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MITTAG-LEFFLER FUNCTION FOR REAL INDEX AND ITS APPLICATION IN SOLVING DIFFERENCE AND DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we derive a Mittag-Leffler function for real index and establish solutions of special type of fractional order differential equations (FDEs). The same concept is extended to discrete case by replacing polynomials into factorial polynomials and differentiation into ℓ-difference operator. Moreover, numerical examples of our results are stated to validate our findings. The acquired results here have the ability to generate a wide range of formulas in relation to newer results.

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1. INTRODUCTION

Fractional calculus, like standard integer calculus, is one of the oldest branches of mathematics. It is nowadays utilized in the vast areas of science, modelling, bioengineering, computational optimization, control analysis of systems. We have different structures of fractional differential equations (FDEs) containing various special functions in their kernels which have main importance in analyzing mathematical models. For instances, the application of these kernels can be seen in some works including [1–3, 5, 10, 18, 21, 22]. In addition to these papers, the Mittag-Leffler function is used as the kernel of some new operators [6, 8, 11]. In fact, in last twenty years, the use of Mittag-Leffler function has gained its momentum in the investigation and solving different boundary problems and dynamical systems in life sciences [13]. The Mittag-Leffler function with two parameters is defined by [24] as

$$E_{\beta,\gamma}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\gamma+\beta j)}, \quad R(\beta), R(\gamma) > 0, \beta, \gamma, z \in \mathbb{C}. \tag{1.1}$$

This function is applicable to solve the fractional Kinetic equations [13]. Regarding Mittag-Leffler function and its properties, we can refer to Hilfer [14] and Saxena

[23]. As exponential function naturally is appeared from solving ordinary differential equations, the Mittag-Leffler function plays a similar role in solving (FDEs) fractional differential equations.

On the other hand, Diaz *et al.* [9] introduced the notion of the fractional difference by the rather usual method which permits the index of difference, in the standard representation of the n-th difference, to be each arbitrary real/complex number. After that, by using Taylor's series, Hirota [15] introduced the difference operator ∇^q in the fractional settings in which $q \in \mathbb{R}$. Nagai [20] indicated another kind of this definition for the mentioned difference operator in view of the modification of the Hirota's definition. Accordingly, Deekshitulu *et al.* [7] changed the definition presented by Nagai in [20] for $q \in (0,1)$ so that the expression for ∇^q does not possess any difference operator. Recently, in the field of discrete fractional calculus, delta (Δ^q) and nabla (∇^q) operators ($q \in \mathbb{R}$) play a key role in the modelings [12, 19].

In this paper, we develop ν^{th} real order Mittag-Leffler function by replacing z by $(\lambda\varkappa)^\nu$, $\gamma = 1, \beta = \nu$ and $\varkappa$ by r in (1.1). The motive of initiating this new type fractional or real order Mittag-Leffler function is for solving the fractional or real order differential and difference equations. The solution of fractional order differential and difference equations helps us to obtain applications in the context of fractional calculus. The applications related to this function clearly yield connection between Binomial expansion and Mittag-Leffler Factorial Function.

2. Mittag-Leffler function in the fractional settings

For this section, we go through some important definitions, concepts of factorial polynomials, Extorial function, Mittag-Leffler function and also necessary theorems can be exploit in the later sections. Here we use the domain set $J_\ell \subset \mathbb{R}$ for which $t \in J_\ell$ gives $t \pm \ell \in J_\ell$ and $\mathbb{N}(a) = \{a, a+1, a+2, \cdots\}$.

Definition 1. [4, 17] The ℓ-Delta operator and the inverse for a function $u : J_\ell \to \ell$ are respectively defined as

$$\Delta_\ell u(\varkappa) = u(\varkappa+\ell) - u(\varkappa),\ \varkappa \in [0,\infty),\ \ell \in (0,\infty), \tag{2.1}$$

$$\text{if } \Delta_\ell v(\varkappa) = u(\varkappa), \text{ then } v(\varkappa) \ = \ \Delta_\ell^{-1} u(\varkappa) + c, \tag{2.2}$$

where the constant c is obtained by substituting suitable value for $\varkappa$.

Definition 2. [17] Let $\ell > 0$, $\nu \in (-\infty, \infty)$ and $\Gamma(\frac{\varkappa}{\ell}+1)$ be the Gamma function. Then, the ℓ - factorial polynomial in $\varkappa$ for real index ν is defined by

$$\varkappa_\ell^{(\nu)} = \ell^\nu \frac{\Gamma(\varkappa/\ell+1)}{\Gamma(\varkappa/\ell+1-\nu)},\ \varkappa/\ell+1, (\varkappa/\ell+1-\nu) \notin -N(0) = \{0,-1,\cdots\}, \tag{2.3}$$

where the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} \mathrm{d}s,\ \ Re(z) > 0.$$

Special Cases:

(i) when $\ell = -1$, $\varkappa_{-1}^{(n)} = \varkappa(\varkappa+1)(\varkappa+2)\cdots(\varkappa+n-1) = \prod_{r=1}^{n}(\varkappa+r-1), n \in \mathbb{Z}^+$.

(ii) when $\ell = 1$, $\varkappa_{1}^{(n)} = \varkappa(\varkappa-1)(\varkappa-2)\cdots(\varkappa-n+1) = \prod_{r=1}^{n}(\varkappa-r+1) = \varkappa^{(n)}, n \in \mathbb{Z}^+$.

Definition 3. The n^{th} order Mittag-Leffler function $e_n(\lambda,\varkappa)$, with $n \in \mathbb{Z}^+$, $\varkappa \in (-\infty,\infty)$ is defined by

$$e_n(\lambda,\varkappa) = 1 + \frac{\lambda^n\varkappa^n}{(n)!} + \frac{\lambda^{2n}\varkappa^{2n}}{(2n)!} + \frac{\lambda^{3n}\varkappa^{3n}}{(3n)!} + \cdots + \infty = \sum_{r=0}^{\infty}\frac{\lambda^{rn}\varkappa^{rn}}{(rn)!}. \tag{2.4}$$

Note that $e_1(\lambda,\varkappa) = e^{\lambda\varkappa}$. Since the function defined in (2.4) is a sub series of $e^{\lambda\varkappa}$ for fixed λ, which is finite.

Definition 4. The ν^{th} order Mittag-Leffler Function, for non integer real number ν, with the condition $r\nu+1 \notin -\mathbb{N}(0)$ define by,

$$e_\nu(\lambda,\varkappa) = \sum_{r=0}^{\infty}\lambda^{r\nu}\frac{\varkappa^{r\nu}}{\Gamma(r\nu+1)}. \tag{2.5}$$

Lemma 1. *[25] We have the identities for any $r \in N(1)$ and $\ell > 0$:*

(i) $\Delta_\ell \varkappa_\ell^{(r)} = r\ell\varkappa_\ell^{(r-1)}$,

(ii) $\Delta_\ell \varkappa_{-\ell}^{(r)} = r\ell(\varkappa+\ell)_{-\ell}^{(r-1)}$,

(iii) $\Delta_\ell \varkappa_\ell^{(-r)} = -r\ell(\varkappa+\ell)_\ell^{-(r+1)}$,

and

(iv) $\Delta_\ell \varkappa_{-\ell}^{(-r)} = -r\ell(\varkappa+\ell)_{-\ell}^{(-(r+1))}$.

We use our n^{th} order Mittag-Leffler Function to solve certain type of linear differential equation.

Theorem 1. *For $\lambda \neq 0$, the function defined by $e_n(\lambda,\varkappa)$ given in (2.4) is considered as a solution of the $(n-1)^{th}$-linear non homogeneous DE*

$$\frac{1}{\lambda^{(n-1)}}\frac{d^{n-1}}{d\varkappa^{n-1}}u(\varkappa) + \frac{1}{\lambda^{(n-2)}}\frac{d^{n-2}}{d\varkappa^{n-2}}u(\varkappa) + \cdots + \frac{1}{\lambda}\frac{d}{d\varkappa}u(\varkappa) + u(\varkappa) = e^{\lambda\varkappa}.$$

Proof. From the Definition 4, we have

$$e_n(\lambda,\varkappa) = 1 + \frac{\lambda^n\varkappa^n}{(n)!} + \frac{\lambda^{2n}\varkappa^{2n}}{(2n)!} + \frac{\lambda^{3n}\varkappa^{3n}}{(3n)!} + \cdots + \infty.$$

Differentiating successively w.r.t $\varkappa$, we obtain

$$\frac{1}{\lambda}\frac{d}{d\varkappa}e_n(\lambda,\varkappa) = \frac{\lambda^{n-1}\varkappa^{n-1}}{(n-1)!} + \frac{\lambda^{2n-1}\varkappa^{2n-1}}{(2n-1)!} + \frac{\lambda^{3n-1}\varkappa^{3n-1}}{(3n-1)!} + \cdots + \infty,$$

$$\frac{1}{\lambda^2}\frac{d^2}{d\varkappa^2}e_n(\lambda,\varkappa)=\frac{\lambda^{n-2}\varkappa^{n-2}}{(n-2)!}+\frac{\lambda^{2n-2}\varkappa^{2n-2}}{(2n-2)!}+\frac{\lambda^{3n-2}\varkappa^{3n-2}}{(3n-2)!}\cdots+\infty.$$

In general,

$$\frac{1}{\lambda^r}\frac{d^r}{d\varkappa^r}e_n(\lambda,\varkappa)=\frac{\lambda^{n-r}\varkappa^{n-r}}{(n-r)!}+\frac{\lambda^{2n-r}\varkappa^{2n-r}}{(2n-r)!}+\frac{\lambda^{3n-r}\varkappa^{3n-r}}{(3n-r)!}\cdots+\infty. \tag{2.6}$$

Adding (2.6) for $r=0,1,2,3,\cdots,n-1, d^0=1$, we get

$$\sum_{r=0}^{n-1}\frac{1}{\lambda^r}\frac{d^r}{d\varkappa^r}e_n(\lambda,\varkappa)=e_1(\lambda,\varkappa),$$

and hence we get the proof by taking $u(\varkappa)=e_n(\lambda,\varkappa)$. □

For the succeeding theorem, we use the notation $\lim_{t\to\infty}\Delta_\ell^{-1}u(t)=\Delta_\ell^{-1}u(-\infty)$.

Theorem 2. *Let $\ell,\nu>0$ and $\varkappa\in\mathbb{R}$ and $u:\mathbb{R}\to\mathbb{R}$. If $\sum_{r=0}^{\infty}(r+\nu-1)$ is convergent, then we have*

$$\Delta_\ell^{-\nu}u(\varkappa)=\sum_{r=0}^{\infty}\frac{(r+\nu-1)_1^{(\nu-1)}}{\Gamma(\nu+1)}u(\varkappa-(r+\nu-1)\ell). \tag{2.7}$$

Proof. Since, $\Delta_\ell^{-1}u(\varkappa)=v(\varkappa)$, we have $u(\varkappa)=v(\varkappa+\ell)-v(\varkappa)$.
Replacing $\varkappa$ by $\varkappa-\ell$, it becomes

$$v(\varkappa)=u(\varkappa-\ell)+v(\varkappa-\ell). \tag{2.8}$$

Again replacing $\varkappa$ by $\varkappa-\ell, \varkappa-2\ell,\ \varkappa-3\ell,\ \cdots,\ \varkappa-(n-1)\ell$ in (2.8), we arrive

$$v(\varkappa)=u(\varkappa-\ell)+u(\varkappa-2\ell)+u(\varkappa-3\ell)+\cdots+u(\varkappa-n\ell)+v(\varkappa-n\ell).$$

Taking $\lim_{n\to\infty}$ in the above equation along with $v(-\infty)=0, u(-\infty)=0$, we obtain

$$\Delta_\ell^{-1}u(\varkappa)=\sum_{r=0}^{\infty}u(\varkappa-(r+1)\ell). \tag{2.9}$$

Taking Δ_ℓ^{-1} on both sides of (2.9), we obtain

$$\Delta_\ell^{-2}u(\varkappa)=\Delta_\ell^{-1}u(\varkappa-\ell)+\Delta_\ell^{-1}u(\varkappa-2\ell)+\Delta_\ell^{-1}u(\varkappa-3\ell)+\Delta_\ell^{-1}u(\varkappa-4\ell)+\cdots.$$

Expanding all the inverse function in above expression by the use of equation (2.9) and then arranging the terms, we obtain

$$\Delta_\ell^{-2}u(\varkappa)=u(\varkappa-2\ell)+2u(\varkappa-3\ell)+3u(\varkappa-4\ell)+4u(\varkappa-5\ell)+5u(\varkappa-6\ell)+\cdots,$$

which is similar to

$$\Delta_\ell^{-2}u(\varkappa)=\sum_{r=0}^{\infty}\frac{(r+1)^{(1)}}{1!}u(\varkappa-(r+2)\ell). \tag{2.10}$$

Proceeding like this, we get

$$\Delta_{\ell}^{-m}u(\varkappa)=\sum_{r=0}^{\infty}\frac{(r+m-1)_1^{(\nu-1)}}{(m-1)!}u(\varkappa-(r+m-1)\ell),m\in\mathbb{N},\ell\in\mathbb{R}. \tag{2.11}$$

For any $\nu>0$, we obtain the result by replacing m by ν. □

Theorem 3. *If* $\Delta_{\ell}^{-1}u(-\infty)=0$ *and* $\sum_{r=0}^{\infty}u(t+r\ell)$ *is convergent, then*

$$\Delta_{-\ell}^{-1}u(t)\Big|_t^{\infty}=\sum_{r=1}^{\infty}u(t+r\ell). \tag{2.12}$$

Proof. Replacing $\varkappa$ by t, ℓ by $-\ell$ in (2.9), we get the proof. □

Corollary 1. *Keeping* $\varkappa$ *as a constant, t as a variable, taking* Δ_{-1}^{-1} *with respect to t, then*

$$\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^{mt}}{T_{-1}^{(mt)}}\Big|_{t=0}=\frac{(\lambda\varkappa)^0}{T_{-1}^{(0)}}+\frac{(\lambda\varkappa)^{m.1}}{T_{-1}^{(m.1)}}+\frac{(\lambda\varkappa)^{m.2}}{T_{-1}^{(m.2)}}+\cdots,\textit{for } T_{-1}^{(mt)}\neq 0. \tag{2.13}$$

Proof. Taking $u(t)=\frac{(\lambda\varkappa)^{mt}}{(T)_{-1}^{(mt)}}$ gives the proof, where

$$T_{-1}^{(m.t)}=T(T+1)\cdots(T+(mt-1))=\sum_{r=0}^{mt-1}(T+r),$$

in (2.12). □

3. Application of fractional order Mittag-Leffler function in numerical methods

This section describes about the relation among Fractional order Mittag-Leffler Function, inverse of Δ_{-1} and raising factorial polynomials and some identities in numerical methods.

Theorem 4. *For* $mt\in\mathbb{N}$, *by denoting* $(s+1)_{-1}^{(mt)}=(s+1)(s+2)\cdots(s+mt)$, *we have the identity*

$$e_1(\lambda,\varkappa)-e_m(\lambda,\varkappa)=\sum_{s=1}^{m-1}\frac{(\lambda\varkappa)^s}{s!}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^{mt}}{(s+1)_{-1}^{(mt)}}\Big|_{t=0}. \tag{3.1}$$

Proof. From the Definition 3 and taking $n=1$, we have

$$e_1(\lambda,\varkappa)=1+\frac{\lambda\varkappa}{1!}+\frac{\lambda^2\varkappa^2}{2!}+\frac{\lambda^3\varkappa^3}{3!}+\frac{\lambda^4\varkappa^4}{4!}+\cdots+\infty.$$

Arranging the terms into m disjoint groups, since the series is absolutely convergent, we obtain

$$\begin{aligned} e_1(\lambda,\varkappa) &= 1+\frac{\lambda^m\varkappa^m}{m!}+\frac{\lambda^{2m}\varkappa^{2m}}{(2m)!}+\frac{\lambda^{3m}\varkappa^{3m}}{(3m)!}+\cdots+\infty \\ &+\frac{\lambda\varkappa}{1!}+\frac{\lambda^{m+1}\varkappa^{m+1}}{(m+1)!}+\frac{\lambda^{2m+1}\varkappa^{2m+1}}{(2m+1)!}+\cdots+\infty \\ &+\frac{\lambda^2\varkappa^2}{2!}+\frac{\lambda^{m+2}\varkappa^{m+2}}{(m+2)!}+\frac{\lambda^{2m+2}\varkappa^{2m+2}}{(2m+2)!}+\cdots+\infty \\ &+\cdots\quad\cdots\quad\cdots \\ &+\frac{\lambda^{m-1}\varkappa^{m-1}}{(m-1)!}+\frac{\lambda^{2m-1}\varkappa^{2m-1}}{(2m-1)!}+\frac{\lambda^{3m-1}\varkappa^{3m-1}}{(3m-1)!}+\cdots+\infty, \end{aligned}$$

which is the same as, by taking $n=m$ in (2.4),

$$\begin{aligned} e_1(\lambda,\varkappa) = e_m(\lambda,\varkappa) &+ \frac{\lambda\varkappa}{1!}\left[1+\frac{(\lambda\varkappa)^{m.1}}{2_{-1}^{(m)}}+\frac{(\lambda\varkappa)^{m.2}}{2_{-1}^{(m.2)}}+\cdots\right] \\ &+\frac{(\lambda\varkappa)^2}{2!}\left[1+\frac{(\lambda\varkappa)^{m.1}}{3_{-1}^{(m)}}+\frac{(\lambda\varkappa)^{m.2}}{3_{-1}^{(m.2)}}+\cdots\right] \\ &+\frac{(\lambda\varkappa)^3}{3!}\left[1+\frac{(\lambda\varkappa)^{m.1}}{4_{-1}^{(m)}}+\frac{(\lambda\varkappa)^{m.2}}{4_{-1}^{(m.2)}}+\cdots\right]. \end{aligned} \tag{3.2}$$

Hence to obtain (3.1), apply (2.13) in (3.2), and this ends the proof. □

Theorem 5. *Suppose that* $\nu m = 1$*, where* $m \in N$*. Then*

$$e_\nu(\lambda,\varkappa) - e_1(\lambda,\varkappa) = \sum_{s=1}^{m-1}\frac{(\lambda\varkappa)^{\nu s}}{\Gamma(1+\nu s)}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^{\nu t}}{(1+\nu s)_{-1}^{(\nu t)}}\Big|t=0. \tag{3.3}$$

Proof. By rearranging into four disjoint groups in Theorem 4, we get (3.3). □

The following example is for the demonstration of Theorem 5.

Example 1. From (2.4), by taking $\nu = 0.25, m = 4$ we have,

$$e_{0.25}(\lambda,\varkappa) = 1+\frac{\lambda^{0.25}\varkappa^{0.25}}{\Gamma(1.25)}+\frac{\lambda^{0.5}\varkappa^{0.5}}{\Gamma(1.5)}+\frac{\lambda^{0.75}\varkappa^{0.75}}{\Gamma(1.75)}+\frac{\lambda^{1}\varkappa^{1}}{\Gamma(2)}+\cdots+\infty.$$

$$e_{0.25}(\lambda,\varkappa) = \frac{\lambda^{0.25}\varkappa^{0.25}}{\Gamma 1.25}+\frac{\lambda^{1.25}\varkappa^{1.25}}{\Gamma 2.25}+\frac{\lambda^{2.25}\varkappa^{2.25}}{\Gamma 3.25}+\cdots+\infty$$

$$+\frac{\lambda^{0.5}\varkappa^{0.5}}{\Gamma 1.5}+\frac{\lambda^{1.5}\varkappa^{1.5}}{\Gamma 2.5}+\frac{\lambda^{3.5}\varkappa^{3.5}}{\Gamma 3.5}+\cdots+\infty$$
$$+\frac{\lambda^{0.75}\varkappa^{0.75}}{\Gamma 1.75}+\frac{\lambda^{1.75}\varkappa^{1.75}}{\Gamma 2.75}+\frac{\lambda^{2.75}\varkappa^{2.75}}{\Gamma 3.75}+\cdots+\infty$$
$$+\frac{\lambda^{0}\varkappa^{0}}{\Gamma 1}+\frac{\lambda^{1}\varkappa^{1}}{\Gamma 2}+\frac{\lambda^{2}\varkappa^{2}}{\Gamma 3}+\cdots+\infty$$

which is the same as

$$\begin{aligned} e_{0.25}(\lambda,\varkappa) &= \frac{\lambda^{0.25}\varkappa^{0.25}}{\Gamma 1.25}\left[1+\frac{\lambda\varkappa}{(1.25)}+\frac{\lambda^2\varkappa^2}{(1.25)(2.25)}+\cdots\right] \\ &\quad+\frac{\lambda^{0.5}\varkappa^{0.5}}{\Gamma 1.5}\left[1+\frac{\lambda\varkappa}{(1.5)}+\frac{\lambda^2\varkappa^2}{(1.5)(2.5)}+\cdots\right] \\ &\quad+\frac{\lambda^{0.75}\varkappa^{0.75}}{\Gamma 1.75}\left[1+\frac{\lambda\varkappa}{(1.75)}+\frac{\lambda^2\varkappa^2}{(1.75)(2.75)}+\cdots\right] \\ &\quad+\frac{\lambda^{0}\varkappa^{0}}{\Gamma 1}\left[1+\frac{\lambda\varkappa}{(1)}+\frac{\lambda^2\varkappa^2}{(1)(2)}+\frac{\lambda^3\varkappa^3}{(1)(2)(3)}+\cdots\right] \end{aligned}$$

From the special cases of Definition 2 and taking Δ_{-1}^{-1} with respect to t, we get

$$\begin{aligned} e_{0.25}(\lambda,\varkappa) &= \frac{\lambda^{0.25}\varkappa^{0.25}}{\Gamma 1.25}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1.25)_{-1}^{(t)}}\ |_{t=0}+\frac{\lambda^{0.5}\varkappa^{0.5}}{\Gamma 1.5}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1.5)_{-1}^{(t)}}\ |_{t=0} \\ &\quad+\frac{\lambda^{0.75}\varkappa^{0.75}}{\Gamma 1.75}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1.75)_{-1}^{(t)}}\ |_{t=0}+\frac{\lambda^{0}\varkappa^{0}}{\Gamma 1}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1)_{-1}^{(t)}}\ |_{t=0} \end{aligned}$$

which can be expressed as

$$e_{0.25}(\lambda,\varkappa)-e_1(\lambda,\varkappa)=\sum_{s=1}^{3}+\frac{(\lambda\varkappa)^{(0.25)s}}{\Gamma 1+s(0.25)}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1+0.25s)_{-1}^{(t)}}\ |_{t=0}\,.$$

Theorem 6. *By defining* $s_1(\lambda,\varkappa)=\frac{\lambda\varkappa}{1!}+\frac{(\lambda\varkappa)^3}{3!}+\frac{(\lambda\varkappa)^5}{5!}+\cdots$ *and* $c_1(\lambda,\varkappa)=1+\frac{(\lambda\varkappa)^2}{2!}+\frac{(\lambda\varkappa)^4}{4!}+\cdots$, *we get the following identities*

(i) $e_1(\lambda,\varkappa)=c_1(\lambda,\varkappa)+s_1(\lambda,\varkappa)$.

(ii) $\frac{d}{d\varkappa}e_2(\lambda,\varkappa)=\lambda s_1(\lambda,\varkappa)$.

(iii) $\frac{d^2}{d\varkappa^2}e_2(\lambda,\varkappa)=\lambda^2 c_1(\lambda,\varkappa)$.

Proof. Keeping λ as a constant and taking differentiation with respect to $'\varkappa'$ termwise,

$$\frac{d}{d\varkappa}e_2(\lambda,\varkappa)=\lambda\Big[\frac{\lambda\varkappa}{1!}+\frac{(\lambda\varkappa)^3}{3!}+\frac{(\lambda\varkappa)^5}{5!}+\cdots\Big]=\lambda s_1(\lambda,\varkappa).$$

Again taking derivative with respect to $\varkappa$, we get,

$$\frac{d^2}{d\varkappa^2}e_2(\lambda,\varkappa) = \lambda\Big[\lambda + \frac{\lambda^3\varkappa^2}{2!} + \frac{\lambda^5\varkappa^4}{4!} + \cdots\Big] = \lambda^2 c_1(\lambda,\varkappa).$$

□

4. Fractional order Mittag-Leffler factorial function

Here, we extend the Fractional order Mittag-Leffler Function into Fractional order Mittag-Leffler Factorial Function by replacing the polynomials into factorial polynomials in the Fractional order Mittag-Leffler Function.

Definition 5. [16] For $|\lambda|, |\ell| < 1$ and $c \in [0,1]$, if $\varkappa - (a+\ell) + cj\ell\nu$ is defined for $j\nu + 1 \notin N(0)$, $\varkappa \in \mathbb{R}$, then the extended Mittag-Leffler factorial (EMLF) function is defined as

$$e_\nu(\lambda, \varkappa_\ell, c) = \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j\nu+1)} \Big(\varkappa - (a+\ell) + cj\ell\nu\Big)_\ell^{(j\nu)}. \tag{4.1}$$

$e_\nu(\lambda, \varkappa_{-\ell}, 1)$ is defined for $|\lambda| \geq 1$, $\ell > 0$ whenever $\varkappa - (a+\ell) + cj\ell\nu$ is positive and a multiple of ℓ.

Also, we have two special cases in the following form:

(i) $e_1(1, \varkappa_0, c) = e^{\varkappa}$ if $\ell = 0$, $\lambda = \nu = 1$, $a = -1$.

(i') $e_\nu(1, \varkappa_\ell, 0) = 1 + \dfrac{\varkappa_\ell^{(1\nu)}}{\Gamma(1\nu+1)} + \dfrac{\varkappa_\ell^{(2\nu)}}{\Gamma(2\nu+1)} + \dfrac{\varkappa_\ell^{(3\nu)}}{\Gamma(3\nu+1)} + \cdots + \dfrac{\varkappa_\ell^{(n\nu)}}{\Gamma(n\nu+1)} + \cdots$.
(It is the extorial function).

(i") $e_\nu(\lambda, \varkappa_0, 1)$ is the same Mittag-Leffler factorial function.

Theorem 7. *The extorial function illustrated by* $e_\nu(1, \varkappa_\ell, 0)$ *(special case of Mittag-Leffler Factorial Function) is convergent if* $|\ell| < 1$.

Proof. In view of the definition of extorial function, we have

$$e(1, \varkappa_\ell, 0) = 1 + \frac{\varkappa_\ell^{(1)}}{1!} + \frac{\varkappa_\ell^{(2)}}{2!} + \frac{\varkappa_\ell^{(3)}}{3!} + \cdots + \frac{\varkappa_\ell^{(n)}}{n!} + \cdots.$$

Consider the term $a_n = \dfrac{\varkappa_\ell^{(n)}}{n!} = \dfrac{\varkappa(\varkappa-\ell)\cdots(\varkappa-(n-1)\ell)}{n!}$. Then

$$|a_n| \leq \frac{|\varkappa|(|\varkappa|+|\ell|)(|\varkappa|+2|\ell|)\cdots(|\varkappa|+(n-1)|\ell|)}{n!}.$$

Since $|\ell| < 1$, choose N such that $|\varkappa|(|\varkappa|+|\ell|)\cdots(|\varkappa|+(n-1)|\ell|) < N!$ and hence

$$|a_N| \leq \frac{|\varkappa|(|\varkappa|+|\ell|)\cdots(|\varkappa|+(n-1)|\ell|)}{N!} < \rho_N < 1,$$

which is possible, since in between two real numbers, there exists another real number.

Since $\rho_N > \rho_{N+1} > \cdots$, $\lim_{n\to\infty} \sqrt[n]{|a_n|} < \sqrt[N]{\rho_N} < 1$, thus, by root test, $e(\varkappa_\ell)$ is convergent. □

Theorem 8. *For rational $\varkappa > 0$ and $|\lambda\ell| < 1$, we have $(1+\lambda\ell)^{\varkappa} = e_1(\lambda,(\ell\varkappa)_\ell,0)$, where $e_1(\lambda,(\ell\varkappa)_\ell,0)$ is the Mittag-Leffler Factorial Function (MLFF).*

Proof. From the binomial expansion for rational index,

$$(1+\lambda\ell)^{\varkappa} = 1 + \frac{(\lambda\ell)\varkappa}{1!} + \frac{(\lambda\ell)^2\varkappa(\varkappa-1)}{2!} + \frac{(\lambda\ell)^3\varkappa(\varkappa-1)(\varkappa-2)}{3!} + \cdots,$$

which can be expressed as

$$(1+\lambda\ell)^{\varkappa} = 1 + \lambda\frac{(\ell\varkappa)}{1!} + \lambda^2\frac{(\ell\varkappa)(\ell\varkappa-\ell)}{2!} + \lambda^3\frac{(\ell\varkappa)(\ell\varkappa-\ell)(\ell\varkappa-2\ell)}{3!} + \cdots,$$

and we get the identity from the definition of $e_1(\lambda,(\ell\varkappa)_\ell,0)$. □

Theorem 9. *For positive integer m and for $\varkappa > 0$, we have the identity*

$$\Delta_\ell^{-m} e_1(\lambda,(\ell\varkappa)_\ell,0) = (\lambda\ell)^m e_1(\lambda,(\ell\varkappa)_\ell,0), \textit{if } (\lambda\ell)^{-1} = 0. \tag{4.2}$$

Proof. By Theorem 8, $e_1(\lambda,(\ell\varkappa)_\ell,0) = (1+\lambda\ell)^{\varkappa}$. By taking Δ_ℓ on both sides, we find

$$\Delta_\ell e_1(\lambda,(\ell\varkappa)_\ell,0) = (1+\lambda\ell)^{\varkappa+1} - (1+\lambda\ell)^{\varkappa} = (1+\lambda\ell)^{\varkappa}(1+\lambda\ell-1),$$

which yields

$$\Delta_\ell^{-1}(1+\ell)^{\varkappa} = \frac{e_1(\lambda,(\ell\varkappa)_\ell,0)}{\lambda\ell}.$$

By Definition 1 of inverse difference operator, we have

$$\Delta_\ell^{-1}(e_1(\lambda,(\ell\varkappa)_\ell,0)) = (\lambda\ell)^{-1}e_1(\lambda,(\ell\varkappa)_\ell,0),$$
$$\Delta_\ell^{-2}(e_1(\lambda,(\ell\varkappa)_\ell,0)) = (\lambda\ell)^{-2}e_1(\lambda,(\ell\varkappa)_\ell,0).$$

In general, by induction on m, $\Delta_\ell^{-m}e_1(\lambda,(\ell\varkappa)_\ell,0) = (\lambda\ell)^{-m}e_1(\lambda,(\ell\varkappa)_\ell,0)$.

Now (4.2) is followed by replacing m by $-\nu$ and the proof is ended. □

Corollary 2. *For any $\nu > 0$ and $|\ell| < 1$, if we take $\varkappa = \frac{m}{\ell}$ in (4.2), we have*

$$\Delta_\ell^{-\nu} e_1(\lambda,(\varkappa)_\ell,0) = (\lambda\ell)^{-\nu} e_1(\lambda,(\varkappa)_\ell,0). \tag{4.3}$$

Proof. Replacing $\varkappa$ by $\frac{m}{\ell}$ in (4.2), then relabeling m as $\varkappa$, we get (4.3). □

To illustrate Corollary 2, we give the following example.

Example 2. Consider

$$\Delta_\ell^{-\nu} e_1(\lambda,(\ell\varkappa)_\ell,0) = (\lambda\ell)^{-\nu} e_1(\lambda,(\ell\varkappa)_\ell,0). \tag{4.4}$$

Taking $\ell = 0.2, \nu = 0.5, a = -\ell, c = 0, \varkappa = 20, \lambda = 2, \nu = 1$ and $u(\varkappa) = e(\lambda, \varkappa_\ell, 0)$ in (4.4) and by using Theorem 2 for $\nu = 1$, we get

$$\sum_{r=0}^{\infty} e_1(\lambda, (\ell(\varkappa - r\ell))_\ell, 0) = (\lambda\ell)^{-1} e_1(\lambda, (\ell\varkappa)_\ell, 0),$$

where the generalized Mittag-Leffler factorial function $e_\nu(\lambda, \varkappa_\ell, 0)$ is given in (4.1) and so

$$\begin{aligned}\sum_{r=0}^{\infty} e_1(\lambda, (\ell(\varkappa - r\ell))_\ell, 0) &= \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - \ell)_{0.2}^{(j)}}{j!} + \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 2\ell)_{0.2}^{(j)}}{j!} \\ &+ \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 3\ell)_{0.2}^{(j)}}{j!} + \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 4\ell)_{0.2}^{(j)}}{j!} + \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 5\ell)_{0.2}^{(j)}}{j!} \\ &+ \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 6\ell)_{0.2}^{(j)}}{j!} + \cdots \\ &= 597.61 + 426.82 + 304.79 + 217.75 + 155.55 + 111.10 \\ &\quad + 79.33 + 56.62 + 40.48 + 28.93 + 20.06 + 14.73 + 10.55 + 7.52 \\ &\quad + 5.38 + 3.85 + 2.74 + 1.96 + 1.4 + 1 + 1 + \cdots = 2089.16,\end{aligned}$$

and

$$\begin{aligned}(\lambda\ell)^{-1} e_1(\lambda, (\ell\varkappa)_\ell, 0) &= \frac{1}{0.4}\Big[\sum_{j=0}^{\infty} \frac{(\ell\varkappa)_\ell^{(j)}}{j!}\Big] = 1 + 8 + 30.40 + 72.96 \\ &+ 124.03 + 158.76 + 158.76 + 127.01 + 82.56 + 44.03 \\ &+ 19.37 + 7.04 + 2.11 + 0.52 + \cdots = \frac{1}{0.4}(836.55) = 2091.38.\end{aligned}$$

Corollary 3. *If* $|\ell| < 1$, *then* $\Delta_\ell^{-\nu}(1+\lambda\ell)^{\frac{\varkappa}{\ell}} = (\lambda\ell)^{-\nu}(1+\lambda\ell)^{\frac{\varkappa}{\ell}} = \Delta_\ell^{-\nu} e_1(\lambda, \varkappa_\ell, 0)$.

Proof. We know that

$$e_1(\lambda, \varkappa_\ell, 0) = 1 + \frac{\lambda\varkappa}{1!} + \frac{\lambda^2 \varkappa(\varkappa - \ell)}{2!} + \frac{\lambda^3 \varkappa(\varkappa - \ell)(\varkappa - 2\ell)}{3!} + \cdots.$$

Replacing k by $m\ell$, we obtain

$$\begin{aligned}e_1(\lambda, m\ell_\ell, 0) &= 1 + \frac{\lambda m\ell}{1!} + \frac{\lambda^2 m\ell(m\ell - \ell)}{2!} + \frac{\lambda^3 m\ell(m\ell - \ell)(m\ell - 2\ell)}{3!} + \cdots \\ &= 1 + \frac{\lambda m\ell}{1!} + \frac{\lambda^2 m(m-1)}{2!}\ell^2 + \frac{\lambda^3 m(m-1)(m-2)}{3!}\ell^3 + \cdots \\ &= (1+\lambda\ell)^m = (1+\lambda\ell)^{\frac{\varkappa}{\ell}}.\end{aligned}$$

The proof is concluded by taking $\Delta_\ell^{-\nu}$ on both sides and applying (4.3). □

5. APPLICATION OF THE MITTAG-LEFFLER FACTORIAL FUNCTION

This section, revolves around the solution of a specific kind of higher–order linear non-homogeneous difference equation.

Theorem 10. *For a positive integer n, the Mittag-Leffler Factorial Function* $e_n(\lambda,\varkappa_\ell,0)$ *satisfies the* $(n-1)^{th}$*-linear non homogeneous difference equation*

$$\Delta_\ell^{n-1}\frac{u(\varkappa)}{\ell^{n-1}}+\Delta_\ell^{n-2}\frac{u(\varkappa)}{\ell^{n-2}}+\cdots+\Delta_\ell\frac{u(\varkappa)}{\ell}+u(\varkappa)=e_1(\varkappa_\ell)=(1+\lambda\ell)^{\frac{\varkappa}{\ell}}. \tag{5.1}$$

Proof. From the special case of Definition 5,

$$e_n(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^n\varkappa_\ell^{(n)}}{n!}+\frac{\lambda^{2n}\varkappa_\ell^{(2n)}}{2n!}+\frac{\lambda^{3n}\varkappa_\ell^{(3n)}}{3n!}+\cdots+\infty.$$

Since $\Delta_\ell\varkappa_\ell^{(r)}=r\ell\varkappa_\ell^{r-1}$, we have

$$\frac{1}{\lambda\ell}\Delta_\ell^1 e_n(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^{n-1}\ell\varkappa_\ell^{(n-1)}}{(n-1)!}+\frac{\lambda^{2n-1}\ell\varkappa_\ell^{(2n-1)}}{(2n-1)!}+\frac{\lambda^{3n-1}\ell\varkappa_\ell^{(3n-1)}}{(3n-1)!}+\cdots+\infty,$$

$$\frac{1}{(\lambda\ell)^2}\Delta_\ell^2 e_n(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^{n-2}\ell^2\varkappa_\ell^{(n-2)}}{(n-2)!}+\frac{\lambda^{2n-2}\ell^2\varkappa_\ell^{(2n-2)}}{(2n-2)!}+\frac{\lambda^{3n-2}\ell^2\varkappa_\ell^{(3n-2)}}{(3n-2)!}+\cdots+\infty,$$

and in general, we find

$$\frac{1}{(\lambda\ell)^r}\Delta_\ell^r e_n(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^{n-r}\ell^r\varkappa_\ell^{(n-r)}}{(n-r)!}+\frac{\lambda^{2n-r}\ell^r\varkappa_\ell^{(2n-r)}}{(2n-r)!}+\frac{\lambda^{3n-r}\ell^r\varkappa_\ell^{(3n-r)}}{(3n-r)!}+\cdots+\infty. \tag{5.2}$$

Adding (5.2) for $r=0,1,2,3,\cdots,n-1,\Delta^0=1$, we get

$$\sum_{r=0}^{n-1}\frac{1}{(\lambda\ell)^r}\Delta_\ell^r e_n(\lambda,\varkappa_\ell,0)=e_1(\lambda,\varkappa_\ell,0),$$

and the proof follows $u(\varkappa)=e_n(\lambda,\varkappa_\ell,0)$. □

Example 3. For $n=2$, the equation (5.1) becomes the extorial function $e_2(\lambda,\varkappa_\ell,0)$ which satisfies the difference equation $\frac{1}{\lambda\ell}\Delta_\ell u(\varkappa)+u(\varkappa)=(1+\lambda\ell)^{\frac{\varkappa}{\ell}},|\ell\lambda|<1.$

Proof. By taking Δ_ℓ on $e_2(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^2\varkappa_\ell^{(2)}}{2!}+\frac{\lambda^4\varkappa_\ell^{(4)}}{4!}+\frac{\lambda^6\varkappa_\ell^{(6)}}{6!}+\cdots$ and applying

$$\Delta_\ell\varkappa_\ell^{(m)}=n\ell\varkappa_\ell^{(n-1)},$$

we get

$$\Delta_\ell e_2(\lambda,\varkappa_\ell,0)=\frac{\lambda^2 2\ell\varkappa_\ell^{(1)}}{1!}+\frac{\lambda^4 4\ell\varkappa_\ell^{(3)}}{4!}+\cdots,$$

which is the same as

$$\frac{1}{\lambda\ell}\Delta_{\ell}e_2(\lambda,\varkappa_{\ell},0)=\frac{\lambda\varkappa_{\ell}^{(1)}}{1!}+\frac{\lambda^3\varkappa_{\ell}^{(3)}}{3!}+\cdots,$$

and yields that $e_2(\lambda,\varkappa_{\ell},0)+\frac{1}{\lambda\ell}\Delta_{\ell}e_2(\lambda,\varkappa_{\ell},0)=e_1(\lambda,\varkappa_{\ell},0)=(1+\lambda\ell)^{\frac{\varkappa}{\ell}}$. □

6. Conclusion

Through this research, we derived a special kind of Mittag-Leffler function and Mittag-Leffler Factorial Functions. These functions are applied to find solution of higher order linear differential and difference equations. These solutions will generate a large number of relations in the vast area of fractional calculus.

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