rings, and J-quasipolar rings. We provide some conditions for NJ-semicommutative rings to be reduced. We also observe that if $R/J(R)$ is reduced, then R is NJ-semicommutative, and therefore we provide some conditions for NJ-semicommutative ring R for which $R/J(R)$ is reduced. We also study some extensions of NJ-semicommutative rings wherein, among other results, we prove that the polynomial ring over an NJ-semicommutative ring need not be NJ-semicommutative.

2010 *Mathematics Subject Classification:* 16U80; 16S34, 16S36

Keywords: semicommutative rings, NJ-semicommutative rings, Jacobson radicals

1. Introduction

All rings considered in this paper are associative with identity unless otherwise mentioned. R represents a ring, and all modules are unital. The symbols $Z(R)$, $E(R)$, $J(R)$, $N(R)$, $U(R)$, $T_n(R)$, $M_n(R)$, $N^*(R)$, and $N_*(R)$ respectively denote the set of all central elements of R, the set of all idempotent elements of R, the Jacobson radical of R, the set of all nilpotent elements of R, the set of all units of R, the ring of upper triangular matrices of order $n \times n$ over R, the ring of all $n \times n$ matrices over R, the upper nil radical of R, and the lower nil radical of R. For any $a \in R$, the notation $l(a)$ $(r(a))$ stands for the left (right) annihilator of a.

Recall that R is said to be:

(1) *reduced* if $N(R) = 0$.
(2) *semicommutative* ([8]) if $wh = 0$ implies $wRh = 0$ for any $w,\ h \in R$.
(3) *abelian* if $E(R) \subseteq Z(R)$.
(4) *directly finite* if $xy = 1$ implies $yx = 1$, where $x,\ y \in R$.
(5) *left (right) quasi-duo* ([7]) if every maximal left (right) ideal of R is an ideal of R.
(6) *2-primal* if $N(R) = N_*(R)$.

Let $ME_l(R) = \{e \in E(R) \mid Re \text{ is a minimal left ideal of } R\}$. An element $e \in E(R)$ is said to be *left (right) semicentral* if $re = ere$ $(er = ere)$ for any $r \in R$. Following

[13], R is called *left min-abel* if every element of $ME_l(R)$ is left semicentral in R. R is called left $MC2$ ([13]) if $aRe = 0$ implies $eRa = 0$ for any $a \in R$, $e \in ME_l(R)$.

2. RESULTS

Definition 1. We call R an $NJ-semicommutative$ if $a, b \in R$ and $ab \in N(R)$, then $aRb \subseteq J(R)$.

Evidently, semicommutative rings are NJ-semicommutative. However, the converse is not true (see the following example).

Example 1. In view of Corollary 10, $T_2(\mathbb{Z})$ is NJ-semicommutative, where $\mathbb{Z}$ is the ring of integers. Note that $e_{11}e_{22} = 0$, $e_{11}e_{12}e_{22} = e_{12}$, where e_{ij} denote the matrix units in $T_2(\mathbb{Z})$ whose $(i,j)^{th}$ entry is 1 and zero elsewhere. So, $T_2(\mathbb{Z})$ is not semicommutative.

Theorem 1. *Left (right) quasi-duo rings are NJ-semicommutative.*

Proof. Let R be a left quasi-duo ring, and M be a maximal left ideal of R. Let $w, h \in R$ be such that $wh \in N(R)$. If $w \notin M$, then $M + Rw = R$. This implies that $m + rw = 1$ for some $m \in M, r \in R$. So $mh + rwh = h$. By [14, Lemma 2.3], $wh \in J(R) \subseteq M$. Since M is an ideal, $h \in M$. Therefore, either $w \in M$ or $h \in M$. This yields that $wRh \subseteq M$, and hence $wRh \subseteq J(R)$, that is, R is NJ-semicommutative. Similarly, it can be shown that R is an NJ-semicommutative ring whenever R is a right quasi-duo ring. □

However, the converse is not true (see the following example).

Example 2. By [5, Example 2(ii)], $\mathbb{H}[x]$ is not right quasi-duo, where $\mathbb{H}$ is the Hamilton quaternion over the field of real numbers. Since $\mathbb{H}[x]$ is reduced, it is NJ-semicommutative.

R is said to be *J-clean* if for each $w \in R$, $w = e + j$ for some $e \in E(R)$ and $j \in J(R)$.

Theorem 2. *J-clean rings are NJ-semicommutative.*

Proof. Let R be a J-clean ring, and $w,\ h \in R$ be such that $wh \in N(R)$. By hypothesis, for any $r \in R$, there exists $e \in E(R)$ such that $wrh - e \in J(R)$. We prove that $e = 0$. Observe that

$$(wrh - e)^2 = wrhwrh - wrhe - e(wrh - e). \tag{2.1}$$

Since $wh \in N(R)$, $1 - hw \in U(R)$. As R is J-clean, $1 - hw - e_1 \in J(R)$ for some $e_1 \in E(R)$. Therefore, $1 - (1 - hw)^{-1}e_1 \in J(R)$. This yields that $(1 - hw)^{-1}e_1 \in U(R)$ and hence $e_1 \in U(R)$, that is, $e_1 = 1$. This implies that $hw \in J(R)$. So, from Equation (2.1), $wrhe \in J(R)$. Note that $wrh - e = j$ for some $j \in J(R)$. Hence $e = wrhe - je \in J(R)$, and so $e = 0$. Thus, $wrh \in J(R)$. □

However, the converse of Theorem 2 is not true; for example, any commutative ring which is not J-clean (for example any field with more than two elements).

Following [6], for any $w \in R$, the *commutant* of w is defined by $comm(w) = \{y \in R \mid yw = wy\}$ and $comm^2(w) = \{x \in R \mid yx = xy$ for all $y \in comm(w)\}$ is called the *double commutant* of w. According to [4], R is called $J-quasipolar$, if for any $w \in R$, $w + f \in J(R)$ for some $f^2 = f \in comm^2(w)$.

The proof of the following proposition is similar to that of Theorem 2.

Proposition 1. *J-quasipolar rings are NJ-semicommutative.*

Theorem 3. *If $R/J(R)$ is NJ-semicommutative, then R is NJ-semicommutative.*

Proof. Let a, $b \in R$ and $ab \in N(R)$. Clearly, $\bar{a}\bar{b} \in N(R/J(R))$. Since $R/J(R)$ is NJ-semicommutative , $\bar{a}\bar{r}\bar{b} \in J(R/J(R))$. As $R/J(R)$ is semiprimitive, $arb \in J(R)$. □

The converse of Theorem 3 is not true (see the following example), and hence a homomorphic image of an NJ-semicommutative ring need not be NJ-semicommutative.

Example 3. Let $R = \mathbb{Z}_{(3\mathbb{Z})}$ be the localization of $\mathbb{Z}$ at $3\mathbb{Z}$ and S the set of quaternions over the ring R. Observe that S is a noncommutative domain. So, S is an NJ-semicommutative ring. Observe that $J(S) = 3S$ and $S/3S \cong M_2(\mathbb{Z}_3)$ via the isomorphism Ψ defined by $\Psi((x_0/y_0) + (x_1/y_1)i + (x_2/y_2)j + (x_3/y_3)k + 3S) =$

$$\begin{pmatrix} \overline{x_0y_0^{-1} + x_1y_1^{-1} - x_2y_2^{-1}} & \overline{x_1y_1^{-1} + x_2y_2^{-1} - x_3y_3^{-1}} \\ \overline{x_1y_1^{-1} + x_2y_2^{-1} + x_3y_3^{-1}} & \overline{x_0y_0^{-1} - x_1y_1^{-1} + x_2y_2^{-1}} \end{pmatrix}$$

for any $(x_0/y_0) + (x_1/y_1)i + (x_2/y_2)j + (x_3/y_3)k) + 3S \in S/3S$. Take $A = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$ and $B = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}$. Observe that $A^2 \in N(M_2(\mathbb{Z}_3))$ and $ABA \notin J(M_2(\mathbb{Z}_3))$ as $J(M_2(\mathbb{Z}_3)) = 0$. So, $S/J(S)$ is not NJ-semicommutative.

Proposition 2. *Let R be an NJ-semicommutative ring. Then*

(1) *R is left-min abel.*
(2) *If $ab \in N(R)$ then, either $a \in M$ or $b \in M$ for any maximal left ideal M of R.*
(3) *R is directly finite.*

Proof.

(1) Let $e \in ME_l(R)$ and $a \in R$. Take $w = ae - eae$. Then $ew = 0$, $we = w$ and $w^2 = 0$. Since R is NJ-semicommutative, $wRw \subseteq J(R)$. As $J(R)$ is semiprime, $w \in J(R)$. If $w = 0$, then we are done. Assume, if possible, that $w \neq 0$. Since

Re is minimal left ideal of R, $Re = Rw$. As $w \in J(R)$, $Re = Rw \subseteq J(R)$, a contradiction. Thus, R is left-min abel.

(2) Let M be a maximal left ideal of R, and $ab \in N(R)$. Suppose $a \notin M$. Then, $Ra + M = R$. So, $ra + m = 1$ for some $r \in R$ and $m \in M$. Thus, $bra + bm = b$. Since R is NJ-semicommutative, $bra \in J(R)$. So, $b \in M$.

(3) Let $wh = 1$. Observe that $(1 - hw)w1 \in N(R)$. Since R is NJ-semicommutative, $(1 - hw)wh1 \in J(R)$, that is, $1 - hw \in J(R) \cap E(R)$. Hence $hw = 1$.

□

A left R-module M is called *Wnil-injective* ([12]) if for each $a\ (\neq 0) \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and each left R-homomorphism from Ra^n to M can be extended to one from R to M.

It is evident that reduced rings are NJ-semicommutative. By Example 1, $T_2(\mathbb{Z})$ is NJ-semicommutative, but it is not reduced. In this context, we have the following result.

Proposition 3. *If R is an NJ-semicommutative ring, then R is reduced in each of the following cases:*

(1) *R is semiprimitive.*

(2) *R is a left MC2 NJ-semicommutative ring and each simple singular left R module is Wnil-injective.*

Proof. (1) Suppose $h^2 = 0$. Since R is NJ-semicommutative, $hRh \subseteq J(R)$. As $J(R)$ is semiprime, $h \in J(R) = 0$.

(2) Suppose $h^2 = 0$ for some $h\ (\neq 0) \in R$. Then, $l(h) \subseteq M$ for some maximal left ideal M of R. Assume, if possible, that M is not an essential left ideal of R, then $M = l(e)$ for some $e \in ME_l(R)$. By Proposition 2 (1), R is left min-abel, and since R is left $MC2$, by [13, Theorem 1.8], $e \in Z(R)$. So, $eh = 0$. Thus, $e \in l(h) \subseteq M = l(e)$, a contradiction. Therefore, M is an essential left ideal of R and R/M is simple singular left R module. By hypothesis, R/M is Wnil-injective. Define a left R-homomorphism $\Psi : Rh \to R/M$ via $\Psi(rh) = r + M$. As R/M is Wnil-injective, $1 - ht \in M$ for some $t \in R$. Since R is NJ-semicommutative, and $h^2 = 0$, $hRh \subseteq J(R)$. Therefore, $h \in J(R)$. So $1 - ht \in U(R)$, a contradiction. Hence $h = 0$.

□

R is said to be:

(1) *semiperiodic* ([2]) if for each $w \in R \setminus (J(R) \cup Z(R))$, $w^q - w^p \in N(R)$ for some integers p and q of opposite parity.

(2) *left* (*right*) *SF* ([10]) if all simple left (right) R-modules are flat.

Theorem 4. *If $R/J(R)$ is reduced, then R is NJ-semicommutative. The converse holds if R is:*

(1) *semiperiodic.*
(2) *left SF.*

Proof. Suppose $R/J(R)$ is reduced. Let $ab \in N(R)$. Clearly, $\bar{b}\bar{a} \in N(R/J(R))$. Since $R/J(R)$ is reduced, $ba \in J(R)$. For any $r \in R$, $(arb)^2 = arbarb \in J(R)$, that is, $\bar{a}\bar{r}\bar{b} \in N(R/J(R)) = 0$ and so, $arb \in J(R)$.
Conversely;

(1) Suppose R is an NJ-semicommutative semiperiodic ring. Write $\bar{R} = R/J(R)$ and let $\bar{w} \in \bar{R}$ with $\bar{w}^2 = 0$. Then by [2, Lemma 2.6], $w^2 \in J(R) \subseteq N(R) \cup Z(R)$. If $w^2 \in N(R)$, then $wRw \subseteq J(R)$ (since R is NJ-semicommutative). As $J(R)$ is semiprime, $\bar{w} = 0$. Suppose $w^2 \notin N(R)$, then $w^2 \in Z(R)$. If $w \in Z(R)$, then $\bar{w}\bar{R}\bar{w} = 0$. As $\bar{R}$ is semiprime, $\bar{w} = 0$. Assume, if possible, that $\bar{w} \notin Z(\bar{R})$ then $w \notin J(R) \cup Z(R)$. By [2, Lemma 2.3(iii)], there exist $e \in E(R)$ and a positive integer p satisfying $w^p = w^p e$ and $e = wy$ for some $y \in R$. Hence $e = ewy = ew(1-e)y + ewey = ew(1-e)y + ew^2y^2$. As R is NJ-semicommutative, $eR(1-e) \subseteq J(R)$. Hence $e \in J(R)$, that is, $e = 0$. This yields that $w^p = 0$ and so $w \in N(R)$, a contradiction to $w^2 \notin N(R)$. Therefore $\bar{w} \in Z(\bar{R})$ and so $\bar{w} = 0$. Thus, $\bar{R}$ is reduced.
(2) Suppose R is an NJ-semicommutative left SF ring. By [11, Proposition 3.2], $R/J(R)$ is left SF. Let $w^2 \in J(R)$ such that $w \notin J(R)$. Assume, if possible, $Rr(w) + J(R) = R$, then $1 = x + \sum^{finite} r_i s_i$, $x \in J(R)$, $r_i \in R$, $s_i \in r(w)$. Then $w = xw + \sum^{finite} r_i s_i w$. Observe that $s_i w \in N(R)$. As R is NJ-semicommutative, $s_i R w \in J(R)$. This implies that $w \in J(R)$, a contradiction. Hence $Rr(w) + J(R) \neq R$. There exist some maximal left ideal H satisfying $Rr(w) + J(R) \subseteq H$. Note that $w^2 \in H$. By [11, Lemma 3.14], $w^2 = w^2 x$ for some $x \in H$, that is, $w - wx \in r(w) \subseteq H$. So, $w \in H$. Hence there exists $y \in H$ satisfying $w = wy$, that is, $1 - y \in r(w) \subseteq H$. This implies that $1 \in H$, a contradiction. Therefore, $R/J(R)$ is reduced.

□

Corollary 1. *If R is an NJ-semicommutative semiperiodic ring, then $R/J(R)$ is commutative.*

Proof. Since $R/J(R)$ is semiperiodic, by Theorem 4 (1) and [2, Theorem 4.4], $R/J(R)$ is commutative. □

Corollary 2. *If R is an NJ-semicommutative left SF, then R is strongly regular.*

Proof. By Theorem 4, $R/J(R)$ is reduced. Hence $R/J(R)$ is strongly regular by [11, Remark 3.13]. This implies that R is left quasi-duo, and hence by [11, Theorem 4.10], R is strongly regular. □

As an immediate consequence of Corollary 1 and Corollary 2, the following corollary is obtained.

Corollary 3. *If R is an NJ-semicommutative, semiperiodic, left SF ring, then R is commutative regular ring.*

The proof of the following proposition is trivial.

Proposition 4. *Suppose $\{R_\delta\}_{\delta\in\Delta}$ is a family of rings, and Δ represents an index set. Then $\Pi_{\delta\in\Delta}R_\delta$ is NJ-semicommutative if and only if R_δ is NJ-semicommutative for each $\delta\in\Delta$.*

Corollary 4. *eR and $(1-e)R$ are NJ-semicommutative for some central idempotent $e\in R$ if and only if R is NJ-semicommutative.*

Proposition 5. *R is NJ-semicommutative if and only if eRe is NJ-semicommutative for all $e\in E(R)$.*

Proof. Suppose R is NJ-semicommutative. Let eae, $ebe\in eRe$ with $(eae)(ebe)\in N(eRe)$. Since R is NJ-semicommutative, $(eae)(ere)(ebe)\in J(R)$ for all $r\in R$. Since $eJ(R)e=J(eRe)$, $(eae)(ere)(ebe)\in J(eRe)$. Hence, eRe is NJ-semicommutative. Whereas the converse is trivial. □

Proposition 6. *Let I be an ideal of an NJ-semicommutative ring W and R a subring of W with $I\subseteq R$. If R/I is NJ-semicommutative, then so is R.*

Proof. Let x, $y\in R$ and $xy\in N(R)$. Since W is NJ-semicommutative, $xr_0y\in J(W)$ for any $r_0\in R$. Therefore, for any $r\in R$, $1-xr_0yr\in U(W)$. There exists $w\in W$ such that $w(1-xr_0yr)=1=(1-xr_0yr)w$. Note that $\bar{x}\bar{y}\in N(R/I)$. Since R/I is NJ-semicommutative, $\bar{x}\bar{r_0}\bar{y}\in J(R/I)$. This implies that $\bar{1}-\bar{x}\bar{r_0}\bar{y}\bar{r}\in U(R/I)$. So there exists $\bar{t}\in R/I$ such that $\bar{t}(\bar{1}-\bar{x}\bar{r_0}\bar{y}\bar{r})=\bar{1}$. This implies that $1-t(1-xr_0yr)\in I$. Hence, $w-t(1-xr_0yr)w\in R$, that is, $w\in R$. Hence $xr_0y\in J(R)$. □

Corollary 5. *Let I be an ideal of an NJ-semicommutative ring W and R an NJ-semicommutative subring of W. Then, $I+R$ is NJ-semicommutative.*

Proof. Follows directly from Proposition 6. □

Corollary 6. *Every finite subdirect product of NJ-semicommutative rings is NJ-semicommutative.*

Proof. Let R/K and R/L be NJ-semicommutative rings for some ideals K and L of R with $K\cap L=0$. Define $\Psi: R\to R/K\bigoplus R/L$ via $\Psi(x)=(x+K, x+L)$. So $R\cong Im(\Psi)$. By hypothesis, $Im(\Psi)/\Psi(K)\cong R/K$ is NJ-semicommutative. Observe that $\Psi(K)\subseteq Im(\Psi)\subseteq R/K\bigoplus R/L$. By Proposition 6, R is NJ-semicommutative. □

Corollary 7. *Let K and L be ideals of R such that R/K and R/L are NJ-semicommutative. Then, $R/(K\cap L)$ is NJ-semicommutative.*

Proof. Define $\Psi: R/(K\cap L)\to R/K$ and $\Phi: R/(K\cap L)\to R/L$ via $\Psi(r+K\cap L)=r+K$ and $\Phi(r+K\cap L)=r+L$, respectively. Clearly, Ψ and Φ are epimorphism with $ker(\Psi)\cap ker(\Phi)=0$. So, $R/(K\cap L)$ is the subdirect product of R/K and R/L. By Corollary 6, $R/(K\cap L)$ is NJ-semicommutative. □

Lemma 1. *Let I be a nil ideal of R such that R/I is an NJ-semicommutative ring. Then, R is NJ-semicommutative.*

Proof. Let $w,\ h \in R$ and $wh \in N(R)$. Clearly, $\bar{w}\bar{h} \in N(R/I)$. Since R/I is NJ-semicommutative, $\bar{w}\bar{r}_0\bar{h} \in J(R/I)$ for any $r_0 \in R$. So $\bar{1} - \bar{w}\bar{r}_0\bar{h}\bar{r} \in U(R/I)$ for all $r \in R$. So $(\bar{1} - \bar{w}\bar{r}_0\bar{h}\bar{r})\bar{s} = \bar{1} = \bar{s}(\bar{1} - \bar{w}\bar{r}_0\bar{h}\bar{r})$ for some $s \in R$. This implies that $1 - (1 - wr_0hr)s \in I$. Since I is nil, $(1 - wr_0hr)s \in U(R)$ and hence $1 - wr_0hr \in U(R)$. Therefore $wr_0h \in J(R)$, that is, R is NJ-semicommutative. □

Proposition 7. *If K and L are ideals of R such that R/K and R/L are NJ-semicommutative, then R/KL is NJ-semicommutative.*

Proof. Observe that $KL \subseteq K \cap L$ and $R/(K\cap L) \cong (R/KL)/((K\cap L)/KL)$. Clearly, $((K\cap L)/KL)^2 = 0$, and by Corollary 7, $R/(K\cap L)$ is NJ-semicommutative. By Lemma 1, R/KL is NJ-semicommutative. □

The following result is an immediate consequence of Proposition 7.

Corollary 8. *The following are equivalent for an ideal I of R.*

(1) *R/I is NJ-semicommutative.*
(2) *R/I^n is NJ-semicommutative for all positive integer n.*

A *Morita context* ([9]) is a 4-tuple $\begin{pmatrix} R_1 & M \\ P & R_2 \end{pmatrix}$, where R_1, R_2 are rings, M is (R_1,R_2)-bimodule and P is (R_2,R_1)-bimodule, and there exists a context product $M\times P \to R_1$ and $P \times M \to R_2$ written multiplicatively as $(m,p) \mapsto mp$ and $(p,m) \mapsto pm$. Clearly, $\begin{pmatrix} R_1 & M \\ P & R_2 \end{pmatrix}$ is an associative ring with the usual matrix operations.

A Morita context $\begin{pmatrix} R_1 & M \\ P & R_2 \end{pmatrix}$ is said to be trivial if the context products are trivial, that is, $MP = 0$ and $PM = 0$.

Proposition 8. *Suppose $R = \begin{pmatrix} R_1 & M \\ P & R_2 \end{pmatrix}$ is a trivial Morita context. Then R is NJ-semicommutative if and only if R_1 and R_2 are NJ-semicommutative.*

Proof. Suppose R is NJ-semicommutative. By Proposition 5, eRe is NJ-semicommutative. So R_1 and R_2 are NJ-semicommutative. Conversely, assume that R_1 and R_2 are NJ-semicommutative and $\alpha = \begin{pmatrix} a_1 & m_1 \\ p_1 & b_1 \end{pmatrix}, \beta = \begin{pmatrix} a_0 & m_0 \\ p_0 & b_0 \end{pmatrix} \in R$ be such that $\alpha\beta \in N(R)$. Then $a_1a_0 \in N(R_1)$ and $b_1b_0 \in N(R_2)$. Let $\gamma = \begin{pmatrix} a & m \\ p & b \end{pmatrix}$ be any element of R. Since R_1 and R_2 are NJ-semicommutative rings, $a_1aa_0 \in J(R_1)$ and $b_1bb_0 \in J(R_1)$. Therefore $\alpha\beta\gamma \in J(R)$. Hence R is NJ-semicommutative. □

Let R_1 and R_2 be any rings, M a (R_1,R_2)-bimodule and $R=\begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$, the formal triangular matrix ring. It is well known that $J(R)=\begin{pmatrix} J(R_1) & M \\ 0 & J(R_2) \end{pmatrix}$.

Corollary 9. *Let R_1 and R_2 be any rings and M a (R_1,R_2)- bimodule. Then $\begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$ is NJ-semicommutative if and only if R_1 and R_2 are NJ-semicommutative.*

Corollary 10. *R is NJ-semicommutative if and only if $T_n(R)$ is NJ-semicommutative.*

Proposition 9. *The following are equivalent.*

(1) *R is NJ-semicommutative.*

(2) $R_n=\left\{\begin{pmatrix} a & a_{12} & \dots & a_{1(n-1)} & a_{1n} \\ 0 & a & \dots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a & a_{(n-1)n} \\ 0 & 0 & \dots & 0 & a \end{pmatrix} : a, a_{ij}\in R,\ i<j\right\}$ *is NJ-semicommutative.*

Proof. (1) $\implies$ (2) Let $I=\left\{\begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} : a_{ij}\in R\right\}\subseteq R_n$. Note that I is an ideal of R_n and $I^n=0$. Also $R_n/I\cong R$. By Lemma 1, R_n is NJ-semicommutative.

(2) $\implies$ (1) It follows from Proposition 5. □

Corollary 11. *The following are equivalent.*

(1) *R is NJ-semicommutative.*

(2) *$R[x]/<x^n>$ is NJ-semicommutative for any positive integer n, where $<x^n>$ is the ideal generated by x^n in $R[x]$.*

Proof. Observe that

$R[x]/<x^n>\cong\left\{\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{pmatrix} : a_i\in R\right\}$. So, the proof follows from the proof of Proposition 9. □

Let A be a (R,R)-bimodule which is a general ring (not necessarily with unity) in which $(aw)r=a(wr)$, $(ar)w=a(rw)$ and $(ra)w=r(aw)$ hold for all $a,w\in A$ and $r\in R$. Then *ideal-extension* (also called *Dorroh extension*) $I(R;A)$ of R by A is defined to be the additive abelian group $I(R;A) = R \oplus A$ with multiplication $(r,a)(s,w) = (rs, rw + as + aw)$.

Proposition 10. *Let A be an (R,R)-bimodule which is a general ring (not necessarily with unity) in which $(aw)r = a(wr)$, $(ar)w = a(rw)$ and $(ra)w = r(aw)$ hold for all $a,w \in A$ and $r \in R$. Suppose that for any $a \in A$ there exists $w \in A$ such that $a + w + aw = 0$. Then the following are equivalent.*

(1) *R is NJ-semicommutative.*
(2) *Dorroh extension $S = I(R;A)$ is NJ-semicommutative.*

Proof. (1) $\Longrightarrow$ (2) Suppose R is NJ-semicommutative and $\alpha = (r,v)$, $\beta = (p,w) \in S$ be such that $\alpha\beta \in N(S)$. Let $\gamma = (s,u)$ be any element of S. Since R is NJ-semicommutative $rsp \in J(R)$. We claim that $\alpha\gamma\beta \in J(S)$. Now, let $(0,a) \in (0,A)$. For any $(r_1,a_1) \in S$, we have, $(1,0) - (0,a)(r_1,a_1) = (1, -ar_1 - aa_1)$. By hypothesis, there exists $a_2 \in A$ such that $(1, -ar_1 - aa_1)(1,a_2) = (1,0)$. Therefore $(0,A) \subseteq J(S)$. Note that $\alpha\gamma\beta = (rsp, w)$ for some $w \in A$. So if we show $(rsp,0) \in J(S)$ then we are done. Let (r_1,v_1) be any element of S. Then $(1,0) - (r_1,v_1)(rsp,0) = (1 - r_1rsp, -v_1rsp) \in U(S)$, as $(1 - r_1rsp, -v_1rsp) = (1 - r_1rsp, 0)(1, (1 - r_1rsp)^{-1}(-v_1rsp))$ and $(1, (1 - r_1rsp)^{-1}(-v_1rsp)) = (1,0) + (0, (1 - r_1rsp)^{-1}(-v_1rsp)) \in U(S)$. Thus $(rsp,0) \in J(S)$ and hence $\alpha\gamma\beta \in J(S)$. Therefore S is NJ-semicommutative.

(2) $\Longrightarrow$ (1) Let a, $b \in R$ and $ab \in N(R)$. Clearly, $(a,0)(b,0) \in N(S)$. Since S is NJ-semicommutative ring, $(a,0)(r,0)(b,0) \in J(S)$ for all $r \in R$. Hence $arb \in J(R)$, that is, R is an NJ-semicommutative ring. □

Let $\Psi : R \to R$ be a ring homomorphism , $R[[x,\Psi]]$ represents the ring of skew formal power series over R, that is, all formal power series in x with coefficients from R and multiplication is defined with respect to the rule $xr = \Psi(r)x$ for all $r \in R$. It is well known that $J(R[[x,\Psi]]) = J(R) + <x>$, $<x>$ is the ideal of $R[[x,\Psi]]$ generated by x. Since $R[[x,\Psi]] \cong I(R; <x>)$, the following result is an immediate consequence of Proposition 10.

Corollary 12. *Let $\Psi : R \to R$ be a ring homomorphism. Then the following are equivalent.*

(1) *R is NJ-semicommutative.*
(2) *$R[[x,\Psi]]$ is NJ-semicommutative.*

Corollary 13. *Then the following are equivalent.*

(1) *R is NJ-semicommutative.*
(2) *$R[[x]]$ is NJ-semicommutative.*

It is a natural question to ask whether the polynomial ring over an NJ-semicommutative ring is NJ-semicommutative. However, the following example gives the answer in negative.

Example 4. For any countable field K, there exists a nonzero nil algebra S over K such that $N^*(S[x]) = 0$ (see the proof of Lemma 3.7 in [3]). Let $R = K + S$. Observe that R is a local ring with $J(R) = S$. Hence, R is an NJ-semicommutative ring and $N^*(R[x]) = N^*(S[x])$. If $R[x]$ is not NJ-semicommutative, then we are done. If $R[x]$ is NJ-semicommutative, then we show that $(R[x])[y]$ is not NJ-semicommutative. Assume, if possible, that $(R[x])[y]$ is NJ-semicommutative. By [1, Theorem 1], $J((R[x])[y]) = I[y]$ for some nil ideal I of $R[x]$ which is $N^*(R[x]) = N^*(S[x]) = 0$. Therefore, $J((R[x])[y]) = 0$. So, $(R[x])[y]$ is semicommutative, which further implies that $R[x]$ is a semicommutative ring. Hence $R[x]$ is 2-primal, and so, $N(R[x]) = N_*(R[x])$. But, this is a contradiction to the fact that $0 \neq N(R) = S \subseteq N(R[x])$ and $N_*(R[x]) \subseteq N^*(R[x]) = N^*(S[x]) = 0$.

REFERENCES

[1] S. A. Amitsur, "Radicals of polynomial rings," *Canadian Journal of Mathematics*, vol. 8, pp. 355–361, 1956, doi: 10.4153/CJM-1956-040-9.

[2] H. E. Bell and A. Yaqub, "On commutativity of semiperiodic rings," *Results in Mathematics*, vol. 53, no. 1, pp. 19–26, 2009, doi: 10.1007/s00025-008-0305-5.

[3] W. Chen, "On linearly weak armendariz rings," *Journal of Pure and Applied Algebra*, vol. 219, no. 4, pp. 1122–1130, 2015, doi: 10.1016/j.jpaa.2014.05.039.

[4] J. Cui and J. Chen, "A class of quasipolar rings," *Communications in Algebra*, vol. 40, no. 12, pp. 4471–4482, 2012, doi: 10.1080/00927872.2011.610854.

[5] C. Huh, S.-H. Jang, C.-O. Kim, and Y. Lee, "Ring whose maximal one-sided ideals are two-sided," *Bulletin of the Korean Mathematical Society*, vol. 39, no. 3, pp. 411–422, 2002, doi: 10.4134/BKMS.2002.39.3.411.

[6] J. J. Koliha and P. Patricio, "Elements of rings with equal spectral idempotents," *Journal of the Australian Mathematical Society*, vol. 72, no. 1, pp. 137–152, 2002, doi: 10.1017/S1446788700003657.

[7] T. Lam and A. S. Dugas, "Quasi-duo rings and stable range descent," *Journal of pure and applied algebra*, vol. 195, no. 3, pp. 243–259, 2005, doi: 10.1016/j.jpaa.2004.08.011.

[8] L. Liang, L. Wang, and Z. Liu, "On a generalization of semicommutative rings," *Taiwanese Journal of Mathematics*, vol. 11, no. 5, pp. 1359–1368, 2007, doi: 10.11650/twjm/1500404869.

[9] K. Morita, "Duality for modules and its applications to the theory of rings with minimum condition," *Science Reports of the Tokyo Kyoiku Daigaku, Section A*, vol. 6, no. 150, pp. 83–142, 1958, doi: https://www.jstor.org/stable/43698445.

[10] V. Ramamurthi, "On the injectivity and flatness of certain cyclic modules," *Proceedings of the American Mathematical Society*, vol. 48, no. 1, pp. 21–25, 1975, doi: 10.1090/S0002-9939-1975-0354779-3.

[11] M. Rege, "On von neumann regular rings and sf-rings," *Mathematica Japonica*, vol. 31, no. 6, pp. 927–936, 1986.

[12] J. C. Wei and J. H. Chen, "nil- injective rings," *International Electronic Journal of Algebra*, vol. 2, no. 2, pp. 1–21, 2007.

[13] J. Wei, "Certain rings whose simple singular modules are nil-injective," *Turkish Journal of Mathematics*, vol. 32, no. 4, pp. 393–408, 2008.

[14] H.-P. Yu, "On quasi-duo rings," *Glasgow Mathematical Journal*, vol. 37, no. 1, pp. 21–31, 1995, doi: 10.1017/S0017089500030342.

Authors' addresses

Sanjiv Subba
National Institute of Technology Meghalaya, Department of Mathematics, Bijni Complex, 793003 Shillong, India
E-mail address: sanjivsubba59@gmail.com

Tikaram Subedi
(**Corresponding author**) National Institute of Technology Meghalaya, Department of Mathematics, Bijni Complex, 793003 Shillong, India
Current address: National Institute of Technology Meghalaya, Department of Mathematics, Bijni Complex, 793003 Shillong, India
E-mail address: tikaram.subedi@nitm.ac.in