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ON THE NEW FRACTIONAL OPERATORS GENERATING MODIFIED GAMMA AND BETA FUNCTIONS

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Abstract. In this paper, we introduce three new fractional operators, MRL fractional integral, MRL fractional derivative and MC fractional derivative operators, which including generalized M-series in their kernels and give some of their fundamental properties. Then we apply Laplace, Mellin and beta integral transformations to the new fractional operators and obtain conclusions involving classical gamma and beta and modified gamma and beta functions. As examples, we also obtain similar conclusions by applying new fractional operators to the functions z^{λ} and $(1-az)^{-\lambda}$. Furthermore, we present the relationships of the new fractional operators with other fractional operators in the literature. Finally, we compare the behavior of the function z^2 in classical Riemann-Liouville and Caputo fractional operators and MRL and MC fractional operators for orders $\varepsilon = 0.2, 0.4, 0.6, 0.8$.

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1. Introduction

It is known how to take the integer integral and derivative of a function such as the first, second, third, fourth. Is it possible to take the fractional integral and derivative of the same function from orders 0.2, 0.4, 0.6, 0.8 etc? First, in the correspondence between Leibniz and L'Hospital in 1695, "Can integer derivatives be extended to fractional derivatives?" [11, 13] the question arose and caught the attention of many famous mathematicians (Riemann, Grünwald, Letnikov, Liouville, Caputo, Euler, Abel, Fourier, Kobel, Erdelyi, Hadamard, Riesz and Laplace).

The fractional integral and derivative has been one of the interesting topics in mathematics for a long time, and this interest is increasing day by day. In the literature, there are many studies by researchers on fractional operators (see for example [1–4, 7–10, 12, 14–19, 22, 23] and reference therein).

In this paper, we define three new fractional operators with generalized M-series in their kernels. Then we apply Laplace, Mellin and beta integral transformations to the new fractional operators and give some examples. Furthermore, we present the relationships of the new fractional operators with other fractional operators in the literature. Finally, we examine the behavior of the function z^2 in classical and new fractional operators, see Figure 1 and Figure 2.

2. Preliminaries

In this section, we give relevant material which will be used throughout the paper. The gamma function [5] for $\Re(x) > 0$ is defined by

$$\Gamma(x) = \int_0^\infty \Delta^{x-1} \exp(-\Delta) d\Delta.$$

The beta function [5] for $\Re(x) > 0$ and $\Re(y) > 0$ is given by

$$B(x,y) = \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} d\Delta.$$

The Pochhammer symbol [5] for $\Re(\lambda) > -n$, $n \in \mathbb{N}_0$ and $\lambda \neq 0,-1,-2,\dots$ is defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad (\lambda)_0 \equiv 1.$$

The generalized M-series [20] for $\Re(\alpha) > 0$, $\xi_1,\dots,\xi_p,\mu_1,\dots,\mu_q \neq 0,-1,-2,\dots$ is given by

$${}_p^{\alpha}M_q^{\beta}(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;z) = \sum_{n=0}^{\infty} \frac{(\xi_1)_n \dots (\xi_p)_n}{(\mu_1)_n \dots (\mu_q)_n} \frac{z^n}{\Gamma(\alpha n+\beta)}.$$

The modified gamma function [6] for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(x) > 0$, $\xi_1,\dots,\xi_p$, $\mu_1,\dots,\mu_q \neq 0,-1,-2,\dots$ is given by

$$\begin{aligned} {}^M\Gamma_{p,q}^{(\alpha,\beta)}(x;\rho) &= {}^M\Gamma_{p,q}^{(\alpha,\beta)}(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;x;\rho) \\ &= \int_0^\infty \Delta^{x-1}\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;-\Delta-\frac{\rho}{\Delta}\right) d\Delta. \end{aligned}$$

The modified beta function [6] for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(x) > 0$, $\Re(y) > 0$, $\xi_1,\dots,\xi_p$, $\mu_1,\dots,\mu_q \neq 0,-1,-2,\dots$ is defined by

$$\begin{aligned} {}^M B_{p,q}^{(\alpha,\beta)}(x,y;\rho) &= {}^M B_{p,q}^{(\alpha,\beta)}(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;x,y;\rho) \\ &= \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1}\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta. \end{aligned}$$

The Mellin transform [11] for $\rho \in \mathbb{C}$ is defined by

$$\mathfrak{M}\{f(w);\rho\} = \int_0^\infty w^{\rho-1} f(w) dw.$$

The Laplace transform [11] for $\Re(s) > 0$ is given by

$$\mathfrak{L}\{f(w);s\} = \int_0^\infty \exp(-sw) f(w) dw.$$

The beta transform [21] for $\Re(x) > 0$ and $\Re(y) > 0$ is defined by

$$\mathfrak{B}\{f(w);x,y\} = \int_0^1 w^{x-1}(1-w)^{y-1} f(w)dw.$$

The Riemann-Liouville fractional integral [13] for $\Re(\varepsilon) > 0$ is given by

$$I_z^{\varepsilon}\{f(z)\} = \frac{1}{\Gamma(\varepsilon)} \int_0^z (z-\Delta)^{\varepsilon-1} f(\Delta)d\Delta. \tag{2.1}$$

The Riemann-Liouville fractional derivative [13] for $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$ is defined by

$$D_z^{\varepsilon}\{f(z)\} = \frac{1}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \int_0^z (z-\Delta)^{m-\varepsilon-1} f(\Delta)d\Delta. \tag{2.2}$$

The Caputo fractional derivative [13] for $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$ is given by

$${}^{c}D_z^{\varepsilon}\{f(z)\} = \frac{1}{\Gamma(m-\varepsilon)} \int_0^z (z-\Delta)^{m-\varepsilon-1} f^{(m)}(\Delta)d\Delta. \tag{2.3}$$

3. New fractional operators

In this section, we introduce three new fractional operators in the classical Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative operators sensitivity and give some of their basic properties. Also, we call new Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative operators as MRL fractional integral, MRL fractional derivative and MC fractional derivative operators, respectively.

Definition 3.1. New Riemann-Liouville fractional integral operator for $\Re(\alpha) > 0$, $\Re(\varepsilon) > 0$, $\xi_1, \ldots, \xi_p, \mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is defined as:

$$\begin{aligned}\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} &= \widehat{I}_z^{\varepsilon}\{f(z);\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\alpha,\beta;w\} \\ &:= \frac{1}{\Gamma(\varepsilon)} \int_0^z (z-\Delta)^{\varepsilon-1} f(\Delta)\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta.\end{aligned}$$

Definition 3.2. New Riemann-Liouville fractional derivative operator for $\Re(\alpha) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$, $\xi_1, \ldots, \xi_p, \mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is defined as:

$$\begin{aligned}&\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \widehat{D}_z^{\varepsilon}\{f(z);\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\alpha,\beta;w\} \\ &:= \frac{1}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \int_0^z (z-\Delta)^{m-\varepsilon-1} f(\Delta)\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta.\end{aligned}$$

Definition 3.3. New Caputo fractional derivative operator for $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\dots,\xi_p$, $\mu_1,\dots,\mu_q\neq 0,-1,-2,\dots$ is defined as:

$$
\begin{aligned}
&{}^{c}\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\}={}^{c}\widehat{D}_z^{\varepsilon}\{f(z);\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\alpha,\beta;w\}\\
&:=\frac{1}{\Gamma(m-\varepsilon)}\int_0^z(z-\Delta)^{m-\varepsilon-1}f^{(m)}(\Delta)\,{}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta.
\end{aligned}
$$

Remark 3.1. If we put $w=0$ and $\beta=1$ to the new fractional operators, we get the classical fractional operators (2.1), (2.2) and (2.3).

Theorem 3.1. *The MRL fractional integral for* $\lambda_1,\lambda_2\in\mathbb{R}$, $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $\xi_1,\dots,\xi_p,\mu_1,\dots,\mu_q\neq 0,-1,-2,\dots$ *is linear:*

$$
\widehat{I}_z^{\varepsilon}\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\}=\lambda_1\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\}+\lambda_2\widehat{I}_z^{\varepsilon}\{g(z);\alpha,\beta;w\}.
$$

Proof. By using MRL fractional integral operator, we have

$$
\begin{aligned}
&\widehat{I}_z^{\varepsilon}\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\}\\
&=\frac{\lambda_1}{\Gamma(\varepsilon)}\int_0^z(z-\Delta)^{\varepsilon-1}f(\Delta)\,{}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&\quad+\frac{\lambda_2}{\Gamma(\varepsilon)}\int_0^z(z-\Delta)^{\varepsilon-1}g(\Delta)\,{}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&=\lambda_1\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\}+\lambda_2\widehat{I}_z^{\varepsilon}\{g(z);\alpha,\beta;w\}.
\end{aligned}
$$

□

Theorem 3.2. *The MRL fractional derivative for* $\lambda_1,\lambda_2\in\mathbb{R}$, $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\dots,\xi_p,\mu_1,\dots,\mu_q\neq 0,-1,-2,\dots$ *is linear:*

$$
\widehat{D}_z^{\varepsilon}\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\}=\lambda_1\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\}+\lambda_2\widehat{D}_z^{\varepsilon}\{g(z);\alpha,\beta;w\}.
$$

Proof. By using MRL fractional derivative operator, we have

$$
\begin{aligned}
&\widehat{D}_z^{\varepsilon}\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\}\\
&=\frac{\lambda_1}{\Gamma(m-\varepsilon)}\frac{d^m}{dz^m}\int_0^z(z-\Delta)^{m-\varepsilon-1}f(\Delta)\,{}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&\quad+\frac{\lambda_2}{\Gamma(m-\varepsilon)}\frac{d^m}{dz^m}\int_0^z(z-\Delta)^{m-\varepsilon-1}g(\Delta)\,{}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&=\lambda_1\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\}+\lambda_2\widehat{D}_z^{\varepsilon}\{g(z);\alpha,\beta;w\}.
\end{aligned}
$$

□

Theorem 3.3. *The MC fractional derivative for* $\lambda_1,\lambda_2\in\mathbb{R}$, $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\dots,\xi_p,\mu_1,\dots,\mu_q\neq 0,-1,-2,\dots$ *is linear:*

$$
{}^{c}\widehat{D}_z^{\varepsilon}\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\}=\lambda_1{}^{c}\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\}+\lambda_2{}^{c}\widehat{D}_z^{\varepsilon}\{g(z);\alpha,\beta;w\}.
$$

Proof. By using MC fractional derivative operator, we have

$$
{}^{c}\widehat{D}_z^{\varepsilon}\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\}
$$

$$= \frac{\lambda_1}{\Gamma(m-\varepsilon)} \int_0^z (z-\Delta)^{m-\varepsilon-1} f^{(m)}(\Delta)\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$
$$+ \frac{\lambda_2}{\Gamma(m-\varepsilon)} \int_0^z (z-\Delta)^{m-\varepsilon-1} g^{(m)}(\Delta)\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$
$$= \lambda_1 {}^c\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} + \lambda_2 {}^c\widehat{D}_z^{\varepsilon}\{g(z);\alpha,\beta;w\}. \qquad \square$$

Theorem 3.4. *Let $f(z)=\sum_{n=0}^{\infty} a^n z^n$ analytic function for $|z|<r$. Then, the MRL fractional integral for $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ is provided as:*

$$\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} a^n\, \widehat{I}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}.$$

Proof. By using MRL fractional integral operator, we have

$$\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(\varepsilon)} \int_0^z (z-\Delta)^{\varepsilon-1}\Delta^n\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$
$$= \sum_{n=0}^{\infty} a^n\, \widehat{I}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}. \qquad \square$$

Theorem 3.5. *Let $f(z)=\sum_{n=0}^{\infty} a^n z^n$ analytic function for $|z|<r$. Then, the MRL fractional derivative for $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ is provided as:*

$$\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} a^n\, \widehat{D}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}.$$

Proof. By using MRL fractional derivative operator, we have

$$\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \int_0^z (z-\Delta)^{m-\varepsilon-1}\Delta^n$$
$$\times {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$
$$= \sum_{n=0}^{\infty} a^n\, \widehat{D}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}. \qquad \square$$

Theorem 3.6. *Let $f(z)=\sum_{n=0}^{\infty} a^n z^n$ analytic function for $|z|<r$. Then, the MC fractional derivative for $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ is provided as:*

$${}^c\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} a^n\, {}^c\widehat{D}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}.$$

Proof. By using MC fractional derivative operator, we have

$$\begin{aligned} {}^{c}\widehat{D}_{z}^{\varepsilon}\{f(z);\alpha,\beta;w\} &= \sum_{n=0}^{\infty}\frac{a^{n}}{\Gamma(m-\varepsilon)}\int_{0}^{z}(z-\Delta)^{m-\varepsilon-1} \\ &\quad \times {}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_{1},\dots,\xi_{p};\mu_{1},\dots,\mu_{q};\frac{-wz^{2}}{\Delta(z-\Delta)}\right)\frac{d^{m}}{d\Delta^{m}}(\Delta^{n})\,d\Delta \\ &= \sum_{n=0}^{\infty}a^{n}\,{}^{c}\widehat{D}_{z}^{\varepsilon}\{z^{n};\alpha,\beta;w\}. \end{aligned}$$

□

4. Application of integral transforms

In this section, we apply Laplace, Mellin and beta transforms to the MRL fractional integral, MRL fractional derivative and MC fractional derivative operators.

4.1. *Application of Laplace transform*

Theorem 4.1. *The Laplace transform of MRL fractional integral for $\Re(\alpha)>0$, $\Re(s)>0$, $\Re(\varepsilon)>0$, $\xi_1,\dots,\xi_p,\mu_1,\dots,\mu_q\neq 0,-1,-2,\dots$ is obtained as:*

$$\mathfrak{L}\left\{\widehat{I}_{z}^{\varepsilon}\{f(z);\alpha,\beta;w\};s\right\}=\frac{1}{s}\,\widehat{I}_{z}^{\varepsilon}\left\{f(z);\xi_{1},\dots,\xi_{p},1;\mu_{1},\dots,\mu_{q};\alpha,\beta;\frac{1}{s}\right\}.$$

Proof. By using MRL fractional integral operator and Laplace transform, we have

$$\begin{aligned} &\mathfrak{L}\left\{\widehat{I}_{z}^{\varepsilon}\{f(z);\alpha,\beta;w\};s\right\} \\ &=\frac{1}{s}\frac{1}{\Gamma(\varepsilon)}\int_{0}^{z}(z-\Delta)^{\varepsilon-1}f(\Delta)\,{}_{p+1}^{\alpha}M_{q}^{\beta}\left(\xi_{1},\dots,\xi_{p},1;\mu_{1},\dots,\mu_{q};\frac{-\left(s^{-1}\right)z^{2}}{\Delta(z-\Delta)}\right)d\Delta \\ &=\frac{1}{s}\,\widehat{I}_{z}^{\varepsilon}\left\{f(z);\xi_{1},\dots,\xi_{p},1;\mu_{1},\dots,\mu_{q};\alpha,\beta;\frac{1}{s}\right\}. \end{aligned}$$

□

Corollary 4.1. *The Laplace transform of the MRL fractional integral operator yields the following result in relation to the modified beta function for $f(z)=z^{\lambda}$:*

$$\mathfrak{L}\left\{\widehat{I}_{z}^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};s\right\}=\frac{1}{s}\frac{z^{\varepsilon+\lambda}}{\Gamma(\varepsilon)}\,{}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\xi_{1},\dots,\xi_{p},1;\mu_{1},\dots,\mu_{q};\lambda+1,\varepsilon;\frac{1}{s}\right).$$

Theorem 4.2. *The Laplace transform of MRL fractional derivative for $\Re(\alpha)>0$, $\Re(s)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\dots,\xi_p,\mu_1,\dots,\mu_q\neq 0,-1,-2,\dots$ is obtained as:*

$$\mathfrak{L}\left\{\widehat{D}_{z}^{\varepsilon}\{f(z);\alpha,\beta;w\};s\right\}=\frac{1}{s}\widehat{D}_{z}^{\varepsilon}\left\{f(z);\xi_{1},\dots,\xi_{p},1;\mu_{1},\dots,\mu_{q};\alpha,\beta;\frac{1}{s}\right\}.$$

Proof. By using the MRL fractional derivative operator and the Laplace transform, the desired result can be easily achieved with similar calculations. □

Corollary 4.2. *The Laplace transform of the MRL fractional derivative operator yields the following result in relation to the modified beta function for* $f(z) = z^{\lambda}$ *:*

$$\mathfrak{L}\left\{\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};s\right\} = \frac{1}{s}\frac{\Gamma(m+\lambda+1-\varepsilon)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-\varepsilon)} \times {}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\mu_1,\ldots,\mu_q;\lambda+1,m-\varepsilon;\frac{1}{s}\right).$$

Theorem 4.3. *The Laplace transform of MC fractional derivative for* $\Re(\alpha) > 0$, $\Re(s) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{L}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};s\right\} = \frac{1}{s}{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p,1;\mu_1,\ldots,\mu_q;\alpha,\beta;\frac{1}{s}\right\}.$$

Proof. By using the MC fractional derivative operator and the Laplace transform, the desired result can be easily achieved with similar calculations. □

Corollary 4.3. *The Laplace transform of the MC fractional derivative operator yields the following result in relation to the modified beta function for* $f(z) = z^{\lambda}$ *:*

$$\mathfrak{L}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};s\right\} = \frac{1}{s}\frac{\Gamma(\lambda+1)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-m)} \times {}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\mu_1,\ldots,\mu_q;\lambda+1-m,m-\varepsilon;\frac{1}{s}\right).$$

4.2. *Application of Mellin transform*

Theorem 4.4. *The Mellin transform of MRL fractional integral for* $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\varepsilon) > 0$, $\Re(\varepsilon+\rho) > 0$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{M}\left\{\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{\Gamma(\varepsilon+\rho)}{\Gamma(\varepsilon)}\frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)}{z^{2\rho}}\, I_z^{\varepsilon+\rho}\left\{z^{\rho}f(z)\right\}.$$

Proof. By using MRL fractional integral operator and Mellin transform, we have

$$\mathfrak{M}\left\{\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{1}{\Gamma(\varepsilon)}\int_0^z (z-\Delta)^{\varepsilon-1}f(\Delta) \times \int_0^{\infty} w^{\rho-1}\, {}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)dwd\Delta.$$

By putting $k = \frac{wz^2}{\Delta(z-\Delta)}$ and multiplied by $\frac{\Gamma(\varepsilon+\rho)}{\Gamma(\varepsilon+\rho)}$, we have

$$\mathfrak{M}\left\{\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{\Gamma(\varepsilon+\rho)}{\Gamma(\varepsilon)}\frac{z^{-2\rho}}{\Gamma(\varepsilon+\rho)}\int_0^z (z-\Delta)^{\varepsilon+\rho-1}\Delta^{\rho}f(\Delta) \times \int_0^{\infty} k^{\rho-1}\, {}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;-k\right)dkd\Delta$$

$$= \frac{\Gamma(\varepsilon+\rho)}{\Gamma(\varepsilon)} \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)}{z^{2\rho}} I_z^{\varepsilon+\rho}\left\{z^{\rho} f(z)\right\}. \qquad \square$$

Corollary 4.4. *The Mellin transform of the MRL fractional integral operator yields the following result in relation to the modified gamma function for* $f(z)=z^{\lambda}$ *:*

$$\mathfrak{M}\left\{\widehat{I}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};\rho\right\} = \frac{\Gamma(\varepsilon+\rho)\Gamma(\lambda+\rho+1)z^{\lambda+\varepsilon}}{\Gamma(\varepsilon)\Gamma(\lambda+\varepsilon+2\rho+1)} {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0).$$

Theorem 4.5. *The Mellin transform of MRL fractional derivative for* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $\Re(m-\varepsilon)>0$, $\Re(m-\varepsilon+\rho)>0$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{M}\left\{\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m}\left(\frac{1}{z^{2\rho}} I_z^{m-\varepsilon+\rho}\left\{z^{\rho} f(z)\right\}\right).$$

Proof. By using MRL fractional derivative operator and Mellin transform, we have

$$\begin{aligned}\mathfrak{M}\left\{\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} &= \frac{1}{\Gamma(m-\varepsilon)}\frac{d^m}{dz^m}\int_0^z (z-\Delta)^{m-\varepsilon-1} f(\Delta) \\ &\times \int_0^{\infty} w^{\rho-1}\, {}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) dw d\Delta.\end{aligned}$$

By putting $k=\frac{wz^2}{\Delta(z-\Delta)}$ and multiplied by $\frac{\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon+\rho)}$, we have

$$\begin{aligned}&\mathfrak{M}\left\{\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} \\ &= \frac{\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)}\frac{d^m}{dz^m}\frac{z^{-2\rho}}{\Gamma(m-\varepsilon+\rho)}\int_0^z (z-\Delta)^{m-\varepsilon+\rho-1}\Delta^{\rho} f(\Delta) \\ &\times \int_0^{\infty} k^{\rho-1}\, {}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;-k\right) dk d\Delta \\ &= \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)}\frac{d^m}{dz^m}\left(\frac{1}{z^{2\rho}} I_z^{m-\varepsilon+\rho}\left\{z^{\rho} f(z)\right\}\right). \qquad \square\end{aligned}$$

Corollary 4.5. *The Mellin transform of the MRL fractional derivative operator yields the following result in relation to the modified gamma function for* $f(z)=z^{\lambda}$ *:*

$$\begin{aligned}\mathfrak{M}\left\{\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};\rho\right\} &= \frac{\Gamma(m+\rho-\varepsilon)\Gamma(m+\lambda+1-\varepsilon)\Gamma(\lambda+\rho+1)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-\varepsilon)\Gamma(m+\lambda+2\rho+1-\varepsilon)} \\ &\times {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0).\end{aligned}$$

Theorem 4.6. *The Mellin transform of MC fractional derivative for* $\Re(\alpha)>0$, $\Re(\rho)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $\Re(m-\varepsilon)>0$, $\Re(m-\varepsilon+\rho)>0$, $m\in\mathbb{N}$,

$\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{M}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\};\rho\right\} = \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)z^{2\rho}}\, I_z^{m-\varepsilon+\rho}\left\{z^{\rho}f^{(m)}(z)\right\}.$$

Proof. By using MC fractional derivative operator and Mellin transform, we have

$$\mathfrak{M}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\};\rho\right\} = \frac{1}{\Gamma(m-\varepsilon)}\int_0^z (z-\Delta)^{m-\varepsilon-1}f^{(m)}(\Delta) \times \int_0^{\infty} w^{\rho-1}\, {}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)dw d\Delta.$$

By putting $k=\frac{wz^2}{\Delta(z-\Delta)}$ and multiplied by $\frac{\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon+\rho)}$, we have

$$\begin{aligned}\mathfrak{M}&\left\{{}^{c}\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\};\rho\right\}\\ &= \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)z^{2\rho}}\frac{1}{\Gamma(m-\varepsilon+\rho)}\int_0^z (z-\Delta)^{m-\varepsilon+\rho-1}\Delta^{\rho}f^{(m)}(\Delta)d\Delta\\ &= \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)z^{2\rho}}\, I_z^{m-\varepsilon+\rho}\left\{z^{\rho}f^{(m)}(z)\right\}.\end{aligned}$$ □

Corollary 4.6. *The Mellin transform of the MC fractional derivative operator yields the following result in relation to the modified gamma function for* $f(z)=z^{\lambda}$ *:*

$$\mathfrak{M}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};\rho\right\} = \frac{\Gamma(\lambda+1)\Gamma(m+\rho-\varepsilon)\Gamma(\lambda+\rho+1-m)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-m)\Gamma(\lambda+2\rho+1-\varepsilon)} \times {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0).$$

4.3. *Application of beta transform*

Theorem 4.7. *The beta transform of MRL fractional integral for* $\Re(\alpha)>0$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\varepsilon)>0$, $\xi_1,\ldots,\xi_p,x,\mu_1,\ldots,\mu_q,x+y\neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{B}\left\{\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\};x,y\right\} = B(x,y)\,\widehat{I}_z^{\varepsilon}\{f(z);\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\alpha,\beta;1\}.$$

Proof. By using MRL fractional integral operator and beta transform, we have

$$\begin{aligned}\mathfrak{B}&\left\{\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\};x,y\right\}\\ &= \frac{B(x,y)}{\Gamma(\varepsilon)}\int_0^z (z-\Delta)^{\varepsilon-1}f(\Delta)\, {}_{p+1}^{\alpha}M_{q+1}^{\beta}\left(\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\frac{-z^2}{\Delta(z-\Delta)}\right)d\Delta\\ &= B(x,y)\,\widehat{I}_z^{\varepsilon}\{f(z);\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\alpha,\beta;1\}.\end{aligned}$$ □

Corollary 4.7. *The beta transform of the MRL fractional integral operator yields the following result in relation to the modified beta function for* $f(z)=z^{\lambda}$:

$$\mathfrak{B}\left\{\widehat{I}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};x,y\right\}=\frac{B(x,y)}{\Gamma(\varepsilon)}z^{\varepsilon+\lambda}$$
$$\times\,{}^{M}B^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\lambda+1,\varepsilon;1\right).$$

Theorem 4.8. *The beta transform of MRL fractional derivative for* $\Re(\alpha)>0$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p,x$, $\mu_1,\ldots,\mu_q$, $x+y\neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{B}\left\{\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};x,y\right\}=B(x,y)\;\widehat{D}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\alpha,\beta;1\right\}.$$

Proof. By using the MRL fractional derivative operator and the beta transform, the desired result can be easily achieved with similar calculations. □

Corollary 4.8. *The beta transform of the MRL fractional derivative operator yields the following result in relation to the modified beta function for* $f(z)=z^{\lambda}$:

$$\mathfrak{B}\left\{\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};x,y\right\}=\frac{B(x,y)\Gamma(m+\lambda+1-\varepsilon)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-\varepsilon)}$$
$$\times\,{}^{M}B^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\lambda+1,m-\varepsilon;1\right).$$

Theorem 4.9. *The beta transform of MC fractional derivative for* $\Re(\alpha)>0$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p,x$, $\mu_1,\ldots,\mu_q$, $x+y\neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{B}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};x,y\right\}=B(x,y)\,{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\alpha,\beta;1\right\}.$$

Proof. By using the MC fractional derivative operator and the beta transform, the desired result can be easily achieved with similar calculations. □

Corollary 4.9. *The beta transform of the MC fractional derivative operator yields the following result in relation to the modified beta function for* $f(z)=z^{\lambda}$:

$$\mathfrak{B}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};x,y\right\}=\frac{B(x,y)\Gamma(\lambda+1)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-m)}$$
$$\times\,{}^{M}B^{(\alpha,\beta)}_{p+1,q+1}\left(\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\lambda+1-m,m-\varepsilon;1\right).$$

5. Examples

In this section, we compute the MRL fractional integrals, MRL fractional derivatives and MC fractional derivatives of functions z^{λ} and $(1-az)^{-\lambda}$ and obtain results that generate the classical gamma and beta and modified gamma and beta functions.

Example 5.1. The MRL fractional integral of $f(z)=z^{\lambda}$ for $\Re(\alpha)>0$, $\Re(w)>0$, $\Re(\lambda+1)>0$, $\Re(\varepsilon)>0$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q\neq 0,-1,-2,\ldots$ is found as:

$$\widehat{I}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\}=\frac{z^{\varepsilon+\lambda}}{\Gamma(\varepsilon)}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1,\varepsilon;w).$$

By using MRL fractional integral operator, we have

$$\begin{aligned}\widehat{I}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\}&=\frac{1}{\Gamma(\varepsilon)}\int_0^z(z-\Delta)^{\varepsilon-1}\Delta^{\lambda}\ {}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\&=\frac{z^{\varepsilon+\lambda}}{\Gamma(\varepsilon)}\int_0^1u^{\lambda}(1-u)^{\varepsilon-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-w}{u(1-u)}\right)du\\&=\frac{z^{\varepsilon+\lambda}}{\Gamma(\varepsilon)}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1,\varepsilon;w).\end{aligned}$$

Example 5.2. The MRL fractional derivative of $f(z)=z^{\lambda}$ for $\Re(\alpha)>0$, $\Re(w)>0$, $\Re(\lambda+1)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $\Re(m-\varepsilon)>0$, $\Re(\lambda+1-\varepsilon)>0$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q\neq 0,-1,-2,\ldots$ is found as:

$$\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\}=z^{\lambda-\varepsilon}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1,m-\varepsilon;w)}{B(\lambda+1,m-\varepsilon)}.$$

By using MRL fractional derivative operator, we have

$$\begin{aligned}&\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\}\\&=\frac{1}{\Gamma(m-\varepsilon)}\frac{d^m}{dz^m}\int_0^z(z-\Delta)^{m-\varepsilon-1}\Delta^{\lambda}\ {}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\&=\frac{z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)}\frac{\Gamma(m+\lambda+1-\varepsilon)}{\Gamma(\lambda+1-\varepsilon)}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1)}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1,m-\varepsilon;w)\\&=z^{\lambda-\varepsilon}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1,m-\varepsilon;w)}{B(\lambda+1,m-\varepsilon)}.\end{aligned}$$

Example 5.3. The MC fractional derivative of $f(z)=z^{\lambda}$ for $\Re(\alpha)>0$, $\Re(w)>0$, $\Re(\lambda+1)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $\Re(m-\varepsilon)>0$, $\Re(\lambda+1-\varepsilon)>0$, $\Re(\lambda+1-m)>0$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q\neq 0,-1,-2,\ldots$ is found as:

$${}^{c}\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\}=z^{\lambda-\varepsilon}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1-m,m-\varepsilon;w)}{B(\lambda+1-m,m-\varepsilon)}.$$

By using MC fractional derivative operator, we have

$$
\begin{aligned}
&{}^{c}\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\}\\
&=\frac{1}{\Gamma(m-\varepsilon)}\int_0^z (z-\Delta)^{m-\varepsilon-1}\ {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)\frac{d^m}{d\Delta^m}\left(\Delta^{\lambda}\right)d\Delta\\
&=z^{\lambda-\varepsilon}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-m)\Gamma(m-\varepsilon)}\frac{\Gamma(\lambda+1-\varepsilon)}{\Gamma(\lambda+1-\varepsilon)}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1-m,m-\varepsilon;w)\\
&=z^{\lambda-\varepsilon}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1-m,m-\varepsilon;w)}{B(\lambda+1-m,m-\varepsilon)}.
\end{aligned}
$$

Example 5.4. The MRL fractional integral of $f(z)=(1-az)^{-\lambda}$ for $\Re(\alpha)>0$, $\Re(w)>0$, $\Re(\varepsilon)>0$, $\xi_1,\dots,\xi_p,\mu_1,\dots,\mu_q,\lambda\neq 0,-1,-2,\dots$ is found as:

$$
\widehat{I}_z^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}=\frac{z^{\varepsilon}}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,\varepsilon;w).
$$

By using MRL fractional integral operator, we have

$$
\begin{aligned}
&\widehat{I}_z^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}\\
&=\frac{1}{\Gamma(\varepsilon)}\int_0^z (z-\Delta)^{\varepsilon-1}(1-a\Delta)^{-\lambda}\ {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&=\frac{z^{\varepsilon}}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\int_0^1 u^n(1-u)^{\varepsilon-1}\ {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\dots,\xi_p;\mu_1,\dots,\mu_q;\frac{-w}{u(1-u)}\right)du\\
&=\frac{z^{\varepsilon}}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,\varepsilon;w).
\end{aligned}
$$

Example 5.5. The MRL fractional derivative of $f(z)=(1-az)^{-\lambda}$ for $\Re(\alpha)>0$, $\Re(w)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $\Re(m-\varepsilon)>0$, $\Re(n+1-\varepsilon)>0$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\xi_1,\dots,\xi_p$, $\mu_1,\dots,\mu_q,\lambda\neq 0,-1,-2,\dots$ is found as:

$$
\widehat{D}_z^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,m-\varepsilon;w)}{B(n+1,m-\varepsilon)}.
$$

By using MRL fractional derivative operator, we have

$$
\begin{aligned}
&\widehat{D}_z^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}\\
&=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\frac{\Gamma(m+n+1-\varepsilon)}{\Gamma(n+1-\varepsilon)\Gamma(m-\varepsilon)}\frac{\Gamma(n+1)}{\Gamma(n+1)}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,m-\varepsilon;w)\\
&=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,m-\varepsilon;w)}{B(n+1,m-\varepsilon)}.
\end{aligned}
$$

Example 5.6. The MC fractional derivative of $f(z)=(1-az)^{-\lambda}$ for $\Re(\alpha)>0$, $\Re(w)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $\Re(m-\varepsilon)>0$, $\Re(n+1-\varepsilon)>0$, $n+1-m>0$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q,\lambda\neq 0,-1,-2,\ldots$ is found as:

$$
\begin{aligned}
&{}^{c}\widehat{D}_z^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}\\
&\quad=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1-m,m-\varepsilon;w)}{B(n+1-m,m-\varepsilon)}.
\end{aligned}
$$

By using MC fractional derivative operator, we have

$$
\begin{aligned}
&{}^{c}\widehat{D}_z^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}\\
&=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1-m)\Gamma(m-\varepsilon)}\frac{\Gamma(n+1-\varepsilon)}{\Gamma(n+1-\varepsilon)}\,{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1-m,m-\varepsilon;w)\\
&=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_n(az)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1-m,m-\varepsilon;w)}{B(n+1-m,m-\varepsilon)}.
\end{aligned}
$$

6. CONCLUSIONS

In this paper, MRL fractional integral, MRL fractional derivative and MC fractional derivative operators, which including generalized M-series at their kernels, are defined. Since the generalized M-series has a more general form then most of the special functions, many fractional operators becomes the special cases of the fractional operators (MRL and MC fractional operators) introduced here.

The definitions of some popular fractional operators that have been defined recently and their relationships with the new fractional operators (MRL and MC fractional operators) are given below.

Özarslan and Özergin [17]:

$$
D_z^{\varepsilon,w}\{f(z)\}=\begin{cases}\frac{1}{\Gamma(-\varepsilon)}\int\limits_0^z(z-\Delta)^{-\varepsilon-1}f(\Delta)\exp\left(\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta,\\ \frac{d^m}{dz^m}D_z^{\varepsilon-m,w}\{f(z)\}\end{cases}
$$

and relationships

$$
\widehat{D}_z^{\varepsilon}\{f(z);1;1;1,1;w\}=\frac{d^m}{dz^m}D_z^{\varepsilon-m,w}\{f(z)\}.
$$

Srivastava et al. [22]:

$$
D^{\varepsilon,w}_{z,\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}\{f(z)\}=\begin{cases}\frac{1}{\Gamma(-\varepsilon)}\int_0^z(z-\Delta)^{-\varepsilon-1}f(\Delta)\Theta\left(\{K_l\}_{l\in\mathbf{N_0}};\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta,\\ \frac{d^m}{dz^m}D^{\varepsilon-m,w}_{z,\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}\{f(z)\}\end{cases}
$$

and relationships

$$\widehat{D}_z^{\varepsilon}\{f(z);\xi;\mu;1,1;w\} = \frac{d^m}{dz^m} D^{\varepsilon-m,w}_{z,\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}\{f(z)\}.$$

Parmar [18]:

$$D_z^{\varepsilon,w;k}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\varepsilon)}\int\limits_0^z (z-\Delta)^{-\varepsilon-1} f(\Delta)\exp\left(\frac{-wz^{2k}}{\Delta^k(z-\Delta)^k}\right) d\Delta, \\ \frac{d^m}{dz^m} D_z^{\varepsilon-m,w;k}\{f(z)\} \end{cases}$$

and relationships

$$\widehat{D}_z^{\varepsilon}\{f(z);1;1;1,1;w\} = \frac{d^m}{dz^m} D_z^{\varepsilon-m,w;1}\{f(z)\}.$$

Agarwal et al. [3]:

$$D_z^{\varepsilon,w;\kappa,\eta}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\varepsilon)}\int\limits_0^z (z-\Delta)^{-\varepsilon-1} f(\Delta)\,{}_1F_1\left(\xi;\mu;\frac{-wz^{\kappa+\eta}}{\Delta^{\kappa}(z-\Delta)^{\eta}}\right) d\Delta, \\ \frac{d^m}{dz^m} D_z^{\varepsilon-m,w;\kappa,\eta}\{f(z)\} \end{cases}$$

and relationships

$$\widehat{D}_z^{\varepsilon}\{f(z);\xi;\mu;1,1;w\} = \frac{d^m}{dz^m} D_z^{\varepsilon-m,w;1,1}\{f(z)\}.$$

Kıymaz et al. [15]:

$$D_z^{\varepsilon,w}\{f(z)\} = \frac{1}{\Gamma(m-\varepsilon)}\int_0^z (z-\Delta)^{m-\varepsilon-1} f^{(m)}(\Delta)\exp\left(\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$

and relationships

$${}^c\widehat{D}_z^{\varepsilon}\{f(z);1;1;1,1;w\} = D_z^{\varepsilon,w}\{f(z)\}.$$

Agarwal et al. [2]:

$$D_z^{\varepsilon,w;k}\{f(z)\} = \frac{1}{\Gamma(m-\varepsilon)}\int_0^z (z-\Delta)^{m-\varepsilon-1} f^{(m)}(\Delta)\exp\left(\frac{-wz^{2k}}{\Delta^k(z-\Delta)^k}\right) d\Delta$$

and relationships

$${}^c\widehat{D}_z^{\varepsilon}\{f(z);1;1;1,1;w\} = D_z^{\varepsilon,w;1}\{f(z)\}.$$

Baleanu et al. [8]:

$$D_z^{\varepsilon}\{f(z);p,q\} = \begin{cases} \frac{1}{\Gamma(-\varepsilon)}\int\limits_0^z (z-\Delta)^{-\varepsilon-1} f(\Delta)\exp\left(\frac{-pz}{\Delta}-\frac{qz}{z-\Delta}\right) d\Delta, \\ \frac{d^m}{dz^m} D_z^{\varepsilon-m}\{f(z);p,q\} \end{cases}$$

and relationships

$$\widehat{D}_z^{\varepsilon}\{f(z);1;1;1,1;w\} = \frac{d^m}{dz^m} D_z^{\varepsilon-m}\{f(z);w,w\}.$$

Kıymaz et al. [14]:

$$D_z^{\varepsilon}\{f(z);p,q\}=\frac{1}{\Gamma(m-\varepsilon)}\int_0^z(z-\Delta)^{m-\varepsilon-1}f^{(m)}(\Delta)\exp\left(\frac{-pz}{\Delta}-\frac{qz}{z-\Delta}\right)d\Delta$$

and relationships

$${}^{c}\widehat{D}_z^{\varepsilon}\{f(z);1;1;1,1;w\}=D_z^{\varepsilon}\{f(z);w,w\}.$$

Çetinkaya et al. [10]:

$$D^{\varepsilon;(\xi,\mu;\kappa,\eta;p,q)}\{f(z)\}=\begin{cases}\frac{1}{\Gamma(-\varepsilon)}\int\limits_0^z(z-\Delta)^{-\varepsilon-1}f(\Delta)_1F_1\left(\xi;\mu;-p\left(\frac{z}{\Delta}\right)^{\kappa}-q\left(\frac{z}{z-\Delta}\right)^{\eta}\right)d\Delta,\\ \frac{d^m}{dz^m}D^{\varepsilon-m;(\xi,\mu;\kappa,\eta;p,q)}\{f(z)\},\end{cases}$$

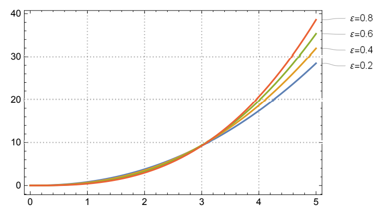
$$\mathbf{D}^{\varepsilon;(\xi,\mu;\kappa,\eta;p,q)}\{f(z)\}=D^{\varepsilon-m;(\xi,\mu;\kappa,\eta;p,q)}\left\{f^{(m)}(z)\right\}$$

and relationships

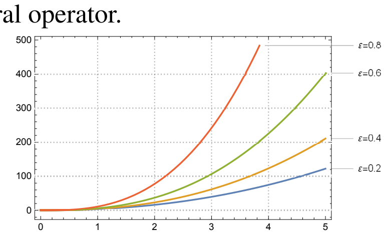
$$\widehat{D}_z^{\varepsilon}\{f(z);\xi;\mu;1,1;w\}=\frac{d^m}{dz^m}D^{\varepsilon-m;(\xi,\mu;1,1;w,w)}\{f(z)\},$$

$${}^{c}\widehat{D}_z^{\varepsilon}\{f(z);\xi;\mu;1,1;w\}=\mathbf{D}^{\varepsilon;(\xi,\mu;1,1;w,w)}\{f(z)\}.$$

Finally, we present the behavior of the function z^2 in both the classical fractional operators and the new fractional operators, see Figure 1 and Figure 2.

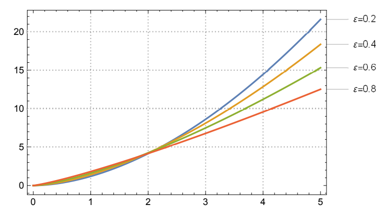


(A) Behavior of the function z^2 in the classical Riemann-Liouville fractional integral operator.

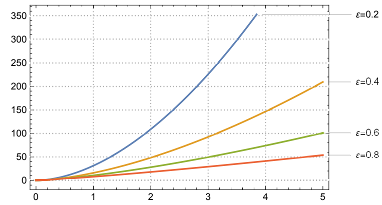


(B) Behavior of the function z^2 in MRL fractional integral operator for values $p=q=\xi_1=\mu_1=1$, $\alpha=\beta=1$, $w=2$ and generalized M-series index $n=0,1$.

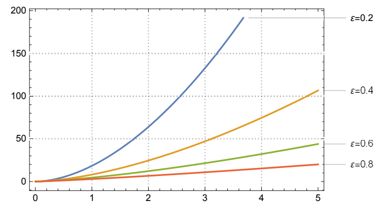
FIGURE 1. Comparison of classical Riemann-Liouville fractional integral operator with MRL fractional integral operator.



(A) Behavior of the function z^2 in the classical Riemann-Liouville and Caputo fractional derivative operators.



(B) Behavior of the function z^2 in MRL fractional derivative operator for values $p = q = \xi_1 = \mu_1 = 1$, $\alpha = \beta = m = 1$, $w = 2$ and generalized M-series index $n = 0, 1$.



(C) Behavior of the function z^2 in MC fractional derivative operator for values $p = q = \xi_1 = \mu_1 = 1$, $\alpha = \beta = m = 1$, $w = 2$ and generalized M-series index $n = 0, 1$.

FIGURE 2. Comparison of classical Riemann-Liouville and Caputo fractional derivative operators with MRL and MC fractional derivative operators.

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