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NEW CURVATURE TENSORS ALONG RIEMANNIAN SUBMERSIONS

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Abstract. In 1966, B. O'Neill [The fundamental equations of a submersion, Michigan Math. J., Volume 13, Issue 4 (1966), 459-469.] defined some fundamental equations and curvature relations between the total space, the base space and the fibres on a submersion. In the present paper, we define new curvature tensors on Riemannian submersions such as Weyl projective curvature tensor, concircular curvature tensor, conharmonic curvature tensor, conformal curvature tensor and $M-$projective curvature tensor, respectively. Finally, we obtain some results in case of the total space of Riemannian submersions has umbilical fibres for any curvature tensors mentioned by the above.

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1. INTRODUCTION AND PRELIMINARIES

In differential geometry, an important tool to define the curvature of $n-$ dimensional spaces (such as Riemannian manifolds) is the Riemannian curvature tensor. The tensor has played an important role both general relativty and gravity. In this manner, Mishra in [13] defined some new curvature tensors on Riemannian manifolds such as concircular curvature tensor, conharmonic curvature tensor, conformal curvature tensor, respectively. Taking into account the paper of Mishra, Pokhariyal and Mishra defined the Wely projective curvature tensor on Riemannian manifolds [17]. Afterwards, Ojha defined $M-$ Projective curvature tensor [15].

Riemannian submersion appears to have been studied and its differential geometry was first defined by O'Neill 1966 and Gray 1967. We note that Riemannian submersions have been studied widely not only in mathematics, but also in theoretical pyhsics because of their applications in the Yang-Mills theory, Kaluza Klein theory, super gravity, relativity and superstring theories (see [3, 4, 9, 10, 14, 18]). Most of the studies related to Riemannian submersion can be found in the books ([5, 6]). In 1966, B. O'Neill has defined a paper related to some fundamental equations of a submersions. In that paper, he has given some curvature relations on Riemannian submersions.

In this study, in addition to the curvature relations previously defined on Riemannian submersion, we investigate new curvature tensors on a Riemannian submersion and the curvature properties of these tensors. In the present paper, in the first part of our study, the basic definitions and theorems that we will use throughout the paper are given. In sections 2-6 include the Weyl projective curvature tensor, concircular curvature tensor, conharmonic curvature tensor, conformal curvature tensor and M-projective curvature tensor relations for a Riemannian submersion respectively. Also various results are obtained by examining the conditions for having total umbilical fibers.

Now, we will give the basic definitions and theorems without proofs that we will use throughout the paper.

Definition 1. Let (M,g) and (N,g_N) be Riemannian manifolds, where-$dim(M) > dim(N)$. A surjective mapping $\pi : (M,g) \to (N,g_N)$ is called a *Riemannian submersion* [16] if:

(S1) The rank of π equals $dim(N)$.
In this case, for each $q \in N$, $\pi^{-1}(q) = \pi_q^{-1}$ is a k-dimensional submanifold of M and called a *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e. , $\pi_*(X_p) = X_{*\pi(p)}$ for all $p \in M$, where π_* is derivative or differential map of π. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\pi_*$, and the horizontal distribution $ker\pi_*^{\perp}$, respectively. As usual, the manifold (M,g) is called *total manifold* and the manifold (N,g_N) is called *base manifold* of the submersion $\pi : (M,g) \to (N,g_N)$.
(S2) π_* preserves the lengths of the horizontal vectors.

These conditions are equivalent to say that the derivative map π_* of π, restricted to $ker\pi_*^{\perp}$, is a linear isometry.

If X and Y are the basic vector fields, π-related to X_N, Y_N, we have the following facts:

(1) $g(X,Y) = g_N(X_N,Y_N) \circ \pi$,
(2) $h[X,Y]$ is the basic vectr field π-related to $[X_N,Y_N]$,
(3) $h(\nabla_X Y)$ is the basic vector field π-related to ${\nabla^N}_{X_N} Y_N$,

for any vertical vector field $\mathcal{V}$, $[X,Y]$ is the vertical.

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{1.1}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{1.2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection, $\mathcal{V}$ and $\mathcal{H}$ are orthogonal projections on vertical and horizontal spaces, respectively.

Now, we are going to give an example for a Riemannian submersion as follows:

Example 1. Let $\phi : (\mathbb{R}^4, g_{\mathbb{R}^4}) \longrightarrow (\mathbb{R}^2, g_{\mathbb{R}^2})$ be a submersion defined by

$$\phi(u_1,u_2,u_3,u_4) = \left(\frac{1}{\sqrt{2}}(u_1-u_2), \sqrt{u_3^2+u_4^2}\right).$$

Then, the Jacobian matrix of ϕ is:

$$\phi_* = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{u_3}{W} & \frac{u_4}{W} \end{bmatrix}$$

where $W = \sqrt{u_3^2+u_4^2}$. The rank of the map equal to 2. It means that the map is a submersion. A straight computations yields

$$\ker\phi_* = \operatorname{span}\left\{V_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2},\ V_2 = u_4\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_4}\right\}$$

and

$$(\ker\phi_*)^\perp = \operatorname{span}\left\{X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right),\ X_2 = \frac{u_3}{W}\frac{\partial}{\partial u_3} + \frac{u_4}{W}\frac{\partial}{\partial u_4}\right\}.$$

Also by direct computations yields

$$\phi_*(X_1) = \partial v_1 \text{ and } \phi_*(X_2) = \partial v_2$$

Thus, it is easy to see that

$$g_{\mathbb{R}^2}(\phi_*(X_i), \phi_*(X_i)) = g_{\mathbb{R}^4}(X_i, X_i),\ \ i = 1,2$$

Hence ϕ is a Riemannian submersion.

Definition 2. Let (M,g) and (G,g') Riemannian manifolds of $n-$ dimensional and the horizontal distribution of (M,g) is $\mathcal{H}$. Let's show $(1,3)$-order curvature tensor field on $X^h(M)$ with R^*. For any $X,Y,Z \in \chi^h(M)$ and $p \in M$

$$R^G_{\pi(p)}(\pi_{*p}X_p, \pi_{*p}Y_p, \pi_{*p}Z_p). \tag{1.3}$$

We now recall the following curvature relations for a Riemannian submersion from [6] and [16].

Theorem 1. *(M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors of M,G and $(\pi^{-1}(x),\hat{g}_x)$ fibre respectively. In this case, there are the following equations for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$

$$\begin{aligned} g(R^M(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_XY,A_ZH) \\ &\quad - g(A_YZ,A_XH) + g(A_XZ,A_YH), \end{aligned} \tag{1.4}$$

$$\begin{aligned} g(R^M(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V) - g(A_XY,T_VZ) \\ &\quad + g(A_YZ,T_VX) - g(A_XZ,T_VY), \end{aligned} \tag{1.5}$$

$$\begin{aligned} g(R^M(X,Y)V,W) &= g((\nabla_V A)_X Y,W) - g((\nabla_W A)_X Y,V) \\ &\quad + g(A_XVA_YW) + g(A_XW,A_YV) \\ &\quad - g(T_VX,T_WY) + g(T_WX,T_VY), \end{aligned} \tag{1.6}$$

$$\begin{aligned} g(R^M(X,V)Y,W) &= g((\nabla_X T)_V W,Y) - g((\nabla_V A)_X Y,W) \\ &\quad - g(T_VX,T_WY) + g(A_XV,A_YW), \end{aligned} \tag{1.7}$$

$$g(R^M(U,V)W,X) = g((\nabla_U T)_V W,X) - g((\nabla_V T)_U W,X) \tag{1.8}$$

and

$$g(R^M(U,V)W,F) = g(\hat{R}(U,V)W,F) + g(T_UW,T_VF) - g(T_VW,T_UF). \tag{1.9}$$

Definition 3. [6] Let (M,g) be a Riemannian manifold and a local orthonormal frame of the vertical distribution ν is $\{U_j\}_{1\le j\le r}$. Then N, the horizontal vector field on (M,g) is locally defined by

$$N = \sum_{j=1}^{r} T_{U_j}U_j.$$

Proposition 1. *Let (M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and $\{X_i,U_j\}$ be a π-compatible frame.

In this case, for any $U,V \in \chi^v(M)$ and $X,Y \in \chi^h(M)$, the Ricci tensor S^M holds the following equations [6]:

$$\begin{aligned} (i) \qquad S^M(U,V) &= \hat{S}(U,V) - g(N,T_UV) \\ &\quad + \sum_i \{g((\nabla_{X_i} T)_U V,X_i) + g(A_{X_i}U,A_{X_i}V)\}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} (ii) \qquad S^M(X,Y) &= S^G(X',Y')\circ\pi + \frac{1}{2}\{g(\nabla_X N,Y) + g(\nabla_Y N,X)\} \\ &\quad - 2\sum_i g(A_XX_i,A_YX_i) - \sum_j g(T_{U_j}X,T_{U_j}Y), \end{aligned} \tag{1.11}$$

$$(iii) \qquad S^M(U,X) = g(\nabla_U N, X) - \sum_j g(\nabla_{U_j} T)_{U_j} U, X) + \sum_i \{g((\nabla_{X_i} A)_{X_i} X, U) - 2g(A_X X_i, T_U X_i)\}. \tag{1.12}$$

Proposition 2. *[6] Let's take the scalar curvatures of $(M,g),(G,g')$ Riemannian manifolds and $x \in G, \pi^{-1}(x)$ fibre with r^M, r^G and $\hat{r}$, respectively. In a*

$$\pi : (M,g) \to (G,g')$$

Riemannian submersion, (M,g) depends on the scalar curvature of the Riemannian manifold r^G and the scalar curve of any lift $\hat{r}$. In this case

$$r^M = \hat{r} + r^G \circ \pi - ||N||^2 - ||A||^2 - ||T||^2 + 2\sum_i g(\nabla_{X_i} N, X_i). \tag{1.13}$$

2. Weyl projective curvature tensor along a Riemannian submersion

In this section, we examine the Weyl projective curvature tensor relations between the total space, the base space and fibres on a Riemannian submersion. We also give a corollary in case of the Riemannian submersion has totally umbilical fibres case.

Definition 4. [13] Let take an n-dimensional differentiable manifold M^n with differentiability class C^∞. In the $n-$dimensional space V_n, the tensor

$$P^*(X,Y)Z = R^M(X,Y)Z - \frac{1}{n-1}\{S^M(Y,Z)X - S^M(X,Z)Y\}.$$

is called Weyl projective curvature tensor, where Ricci tensor of total space denoted by S^M.

Now, we have the following main theorem.

Theorem 2. *Let, (M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors, S^M, S^G and $\hat{S}$ be Ricci tensors of M, G and the fibre respectively. Then for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$, we have the following relations for Weyl projective curvature tensor:

$$\begin{aligned} g(P^*(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_X Y, A_Z H) \\ &\quad - g(A_Y Z, A_X H) + g(A_X Z, A_Y H) \\ &\quad - \frac{1}{n-1}\Bigg\{ g(X,H)\Bigg[S^G(Y',Z')\circ\pi + \frac{1}{2}(g(\nabla_Y N, Z) + g(\nabla_Z N, Y)) \\ &\quad - 2\sum_i g(A_Y X_i, A_Z X_i) - \sum_j g(T_{U_j} Y, T_{U_j} Z)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&- g(Y,H)\left[S^G(X',Z')\circ\pi + \frac{1}{2}\left(g(\nabla_X N,Z)+g(\nabla_Z N,X)\right)\right.\\
&\left.\left. - 2\sum_i g(A_X X_i, A_Z X_i) - \sum_j g(T_{U_j}X, T_{U_j}Z)\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
g(P^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V) - g(A_X Y, T_V Z)\\
&\quad + g(A_Y Z, T_V X) - g(A_X Z, T_V Y),\\
g(P^*(X,Y)V,W) &= g((\nabla_V A)_X Y,W) - g((\nabla_W A)_X Y,V) + g(A_X V, A_Y W)\\
&\quad - g(A_X W, A_Y V) - g(T_V X, T_W Y) + g(T_W X, T_V Y),\\
g(P^*(X,V)Y,W) &= g((\nabla_X T)_V W,Y) + g((\nabla_V A)_X Y,W)\\
&\quad - g(T_V X, T_W Y) + g(A_X V, A_Y W)\\
&\quad + \frac{1}{n-1}\left\{g(V,W)\left[S^G(X',Y')\circ\pi + \frac{1}{2}\{g(\nabla_X N,Y)\right.\right.\\
&\quad \left.\left. + g(\nabla_Y N,X)\} - 2\sum_i g(A_X X_i, A_Y X_i) - \sum_j g(T_{U_j}X, T_{U_j}Y)\right]\right\},\\
g(P^*(U,V)W,X) &= g((\nabla_U T)_V W,X) - g((\nabla_V T)_U W,X)
\end{aligned}
$$

and

$$
\begin{aligned}
g(P^*(U,V)W,F) &= g(\hat{R}(U,V)W,F) + g(T_U W, T_V F)\\
&\quad - g(T_V W, T_U F) - \frac{1}{n-1}\left\{g(F,U)\left[\hat{S}(V,W) - g(N, T_V W)\right.\right.\\
&\qquad \left. + \sum_i \left(g((\nabla_{X_i}T)_V W, X_i) + g(A_{X_i}V, A_{X_i}W)\right)\right]\\
&\quad - g(F,V)\left[\hat{S}(U,W) - g(N, T_U W)\right.\\
&\qquad \left.\left. + \sum_i \left(g((\nabla_{X_i}T)_U W, X_i) + g(A_{X_i}U, A_{X_i}W)\right)\right]\right\}.
\end{aligned}
$$

Proof. We only give the proof of the 1^{st} equation of this theorem. The following equations are obtained inner production with H to P^* and using (1.4) and (1.11) equations.

$$
\begin{aligned}
g(R^M(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_X Y, A_Z H) - g(A_Y Z, A_X H)\\
&\quad + g(A_X Z, A_Y H),
\end{aligned}
$$

$$S^M(Y,Z) = S^G(Y',Z')\circ\pi + \frac{1}{2}\{g(\nabla_Y N,Z)+g(\nabla_Z N,Y)\}$$
$$-2\sum_i g(A_Y Y_i, A_Z Y_i) - \sum_j (T_{U_j}Y, T_{U_j}Z)$$

and

$$S^M(X,Z) = S^G(Y',Z')\circ\pi + \frac{1}{2}\{g(\nabla_X N,Z)+g(\nabla_Z N,X)\}$$
$$-2\sum_i g(A_X X_i, A_Z X_i) - \sum_j (T_{U_j}X, T_{U_j}Z).$$

When these equations are substituted in P^*, the given result is obtained. Other equations are similarly proved by using Theorem 1 and Proposition 1. □

Corollary 1. *Let $\pi:(M,g)\to(G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. If the Riemannian submersion has total umbilical fibres that is $N=0$ and then the Weyl projective curvature tensor is given by*

$$g(P^*(X,Y)Z,H) = g(R^G(X,Y)Z,H)+2g(A_XY,A_ZH)-g(A_YZ,A_XH)+g(A_XZ,A_YH)$$
$$-\frac{1}{n-1}\left\{g(X,H)\left[S^G(Y',Z')\circ\pi - 2\sum_i g(A_YX_i,A_ZX_i)-\sum_j g(T_{U_j}Y,T_{U_j}Z)\right]\right.$$
$$\left.-g(Y,H)\left[S^G(X',Z')\circ\pi - 2\sum_i g(A_XX_i,A_ZX_i)-\sum_j g(T_{U_j}X,T_{U_j}Z)\right]\right\},$$

and

$$g(P^*(U,V)W,F) = g(\hat{R}(U,V)W,F)+g(T_UW,T_VF)-g(T_VW,T_UF)$$
$$-\frac{1}{n-1}\left\{g(F,U)\left[\hat{S}(V,W)+\sum_i (g((\nabla_{X_i}T)_VW,X_i)+g(A_{X_i}V,A_{X_i}W))\right]\right.$$
$$\left.-g(F,V)\left[\hat{S}(U,W)+\sum_i (g((\nabla_{X_i}T)_UW,X_i)+g(A_{X_i}U,A_{X_i}W))\right]\right\}.$$

for any $U,V,W,F\in\chi^v(M)$ and $X,Y,Z,H\in\chi^h(M)$,

3. Concircular curvature tensor along a Riemannian submersion

In this section, curvature relations of concircular curvature tensor in a Riemannian submersion are examined and showing that the Riemannian submersion with concircular curvature tensor has no the totally umbilical fibres.

Definition 5. In the $n-$dimensional space V_n, the tensor

$$C^*(X,Y,Z,H) = R^M(X,Y,Z,H) - \frac{r^M}{n(n-1)}[g(X,H)g(Y,Z)-g(Y,H)g(X,Z)],$$

is called concircular curvature tensor, where scalar tensor denoted by r^M [13].

Now, we have the following main theorem.

Theorem 3. *Let,* (M,g) *and* (G,g') *Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G *and* $\hat{R}$ *be Riemannian curvature tensors,* r^M, r^G *and* $\hat{r}$ *be scalar curvature tensors of* M, G *and the fibre respectively. Then for any* $U,V,W,F \in \chi^v(M)$ *and* $X,Y,Z,H \in \chi^h(M)$*, we have the following relations*

$$\begin{aligned} g(C^*(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_XY,A_ZH) \\ &\quad - g(A_YZ,A_XH) + g(A_XZ,A_YH) \\ &\quad - \frac{r^M}{n(n-1)}\left\{ g(Y,Z)g(X,H) - g(X,Z)g(Y,H) \right\}, \\ g(C^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V) - g(A_XT,T_VZ) \\ &\quad + g(A_YZ,T_VX) - g(A_XZ,T_VY), \\ g(C^*(X,Y)V,W) &= g((\nabla_V A)_X Y,W) - g((\nabla_W A)_X Y,V) + g(A_XV,A_YW) \\ &\quad - g(A_XW,A_YV) - g(T_VX,T_WY) + g(T_WX,T_VY), \\ g(C^*(X,V)Y,W) &= g((\nabla_X T)_V W,Y) + g((\nabla_V A)_X Y,W) - g(T_VX,T_WY) \\ &\quad + g(A_XY,A_YW) - \frac{r^M}{n(n-1)}\{-g(X,Y)g(V,W)\}, \\ g(C^*(U,V)W,X) &= g((\nabla_U T)_V W,X) - g((\nabla_V T)_U W,X) \end{aligned}$$

and

$$\begin{aligned} g(C^*(U,V)W,F) &= g(\hat{R}(U,V)W,F) + g(T_UW,T_VF) - g(T_VW,T_UF) \\ &\quad - \frac{r^M}{n(n-1)}\left\{ g(V,W)g(U,F) - g(U,W)g(V,F) \right\} \end{aligned}$$

where

$$r^M = \hat{r} + r^G \circ \pi - ||A||^2 - ||T||^2.$$

Proof. Let's prove the 2^{nd} equation of this theorem. Taking inner product C^* with V then we have

$$g(C^*(X,Y)Z,V) = g(R(X,Y)Z,V) - \frac{r^M}{n(n-1)}\{g(Y,Z)g(X,V) - g(X,Z)\}.$$

Then using equation (1.5), we get

$$\begin{aligned} g(C^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V) - g(A_XT,T_VZ) \\ &\quad + g(A_YZ,T_VX) - g(A_XZ,T_VY). \end{aligned}$$

which completes the proof of the second equation. Other equations are similarly proved by using Theorem 1, Proposition 1 and Proposition 2. □

Corollary 2. *Let $\pi : (M,g) \to (G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. Then the concircular curvature tensor of Riemannian submersion has no total umbilical fibres.*

4. Conharmonic curvature tensor along a Riemannian submersion

In this section, curvature relations of conharmonic curvature tensor in a Riemannian submersion are examined.

Definition 6. In the $n-$dimensional space V_n, the tensor

$$L^*(X,Y,Z,H) = R^M(X,Y,Z,H) - \frac{1}{n-2}[g(Y,Z)Ric(X,H) - g(X,Z)Ric(Y,H) + g(Y,H)Ric(Y,Z) - g(Y,H)Ric(X,Z)]$$

is called conharmonic curvature tensor, where Ricci tensor denoted by Ric [13].

In a similar way, we have the following main theorem.

Theorem 4. *Let, (M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors, S^M, S^G and $\hat{S}$ be Ricci tensors of M, G and the fibre respectively. Then for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$, we have the following relations

$$\begin{aligned}
g(L^*(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_XY,A_ZH) - g(A_YZ,A_XH) \\
&+ g(A_XZ,A_YH) - \frac{1}{(n-2)}\Bigg\{g(Y,Z)\Bigg[S^G(X',H')\circ\pi \\
&+ \frac{1}{2}(g(\nabla_XN,H) + g(\nabla_HN,X)) - 2\sum_i g(A_XX_i,A_HX_i) \\
&- \sum_j g(T_{U_j}X,T_{U_j}H)\Bigg] - g(X,Z)\Bigg[S^G(Y',H')\circ\pi \\
&+ \frac{1}{2}(g(\nabla_YN,H) + g(\nabla_HN,Y)) - 2\sum_i g(A_YX_i,A_HX_i) \\
&- \sum_j g(T_{U_j}Y,T_{U_j}H)\Bigg] + g(X,H)\Bigg[S^G(Y',Z')\circ\pi \\
&+ \frac{1}{2}(g(\nabla_YN,Z) + g(\nabla_ZN,Y))
\end{aligned}$$

$$
\begin{aligned}
&\quad -2\sum_i g(A_Y X_i, A_Z X_i) - \sum_j g(T_{U_j} Y, T_{U_j} Z)\Big] \\
&\quad - g(Y,H)\Big[S^G(X',Z')\circ\pi + \frac{1}{2}\left(g(\nabla_X N, Z) + g(\nabla_Z N, X)\right) \\
&\quad -2\sum_i g(A_X X_i, A_Z X_i) - \sum_j g(T_{U_j} X, T_{U_j} Z)\Big]\Big\}, \\
g(L^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y, V) - g(A_X T, T_V Z) + g(A_Y Z, T_V X) \\
&\quad - g(A_X Z, T_V Y) - \frac{1}{(n-2)}\Big\{g(Y,Z)\Big[g(\nabla_X N, V) \\
&\quad - \sum_j g((\nabla_{U_j} T)_{U_j} X, V) + \sum_i (g((\nabla_{X_i} A)_{X_i} X, V) \\
&\quad -2g(A_V X_i, T_X X_i))\Big] - g(X,Z)\Big[g(\nabla_Y N, V) \\
&\quad - \sum_j g((\nabla_{U_j} T)_{U_j} Y, V) + \sum_i (g((\nabla_{X_i} A)_{X_i} Y, V) \\
&\quad -2g(A_V X_i, T_Y X_i))\Big]\Big\}, \\
g(L^*(X,Y)V,W) &= g((\nabla_V A)_X Y, W) - g((\nabla_W A)_X Y, V) + g(A_X V, A_Y W) \\
&\quad - g(A_X W, A_Y V) - g(T_V X, T_W Y) + g(T_W X, T_V Y), \\
g(L^*(X,V)Y,W) &= g((\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y) \\
&\quad + g(A_X Y, A_Y W) - \frac{1}{(n-2)}\Big\{-g(V,W)\Big[S^G(X',Y')\circ\pi \\
&\quad + \frac{1}{2}\left(g(\nabla_X N, Y) + g(\nabla_Y N, X)\right) - 2\sum_i g(A_X X_i, A_Y X_i) \\
&\quad - \sum_j g(T_{U_j} X, T_{U_j} Y)\Big] - g(X,Y)\Big[\hat{S}(V,W) - g(N, T_V W) \\
&\quad + \sum_i \left(g((\nabla_{X_i} T)_V W, X_i) + g(A_{X_i} V, A_{X_i} W)\right)\Big]\Big\}, \\
g(L^*(U,V)W,X) &= g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X) \\
&\quad - \frac{1}{(n-2)}\Big\{g(V,W)\Big[g(\nabla_U N, X) - \sum_j g(\nabla_{U_j} T)_{U_j} U, X) \\
&\quad + \sum_i \{g((\nabla_{X_i} A)_{X_i} X, U) - 2g(A_X X_i, T_U X_i)\}\Big]
\end{aligned}
$$

$$-g(U,W)\Big[g(\nabla_V N,X)-\sum_j g(\nabla_{U_j}T)_{U_j}V,X)$$
$$+\sum_i\{g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_X X_i,T_V X_i)\}\Big]\Big\}$$

and

$$\begin{aligned}g(L^*(U,V)W,F) &= g(\hat{R}(U,V)W,F)+g(T_U W,T_V F)-g(T_V W,T_U F)\\ &-\frac{1}{(n-2)}\Big\{g(V,W)\Big[\hat{S}(U,F)-g(N,T_U F)\\ &+\sum_i(g((\nabla_{X_i}T)_U F,X_i)+g(A_{X_i}U,A_{X_i}F))\Big]\\ &-g(U,W)\Big[\hat{S}(V,F)-g(N,T_V F)\\ &+\sum_i(g((\nabla_{X_i}T)_V F,X_i)+g(A_{X_i}V,A_{X_i}F))\Big]\\ &+g(F,U)\Big[\hat{S}(V,W)-g(N,T_V W)\\ &+\sum_i(g((\nabla_{X_i}T)_V W,X_i)+g(A_{X_i}V,A_{X_i}W))\Big]\\ &-g(F,V)\Big[\hat{S}(U,W)-g(N,T_U W)\\ &+\sum_i(g((\nabla_{X_i}T)_U W,X_i)+g(A_{X_i}U,A_{X_i}W))\Big]\Big\}.\end{aligned}$$

Proof. Let's prove the 3^{th} equation of this theorem. The following equations are obtained inner production with W to L^* and by using equation (1.6)

$$\begin{aligned}g(L^*(X,Y)V,W) &= g(R^M(X,Y)V,W)-\frac{1}{n-2}\{g(X,W)S(Y,V)\\ &-g(Y,W)S(X,V)+g(Y,V)S(X,W)-g(X,V)S(Y,W)\}.\end{aligned}$$

One can easily obtain the other equations by using Theorem 1 and Proposition 1. □

Corollary 3. *Let $\pi:(M,g)\to(G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. If the Riemannian submersion has total umbilical fibres that is $N=0$ and then the conharmonic curvature tensor is given by*

$$\begin{aligned}g(L^*(X,Y)Z,H) &= g(R^G(X,Y)Z,H)+2g(A_X Y,A_Z H)-g(A_Y Z,A_X H)\\ &+g(A_X Z,A_Y H)-\frac{1}{(n-2)}\Big\{g(Y,Z)\Big[S^G(X',H')\circ\pi\\ &-2\sum_i g(A_X X_i,A_H X_i)-\sum_j g(T_{U_j}X,T_{U_j}H)\Big]\end{aligned}$$

$$
\begin{aligned}
& - g(X,Z)\Big[S^G(Y',H')\circ\pi - 2\sum_i g(A_YX_i,A_HX_i)\\
& - \sum_j g(T_{U_j}Y,T_{U_j}H)\Big] + g(X,H)\Big[S^G(Y',Z')\circ\pi\\
& - 2\sum_i g(A_YX_i,A_ZX_i) - \sum_j g(T_{U_j}Y,T_{U_j}Z)\Big]\\
& - g(Y,H)\Big[S^G(X',Z')\circ\pi - 2\sum_i g(A_XX_i,A_ZX_i)\\
& - \sum_j g(T_{U_j}X,T_{U_j}Z)\Big]\Big\},\\
g(L^*(X,Y)Z,V) &= -g((\nabla_ZA)_XY,V) - g(A_XT,T_VZ) + g(A_YZ,T_VX)\\
& - g(A_XZ,T_VY) - \frac{1}{(n-2)}\Big\{g(Y,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}X,V)\\
& + \sum_i (g((\nabla_{X_i}A)_{X_i}X,V) - 2g(A_VX_i,T_XX_i))\Big]\\
& - g(X,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}Y,V)\\
& + \sum_i (g((\nabla_{X_i}A)_{X_i}Y,V) - 2g(A_VX_i,T_YX_i))\Big]\Big\},\\
g(L^*(X,V)Y,W) &= g((\nabla_XT)_VW,Y) + g((\nabla_VA)_XY,W) - g(T_VX,T_WY)\\
& + g(A_XY,A_YW) - \frac{1}{(n-2)}\Big\{-g(V,W)\Big[S^G(X',Y')\circ\pi\\
& - 2\sum_i g(A_XX_i,A_YX_i) - \sum_j g(T_{U_j}X,T_{U_j}Y)\Big]\\
& - g(X,Y)\Big[\hat{S}(V,W) + \sum_i (g((\nabla_{X_i}T)_VW,X_i) + g(A_{X_i}V,A_{X_i}W))\Big]\Big\},\\
g(L^*(U,V)W,X) &= g((\nabla_UT)_VW,X) - g((\nabla_VT)_UW,X)\\
& - \frac{1}{(n-2)}\Big\{g(V,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}U,X)\\
& + \sum_i \{g((\nabla_{X_i}A)_{X_i}X,U) - 2g(A_XX_i,T_UX_i)\}\Big]
\end{aligned}
$$

$$-g(U,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}V,X)$$

$$+\sum_i\{g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_XX_i,T_VX_i)\}\Big]\Big\}$$

and

$$g(L^*(U,V)W,F)=g(\hat{R}(U,V)W,F)+g(T_UW,T_VF)$$

$$-g(T_VW,T_UF)-\frac{1}{(n-2)}\Big\{g(V,W)\Big[\hat{S}(U,F)$$

$$+\sum_i(g((\nabla_{X_i}T)_UF,X_i)+g(A_{X_i}U,A_{X_i}F))\Big]$$

$$-g(U,W)\Big[\hat{S}(V,F)+\sum_i(g((\nabla_{X_i}T)_VF,X_i)+$$

$$g(A_{X_i}V,A_{X_i}F))\Big]+g(F,U)\Big[\hat{S}(U,V)$$

$$+\sum_i(g((\nabla_{X_i}T)_UV,X_i)+g(A_{X_i}U,A_{X_i}V))\Big]$$

$$-g(F,V)\Big[\hat{S}(U,W)+\sum_i(g((\nabla_{X_i}T)_UW,X_i)+g(A_{X_i}U,A_{X_i}W))\Big]\Big\}.$$

5. Conformal curvature tensor along a Riemannian submersion

In this section, we find some curvature relations of conformal curvature tensor in a Riemannian submersion and give a corollary in case of the Riemannian submersion has totally umbilical fibres.

Definition 7. In the $n-$dimensional space V_n, the tensor

$$V^*(X,Y,Z,H)=R^M(X,Y,Z,H)-\frac{1}{n-2}[g(X,H)Ric(Y,Z)-g(Y,H)Ric(X,Z)$$
$$+g(Y,Z)Ric(X,H)-g(X,Z)Ric(Y,H)]$$
$$+\frac{r^M}{(n-1)(n-2)}[g(X,H)g(Y,Z)-g(Y,H)g(X,Z)],$$

is called conformal curvature tensor, where Ricci tensor and scalar tensor denoted by Ric and r^M respectively [13].

Theorem 5. *Let,* (M,g) *and* (G,g') *Riemannian manifolds,*

$$\pi:(M,g)\to(G,g')$$

a Riemannian submersion and R^M*,* R^G *and* $\hat{R}$ *be Riemannian curvature tensors,* S^M*,* S^G *and* $\hat{S}$ *be Ricci tensors and* r^M*,* r^G *and* $\hat{r}$ *be scalar curvature tensors of* M*,* G *and* *the fibre respectively. Then for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$, we have the following relations*

$$\begin{aligned}
g(V^*(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_XY,A_ZH) - g(A_YZ,A_XH) \\
&\quad + g(A_XZ,A_YH) - \frac{1}{(n-2)}\Bigg\{ g(X,H)\Bigg[S^G(Y',Z')\circ\pi \\
&\quad + \frac{1}{2}\left(g(\nabla_Y N,Z) + g(\nabla_Z N,Y)\right) - 2\sum_i g(A_YX_i,A_ZX_i) \\
&\quad - \sum_j g(T_{U_j}Y,T_{U_j}Z)\Bigg] - g(Y,H)\Bigg[S^G(X',Z')\circ\pi \\
&\quad + \frac{1}{2}\left(g(\nabla_X N,Z) + g(\nabla_Z N,X)\right) - 2\sum_i g(A_XX_i,A_ZX_i) \\
&\quad - \sum_j g(T_{U_j}X,T_{U_j}Z)\Bigg] + g(Y,Z)\Bigg[S^G(X',H')\circ\pi \\
&\quad + \frac{1}{2}\left(g(\nabla_X N,H) + g(\nabla_H N,X)\right) \\
&\quad - 2\sum_i g(A_XX_i,A_HX_i) - \sum_j g(T_{U_j}X,T_{U_j}H)\Bigg] \\
&\quad - g(X,Z)\Bigg[S^G(Y',H')\circ\pi + \frac{1}{2}\left(g(\nabla_Y N,H) + g(\nabla_H N,Y)\right) \\
&\quad - 2\sum_i g(A_YX_i,A_HX_i) - \sum_j g(T_{U_j}Y,T_{U_j}H)\Bigg]\Bigg\} \\
&\quad + \frac{r^M}{(n-1)(n-2)}\{g(Y,Z)g(X,H) - g(X,Z)g(Y,H)\}, \\
g(V^*(X,Y)Z,V) &= -g((\nabla_Z A)_XY,V) - g(A_XT,T_VZ) + g(A_YZ,T_VX) \\
&\quad - g(A_XZ,T_VY) - \frac{1}{(n-2)}\Bigg\{ g(Y,Z)\Bigg[g(\nabla_X N,V) \\
&\quad - \sum_j g((\nabla_{U_j}T)_{U_j}X,V) + \sum_i (g((\nabla_{X_i}A)_{X_i}V,X) \\
&\quad -2g(A_VX_i,T_XX_i))\Bigg] - g(X,Z)\Bigg[g(\nabla_Y N,V) \\
&\quad - \sum_j g((\nabla_{U_j}T)_{U_j}Y,V) + \sum_i (g((\nabla_{X_i}A)_{X_i}V,Y)
\end{aligned}$$

$$-2g(A_V X_i, T_Y X_i))\Big]\Big\},$$

$$\begin{aligned}
g(V^*(X,Y)V,W) &= g((\nabla_V A)_X Y, W) - g((\nabla_W A)_X Y, V) + g(A_X V, A_Y W) \\
&\quad - g(A_X W, A_Y V) - g(T_V X, T_W Y) + g(T_W X, T_V Y), \\
g(V^*(X,V)Y,W) &= g((\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y) \\
&\quad + g(A_X Y, A_Y W) - \frac{1}{(n-2)}\Big\{ - g(V,W)\Big[S^G(X',Y')\circ\pi \\
&\quad + \frac{1}{2}\left(g(\nabla_X N, Y) + g(\nabla_Y N, X)\right) - 2\sum_i g(A_X X_i, A_Y X_i) \\
&\quad - \sum_j g(T_{U_j} X, T_{U_j} Y)\Big] - g(X,Y)\Big[\hat{S}(V,W) - g(N, T_V W) \\
&\quad + \sum_i \left(g((\nabla_{X_i} T)_V W, X_i) + g(A_{X_i} V, A_{X_i} W)\right)\Big]\Big\} \\
&\quad + \frac{r^M}{(n-1)(n-2)}\{g(X,Y)g(V,W)\}, \\
g(V^*(U,V)W,X) &= g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X) \\
&\quad - \frac{1}{(n-2)}\Big\{g(V,W)\Big[g(\nabla_U N, X) - \sum_j g(\nabla_{U_j} T)_{U_j} U, X) \\
&\quad + \sum_i \{g((\nabla_{X_i} A)_{X_i} X, U) - 2g(A_X X_i, T_U X_i)\}\Big] \\
&\quad - g(U,W)\Big[g(\nabla_V N, X) - \sum_j g(\nabla_{U_j} T)_{U_j} V, X) \\
&\quad + \sum_i \{g((\nabla_{X_i} A)_{X_i} X, V) - 2g(A_X X_i, T_V X_i)\}\Big]\Big\}
\end{aligned}$$

and

$$\begin{aligned}
g(V^*(U,V)W,F) &= g(\hat{R}(U,V)W,F) + g(T_U W, T_V F) - g(T_V W, T_U F) \\
&\quad - \frac{1}{(n-2)}\Big\{g(F,U)\Big[\hat{S}(V,W) - g(N, T_V W) \\
&\quad + \sum_i \left(g(\nabla_{X_i} T)_V W, X_i + g(A_{X_i} V, A_{X_i} W)\right)\Big] \\
&\quad - g(F,V)\Big[\hat{S}(U,W) - g(N, T_U W) + \sum_i \left(g((\nabla_{X_i} T)_U W, X_i)\right. \\
&\quad \left. + g(A_{X_i} U, A_{X_i} W)\right)\Big] + g(V,W)\Big[\hat{S}(U,F) - g(N, T_U F)
\end{aligned}$$

$$+\sum_i (g((\nabla_{X_i}T)_U F, X_i) + g(A_{X_i}U, A_{X_i}F))\Big]$$
$$-g(U,W)\Big[\hat{S}(V,F) - g(N, T_V F) + \sum_i (g((\nabla_{X_i}T)_V F, X_i)$$
$$+ g(A_{X_i}V, A_{X_i}F))\Big]\Big\}$$
$$+\frac{r^M}{(n-1)(n-2)}\{g(V,W)g(U,F) - g(U,W)g(V,F)\}$$

where

$$r^M = \hat{r} + r^G \circ \pi - ||N||^2 - ||A||^2 - ||T||^2 + 2\sum_i g(\nabla_{X_i}N, X_i).$$

Proof. Let's prove the 4^{th} equation of this theorem. The following equations are obtained inner production with W to V^*

$$g(V^*(X,V)Y,W) = g(R^M(X,V)Y,W) - \frac{1}{n-2}\{g(X,W)S^M(V,Y)$$
$$- g(V,W)S^M(X,Y) + g(V,Y)S(X,W) - g(X,Y)S^M(V,W)\}$$
$$+\frac{r^M}{(n-1)(n-2)}\{g(X,W)g(Y,V) - g(X,V)g(Y,W)\}.$$

Then using equations (1.7)-(1.11), we have the desired result. From the Theorem 1, Proposition 1 and Proposition 2 the above equations are obtained. □

Corollary 4. *Let $\pi : (M,g) \to (G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. If the Riemannian submersion has total umbilical fibres that is $N = 0$ and then the conformal curvature tensor is given by*

$$g(V^*(X,Y)Z,H) = g(R^G(X,Y)Z,H) + 2g(A_X Y, A_Z H) - g(A_Y Z, A_X H)$$
$$+ g(A_X Z, A_Y H) - \frac{1}{(n-2)}\Big\{g(X,H)\Big[S^G(Y',Z')\circ\pi$$
$$-2\sum_i g(A_Y X_i, A_Z X_i) - \sum_j g(T_{U_j}Y, T_{U_j}Z)\Big] - g(Y,H)\Big[S^G(X',Z')$$
$$\circ\pi - 2\sum_i g(A_X X_i, A_Z X_i) - \sum_j g(T_{U_j}X, T_{U_j}Z)\Big]$$
$$+ g(Y,Z)\Big[S^G(X',H')\circ\pi - 2\sum_i g(A_X X_i, A_H X_i)$$
$$-\sum_j g(T_{U_j}X, T_{U_j}H)\Big] - g(X,Z)\Big[S^G(Y',H')\circ\pi$$

$$
\begin{aligned}
&\qquad -2\sum_i g(A_YX_i,A_HX_i)-\sum_j g(T_{U_j}Y,T_{U_j}H)\Big]\Bigg\}\\
&\qquad +\frac{r^M}{(n-1)(n-2)}\{g(Y,Z)g(X,H)-g(X,Z)g(Y,H)\},\\
g(V^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V)-g(A_XT,T_VZ)+g(A_YZ,T_VX)-g(A_XZ,T_VY)\\
&\qquad -\frac{1}{(n-2)}\Bigg\{g(Y,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}X,V)\\
&\qquad +\sum_i (g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_VX_i,T_XX_i))\Big]\\
&\qquad -g(X,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}Y,V)\\
&\qquad +\sum_i (g((\nabla_{X_i}A)_{X_i}Y,V)-2g(A_VX_i,T_YX_i))\Big]\Bigg\},\\
g(V^*(X,V)Y,W) &= g((\nabla_X T)_V W,Y)+g((\nabla_V A)_X Y,W)-g(T_VX,T_WY)\\
&\qquad +g(A_XY,A_YW)-\frac{1}{(n-2)}\Bigg\{-g(V,W)\Big[S^G(X',Y')\circ\pi\\
&\qquad -2\sum_i g(A_XX_i,A_YX_i)-\sum_j g(T_{U_j}X,T_{U_j}Y)\Big]-g(X,Y)\\
&\qquad \times\Big[\hat{S}(V,W)+\sum_i (g((\nabla_{X_i}T)_V W,X_i)+g(A_{X_i}V,A_{X_i}W))\Big]\Bigg\},\\
g(V^*(U,V)W,X) &= g((\nabla_U T)_V W,X)-g((\nabla_V T)_U W,X)\\
&\qquad -\frac{1}{(n-2)}\Bigg\{g(V,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}U,X)\\
&\qquad +\sum_i \{g((\nabla_{X_i}A)_{X_i}X,U)-2g(A_XX_i,T_UX_i)\}\Big]\\
&\qquad -g(U,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}V,X)\\
&\qquad +\sum_i \{g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_XX_i,T_VX_i)\}\Big]\Bigg\}
\end{aligned}
$$

and

$$
g(V^*(U,V)W,F)=g(\hat{R}(U,V)W,F)+g(T_UW,T_VF)-g(T_VW,T_UF)-\frac{1}{(n-2)}
$$

$$\times\Big\{g(F,U)\Big[\hat{S}(V,W)+\sum_i(g(\nabla_{X_i}T)_V W,X_i+g(A_{X_i}V,A_{X_i}W))\Big]$$
$$-g(F,V)\Big[\hat{S}(U,W)+\sum_i(g((\nabla_{X_i}T)_U W,X_i)+g(A_{X_i}U,A_{X_i}W))\Big]$$
$$+g(V,W)\Big[\hat{S}(U,F)+\sum_i(g((\nabla_{X_i}T)_U F,X_i)+g(A_{X_i}U,A_{X_i}F))\Big]$$
$$-g(U,W)\Big[\hat{S}(V,F)+\sum_i(g((\nabla_{X_i}T)_V F,X_i)+g(A_{X_i}V,A_{X_i}F))\Big]\Big\}$$
$$+\frac{r^M}{(n-1)(n-2)}\{g(V,W)g(U,F)-g(U,W)g(V,F)\}$$

where

$$r^M=\hat{r}+r^G\circ\pi-||A||^2-||T||^2.$$

Finally, we investigate the $M-$projective curvature tenson on a Riemannian submersion and give a corollary in case of the totall umbilical fibres.

6. M-PROJECTIVE CURVATURE TENSOR ALONG A RIEMANNIAN SUBMERSION

In this section, curvature relations of M-projective curvature tensor in a Riemannian submersion are examined and obtain a corollary using the curvature tensor.

Definition 8. Let take an n-dimensional differentiable manifold $M^n$with differentiability class C^∞. In 1971 on a n -dimensional Riemannian manifold, ones [17] defined a tensor field W^* as

$$W^*(X,Y)Z=R^M(X,Y)Z-\frac{1}{2(n-1)}[S^M(Y,Z)X$$
$$-S^M(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]$$

tensor W^* as M-projective curvature tensor.

In addition, on an $n-$dimensional Riemannian manifold M^n the Ricci operator Q is defined by

$$S^M(X,Y)=g(QX,Y).$$

Theorem 6. *Let, (M,g) and (G,g') Riemannian manifolds,*

$$\pi:(M,g)\to(G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors, S^M, S^G and $\hat{S}$ be Ricci tensors of M,G and the fibre respectively. Then for any $U,V,W,F\in\chi^v(M)$ and $X,Y,Z,H\in\chi^h(M)$, we have the following relations for M-projective curvature tensor:

$$g(W^*(X,Y)Z,H)=g(R^G(X,Y)Z,H)+2g(A_XY,A_ZH)-g(A_YZ,A_XH)$$

$$
\begin{aligned}
&+g(A_XZ,A_YH)-\frac{1}{2(n-1)}\Bigg\{g(X,H)\Bigg[S^G(Y',Z')\circ\pi\\
&+\frac{1}{2}\left(g(\nabla_YN,Z)+g(\nabla_ZN,Y)\right)-2\sum_i g(A_YX_i,A_ZX_i)\\
&-\sum_j g(T_{U_j}Y,T_{U_j}Z)\Bigg]-g(Y,H)\Bigg[S^G(X',Z')\circ\pi+\frac{1}{2}\left(g(\nabla_XN,Z)\right.\\
&\left.+g(\nabla_ZN,X)\right)-2\sum_i g(A_XX_i,A_ZX_i)-\sum_j g(T_{U_j}X,T_{U_j}Z)\Bigg]\\
&+g(Y,Z)\Bigg[S^G(X',H')\circ\pi+\frac{1}{2}\left(g(\nabla_XN,H)+g(\nabla_HN,X)\right)\\
&-2\sum_i g(A_XX_i,A_HX_i)-\sum_j g(T_{U_j}X,T_{U_j}H)\Bigg]\\
&-g(X,Z)\Bigg[S^G(Y',H')\circ\pi+\frac{1}{2}\left(g(\nabla_YN,H)+g(\nabla_HN,Y)\right)\\
&-2\sum_i g(A_YX_i,A_HX_i)-\sum_j g(T_{U_j}Y,T_{U_j}H)\Bigg]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
g(W^*(X,Y)Z,V)=&-g((\nabla_ZA)_XY,V)-g(A_XY,T_VZ)+g(A_YZ,T_VX)-g(A_XZ,T_VY)\\
&-\frac{1}{2(n-1)}\Bigg\{g(Y,Z)\Bigg[g(\nabla_XN,V)-\sum_j g((\nabla_{U_j}T)_{U_j}X,V)\\
&+\sum_i\left(g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_VX_i,T_XX_i)\right)\Bigg]\\
&-g(X,Z)\Bigg[g(\nabla_YN,V)-\sum_j g((\nabla_{U_j}T)_{U_j}Y,V)\\
&+\sum_i\left(g((\nabla_{X_i}A)_{X_i}Y,V)-2g(A_VX_i,T_YX_i)\right)\Bigg]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
g(W^*(X,Y)V,W)=&\,g((\nabla_VA)_XY,W)-g((\nabla_WA)_XY,V)+g(A_XV,A_YW)\\
&-g(A_XW,A_YV)-g(T_VX,T_WY)+g(T_WX,T_VY),
\end{aligned}
$$

$$
\begin{aligned}
g(W^*(X,V)Y,W)=&\,g((\nabla_XT)_VW,Y)+g((\nabla_VA)_XY,W)-g(T_VX,T_WY)\\
&+g(A_XY,A_YW)-\frac{1}{2(n-1)}\Bigg\{-g(V,W)\Big[S^G(X',Y')\circ\pi\\
&+\frac{1}{2}\left(g(\nabla_XN,Y)+g(\nabla_YN,X)\right)
\end{aligned}
$$

$$-2\sum_i g(A_X X_i, A_Y X_i) - \sum_j g(T_{U_j}X, T_{U_j}Y)\Big] - g(X,Y)\Big[\hat{S}(V,W)$$
$$- g(N, T_V W) + \sum_i (g((\nabla_{X_i}T)_V W, X_i) + g(A_{X_i}V, A_{X_i}W))\Big]\Big\},$$

$$g(W^*(U,V)W,X) = g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X)$$
$$- \frac{1}{2(n-1)}\Big\{g(V,W)\Big[g(\nabla_U N, X) - \sum_j g(\nabla_{U_j}T)_{U_j}U, X)$$
$$+ \sum_i \{g((\nabla_{X_i}A)_{X_i}X, U) - 2g(A_X X_i, T_U X_i)\}\Big]$$
$$- g(U,W)\Big[g(\nabla_V N, X) - \sum_j g(\nabla_{U_j}T)_{U_j}V, X)$$
$$+ \sum_i \{g((\nabla_{X_i}A)_{X_i}X, V) - 2g(A_X X_i, T_V X_i)\}\Big]\Big\}$$

and

$$g(W^*(U,V)W,F) = g(\hat{R}(U,V)W,F) + g(T_U W, T_V F) - g(T_V W, T_U F)$$
$$- \frac{1}{2(n-1)}\Big\{g(F,U)\Big[\hat{S}(V,W) - g(N, T_V W)$$
$$+ \sum_i (g(\nabla_{X_i}T)_V W, X_i + g(A_{X_i}V, A_{X_i}W))\Big]$$
$$- g(F,V)\Big[\hat{S}(U,W) - g(N, T_U W)$$
$$+ \sum_i g((\nabla_{X_i}T)_U W, X_i) + g(A_{X_i}U, A_{X_i}W)\Big]$$
$$+ g(V,W)\Big[\hat{S}(U,F) - g(N, T_U F)$$
$$+ \sum_i (g((\nabla_{X_i}T)_U F, X_i) + g(A_{X_i}U, A_{X_i}F))\Big]$$
$$- g(U,W)\Big[\hat{S}(V,F) - g(N, T_V F)$$
$$+ \sum_i (g((\nabla_{X_i}T)_V F, X_i) + g(A_{X_i}V, A_{X_i}F))\Big]\Big\}.$$

Proof. Let's prove the 6^{th} equation of this theorem. The following equations are obtained inner production with F to W^* and using (1.9) and (1.10) equations.

$$g(W^*(U,V)W,F) = g(R^M(U,V)W,F) - \frac{1}{2(n-1)}\Big\{g(F,U)S^M(U,V)$$

$$-g(F,V)S^M(U,W)+g(V,W)S^M(U,F)-g(U,W)S^M(V,F)\Big\},$$

$$g(R^M(U,V)W,F)=g(\hat{R}(U,V)W,F)+g(T_UW,T_VF)-g(T_VW,T_UF)$$

and

$$S^M(U,V)=\hat{S}(U,V)-g(N,T_UV)+\sum_i\{g((\nabla_{X_i}T)_UV,X_i)+g(A_{X_i}U,A_{X_i}V)\}.$$

When these equations are substituted in W^*, the given result is obtained. Other equations are similarly proved by using Theorem 1 and Proposition 1. □

Corollary 5. *Let $\pi:(M,g)\to(G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. If the Riemannian submersion has total umbilical fibres that is $N=0$ and then the M-projective curvature tensor is given by*

$$\begin{aligned}
g(W^*(X,Y)Z,H)&=g(R^G(X,Y)Z,H)+2g(A_XY,A_ZH)-g(A_YZ,A_XH)\\
&\quad+g(A_XZ,A_YH)-\frac{1}{2(n-1)}\Big\{g(X,H)\Big[S^G(Y',Z')\circ\pi\\
&\quad-2\sum_i g(A_YX_i,A_ZX_i)-\sum_j g(T_{U_j}Y,T_{U_j}Z)\Big]\\
&\quad-g(Y,H)\Big[S^G(X',Z')\circ\pi-2\sum_i g(A_XX_i,A_ZX_i)\\
&\qquad-\sum_j g(T_{U_j}X,T_{U_j}Z)\Big]\\
&\quad+g(Y,Z)\Big[S^G(X',H')\circ\pi-2\sum_i g(A_XX_i,A_HX_i)\\
&\qquad-\sum_j g(T_{U_j}X,T_{U_j}H)\Big]\\
&\quad-g(X,Z)\Big[S^G(Y',H')\circ\pi-2\sum_i g(A_YX_i,A_HX_i)\\
&\qquad-\sum_j g(T_{U_j}Y,T_{U_j}H)\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
g(W^*(X,Y)Z,V)&=-g((\nabla_ZA)_XY,V)-g(A_XY,T_VZ)+g(A_YZ,T_VX)\\
&\quad-g(A_XZ,T_VY)-\frac{1}{2(n-1)}\Big\{g(Y,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}X,V)\\
&\quad+\sum_i(g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_VX_i,T_XX_i))\Big]
\end{aligned}$$

$$-g(X,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}Y,V)$$
$$+\sum_i (g((\nabla_{X_i}A)_{X_i}Y,V)-2g(A_VX_i,T_YX_i))\Big]\Big\},$$
$$g(W^*(X,V)Y,W)=g((\nabla_XT)_VW,Y)+g((\nabla_VA)_XY,W)-g(T_VX,T_WY)$$
$$+g(A_XY,A_YW)-\frac{1}{2(n-1)}\Big\{-g(V,W)\Big[S^G(X',Y')\circ\pi$$
$$-2\sum_i g(A_XX_i,A_YX_i)-\sum_j g(T_{U_j}X,T_{U_j}Y)\Big]$$
$$-g(X,Y)\Big[\hat{S}(V,W)+\sum_i (g((\nabla_{X_i}T)_VW,X_i)+g(A_{X_i}V,A_{X_i}W))\Big]\Big\},$$
$$g(W^*(U,V)W,X)=g((\nabla_UT)_VW,X)-g((\nabla_VT)_UW,X)$$
$$-\frac{1}{2(n-1)}\Big\{g(V,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}U,X)$$
$$+\sum_i\{g((\nabla_{X_i}A)_{X_i}X,U)-2g(A_XX_i,T_UX_i)\}\Big]$$
$$-g(U,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}V,X)$$
$$+\sum_i\{g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_XX_i,T_VX_i)\}\Big]\Big\}$$

and

$$g(W^*(U,V)W,F)=g(\hat{R}(U,V)W,F)+g(T_UW,T_VF)-g(T_VW,T_UF)-\frac{1}{2(n-1)}$$
$$\times\Big\{g(F,U)\Big[\hat{S}(V,W)+\sum_i (g(\nabla_{X_i}T)_VW,X_i+g(A_{X_i}V,A_{X_i}W))\Big]$$
$$-g(F,V)\Big[\hat{S}(U,W)+\sum_i g((\nabla_{X_i}T)_UW,X_i)+g(A_{X_i}U,A_{X_i}W)\Big]$$
$$+g(V,W)\Big[\hat{S}(U,F)+\sum_i (g((\nabla_{X_i}T)_UF,X_i)+g(A_{X_i}U,A_{X_i}F))\Big]$$
$$-g(U,W)\Big[\hat{S}(V,F)+\sum_i (g((\nabla_{X_i}T)_VF,X_i)+g(A_{X_i}V,A_{X_i}F))\Big]\Big\}.$$

7. Conclusion

The authors investigate new curvature tensors along Riemannian submersions and obtain some results by using totally umbilical fibres. Therefore, it will be worth examining new curvature tensors along Riemannian submersions. Based on this study, pseudo-projective curvature tensor and quasi-conformal curvature tensor for a Riemannian submersion have been studied in [2]. Again other features such as flatness, symmetry conditions, and a variety of specific conditions on these curvature tensors can be investigated. A sequence of inequalities for Riemannian submersions and various applications between Riemannian submersions and Riemannian manifolds can also be established using some special functions [7, 8].

Riemannian submersions have applications in theoretical pyhsics, too. Other example is in robotic theory, for the modeling and control of certain types of redundant robotic chains [1] (see: https://ieeexplore.ieee.org/document/1284418). Moreover, tensor analysis performed in this work has potential applications in dynamics of rigid bodies, electricity and magnetism, as well as in special theory of relativity, due to the fact that the tensors new considered are of prime interest in the fields of research [11, 12].

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