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COINCIDENCE POINT RESULTS ON A METRIC SPACE ENDOWED WITH A LOCALLY T-TRANSITIVE BINARY RELATION EMPLOYING COMPARISON FUNCTIONS

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Abstract. In this paper, we prove coincidence point results for a pair of self mappings defined on a metric space employing a comparison function and a locally T-transitive binary relation. Our results extend and generalize several known results especially those are contained in Alam and Imdad [Filomat 31 (14) (2017) 4421-4439] and Arif *et al*. [Miskolc Math. Notes 23 (1) (2022) 71-83]. Finally, we construct some examples to demonstrate the accomplished improvements in our newly proved results in this paper.

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1. Introduction and Preliminaries

Banach contraction principle [or, in short BCP] (cf. [7]) is a pioneer and noted result of metric fixed point theory. In the course of last several years, numerous researchers extended this result by weakening the contraction conditions besides enlarging the class of underlying metric spaces. In the existing literature, one of the weaker form of contraction conditions is nonlinear contraction. A self-mapping φ defined on $[0,\infty)$ satisfying $\varphi(s) < s$ for each $s > 0$ is control function. A self-mapping T defined on a metric space (X,d) is said to be a nonlinear contraction with respect to control function φ (or, in short, φ-contraction) if $d(Tx,Ty) \leq \varphi(d(x,y))$ for all $x,y \in X$. In 1968, Browder [10] established a nonlinear version of BCP, wherein author assumed φ to be increasing control function and right continuous. Thereafter, numerous mathematicians generalized the BCP by utilizing different types of control functions such as: Boyd-Wong [9] and Matkowski [16] contractions. In 1986, Turinici [24] discovered the idea of order-theoretic fixed points results. In 2004, Ran and Reurings [20] reformulated a comparatively more natural order-theoretic version of classical BCP. There exists a vast literature on relation-theoretic results on fixed, coincidence and common fixed point (e.g., Ran and Reurings [20] and Nieto and Rodríguez-López [18], Agarwal et al. [1], Alam et al. [2], O'Regan and Petruşel [19], Turinici [25], Ghods et al. [11], Ben-El-Mechaiekh [8], etc). Later, Alam and Imdad [3,4] established relation-theoretic analogues of BCP employing amorphous relation which in turn unify the several well known relevant order-theoretic fixed point theorems. Recently, Arif et al. [6] proved a Matkowski type nonlinear fixed point result using a locally T-transitive binary relation.

Our aim in this article is to establish some coincidence and common fixed point results for Matkowski type nonlinear contractions on a metric space endowed with locally T-transitive binary relation. Newly proved results generalize and extend the main results of Arif et al. [6], Alam and Imdad [4] and several others. Moreover, a corollary to one of our main results remains a sharpened version of a corresponding earlier known result of the existing literature. In order to demonstrate the validity of the hypotheses and degree of generality of our results, we also furnish some examples.

2. Preliminaries

For the sake of simplicity, we collect some basic notions and propositions for our subsequent discussion.

Definition 1 ([12,14]). Let (T,g) be a pair of self-mappings defined on a nonempty set X. Then

- (i) a point $x \in X$ is said to be a coincidence point of the pair (T,g) if $Tx = gx$,
- (ii) a point $y \in X$ is said to be a point of coincidence of the pair (T,g) if there exists $x \in X$ such that $y = Tx = gx$,
- (iii) a coincidence point $x \in X$ of the pair (T,g) is said to be a common fixed point if $x = Tx = gx$,
- (iv) a pair (T,g) is called commuting if $T(gx) = g(Tx), \forall\, x \in X$,
- (v) a pair (T,g) is called weakly compatible if T and g commutes at their coincidence point.

Definition 2 ([13,22,23]). Let (T,g) be a pair of self-mappings defined on a metric space (X,d). Then

- (i) (T,g) is said to be weakly commuting if for all $x \in X$, $d(T(gx),g(Tx)) \leq d(Tx,gx)$,
- (ii) (T,g) is said to be compatible if $\lim_{n\to\infty} d(T(gx_n),g(Tx_n)) = 0$ whenever $\{x_n\} \subset X$ is a sequence such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} Tx_n$,
- (iii) T is said to be g-continuous at $x \in X$ if $gx_n \xrightarrow{d} gx$, for all sequence $\{x_n\} \subset X$, we have $Tx_n \xrightarrow{d} Tx$. Moreover, T is said to be g-continuous if it is continuous at every point of X.

Definition 3 ([15])**.** A subset $\mathcal{R}$ of $X \times X$ is called a binary relation on X. We say that "x relates y under $\mathcal{R}$" if and only if $(x,y) \in \mathcal{R}$.

Throughout this paper, $\mathcal{R}$ stands for a 'non-empty binary relation' (i.e., $\mathcal{R} \neq \varnothing$) instead of 'binary relation' while $\mathbb{N}$, the set of natural numbers (also, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$, (where $\mathcal{R}^{-1} := \{(y,x) \in X^2 : (x,y) \in \mathcal{R}\}$) and $\mathcal{R}|_Z := \mathcal{R} \cap Z^2$, $Z \subseteq X$. Indeed, $\mathcal{R}|_Z$ is a relation on Z induced by $\mathcal{R}$. Although, $\mathcal{R} = \varnothing$ denotes the null relation, while $\mathcal{R} = X^2$ the universal relation.

Definition 4. [5] Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined on X. Then $\mathcal{R}$ is called "T-transitive" if for any $x,y,z \in X$,

$$(Tx,Ty),(Ty,Tz) \in \mathcal{R} \Rightarrow (Tx,Tz) \in \mathcal{R}.$$

Definition 5 ([3])**.** Let X be a non-empty set X endowed with a binary relation $\mathcal{R}$. Then a sequence $\{x_n\} \subset X$ is said to be $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}, \ \forall\, n \in \mathbb{N}_0$.

Definition 6 ([5])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is called "locally transitive" if for each (effectively) $\mathcal{R}$-preserving sequence $\{x_n\} \subset X$ (with range $D = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_D$ is transitive.

Definition 7 ([5])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined X. Then $\mathcal{R}$ is called locally T-transitive if for each (effectively) $\mathcal{R}$-preserving sequence $\{x_n\} \subset T(X)$ (with range $D = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_D$ is transitive.

Proposition 1. *Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let (T,g) be a pair of self-mappings defined on X. Then*

(i) *$\mathcal{R}$ is T-transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is transitive,*
(ii) *$\mathcal{R}$ is locally T-transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is locally transitive,*
(iii) *$\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is locally transitive $\Rightarrow \mathcal{R}$ is locally T-transitive,*
(iv) *$\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is T-transitive $\Rightarrow \mathcal{R}$ is locally T-transitive,*
(iv) *if $T(X) \subseteq g(X)$, then g-transitivity of $\mathcal{R} \Rightarrow \mathcal{R}$ is T-transitive and locally g-transitivity of $\mathcal{R} \Rightarrow \mathcal{R}$ is locally T-transitive but not conversely.*

Definition 8 ([15])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is called complete if for all x,y in X, either $(x,y) \in \mathcal{R}$ or $(y,x) \in \mathcal{R}$ which is denoted by $[x,y] \in \mathcal{R}$.

Proposition 2 ([3])**.** *Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then $(x,y) \in \mathcal{R}^s$ if and only if $[x,y] \in \mathcal{R}$.*

Definition 9 ([3])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined X. Then $\mathcal{R}$ is called T-closed if for all $x,y \in X$, $(x,y) \in \mathcal{R} \Rightarrow (Tx,Ty) \in \mathcal{R}$.

Definition 10 ([4])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let (T,g) be a pair of self-mappings defined on X. Then $\mathcal{R}$ is called (T,g)-closed if for all $x,y \in X$, $(gx,gy) \in \mathcal{R} \Rightarrow (Tx,Ty) \in \mathcal{R}$.

Definition 11 ([4])**.** Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$. Then (X,d) is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges to a point in X.

Remark 1 ([4]). Every complete metric space is $\mathcal{R}$-complete, where $\mathcal{R}$ denotes a binary relation. Moreover, if $\mathcal{R} = X^2$, then notions of $\mathcal{R}$-completeness and completeness are the same.

Definition 12 ([4])**.** Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined on X. Then T is said to be $\mathcal{R}$-continuous at x if $x_n \xrightarrow{d} x$, for any $\mathcal{R}$-preserving sequence $\{x_n\} \subset X$, we have $Tx_n \xrightarrow{d} Tx$. Moreover, T is said to be $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at every point of X.

Definition 13 ([4])**.** Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ and let (T,g) be a pair of self-mappings defined on X. Then T is said to be $(g,\mathcal{R})$-continuous at x if $gx_n \xrightarrow{d} gx$, for any $\mathcal{R}$-preserving sequence $\{gx_n\} \subset X$, we have $Tx_n \xrightarrow{d} Tx$. Moreover, T is called $(g,\mathcal{R})$-continuous if it is $(g,\mathcal{R})$-continuous at every point of X.

Remark 2. Every continuous mapping is $\mathcal{R}$-continuous, where $\mathcal{R}$ denotes a binary relation. Moreover, if $\mathcal{R} = X^2$, then notions of continuity and $\mathcal{R}$-continuity are the same.

Definition 14 ([3])**.** Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is said to be d-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N}$.

Definition 15 ([4])**.** Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ an let g be a self-mapping defined on X. Then $\mathcal{R}$ is said to be (g,d)-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[gx_{n_k}, gx] \in \mathcal{R}$, for all $k \in \mathbb{N}$.

Definition 16 ([21])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then a subset D of X is said to be $\mathcal{R}$-directed if for every pair of points x,y in D, there is w in X such that $(x,w) \in \mathcal{R}$ and $(y,w) \in \mathcal{R}$.

Definition 17 ([4])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let g be a self-mapping defined on X. Then a subset D of X is said to be $(g,\mathcal{R})$-directed if all x,y in D, there is 'w' in X such that $(x,gw) \in \mathcal{R}$ and $(y,gw) \in \mathcal{R}$.

Remark 3. On setting $g = I$, (the identity mapping on X) in Definitions 10, 13, 15 and 17 reduces to Definitions 9, 12, 14 and 16, respectively.

Definition 18 ([15])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Given $x, y \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{w_0, w_1, w_2, \ldots, w_k\} \subset X$ satisfying the following:

(i) $w_0 = x$ and $w_k = y$,

(ii) $(w_i, w_{i+1}) \in \mathcal{R}$ for each i $(0 \leq i \leq k-1)$.

Moreover, a path of length k involves $k+1$ points of X, although they may or may not be distinct.

Definition 19 ([4])**.** Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then a subset D of X is called $\mathcal{R}$-connected if for each pair $x, y \in D$, there exists a path in $\mathcal{R}$ from x to y.

Definition 20 ([4])**.** Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and let (T, g) be a pair of self-mappings defined on X. Then (T, g) is said to be $\mathcal{R}$-compatible if $\lim_{n\to\infty} d(g(Tx_n), T(gx_n)) = 0$, whenever $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} T(x_n)$, for any sequence $\{x_n\} \subset X$ such that $\{Tx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$-preserving.

For a given non-empty set X together with a binary relation $\mathcal{R}$ on X and a pair of self-mappings (T, g) on X, we use the following notations:

- $X(T, g, \mathcal{R}) := \{x \in X : (gx, Tx) \in \mathcal{R}\}$;
- $C(T, g)$: the collection of all coincidence points of (T, g);
- $\overline{C}(T, g)$: the collection of all points of coincidence of (T, g);
- $F(T, g)$: the collection of all common fixed points of (T, g);
- $F(T)$: the collection of all fixed points of T;
- $X(T, \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$.

In fact, the idea of a comparison function was initiated by Matkowski [16] in 1975, we denote the set of all comparison functions by Φ. A mapping $\varphi\colon [0, \infty) \to [0, \infty)$ is said to be a comparison function if φ satisfies the following properties:

(Φ_1) φ is increasing;

(Φ_2) $\lim_{n\to\infty} \varphi^n(s) = 0$ for each $s > 0$, where, for each n, φ^n is the n-th iterate of φ.

Now, we propose the two main properties of the comparison function:

Proposition 3 ([17])**.** *If φ is a comparison function, then $\varphi(s) < s$, for each $s > 0$.*

Proof. On contrary suppose that exists $s_0 > 0$ such that $s_0 \leq \varphi(s_0)$. Since φ is increasing, $\phi(s_0) \leq \varphi^2(s_0)$, it follows that $s_0 \leq \varphi(s_0) \leq \varphi^2(s_0)$. Thus, inductively for all $n \in \mathbb{N}$, we obtain $s_0 \leq \varphi^n(s_0)$ which on letting $n \to \infty$, gives rise, $s_0 \leq 0$, which is a contradiction. Hence $\varphi(s) < s$. $\square$

Proposition 4 ([17])**.** *If φ is a comparison function, then $\varphi(0) = 0$.*

Proof. On contrary suppose that $\varphi(0) = s$ for some $s > 0$. Since $0 < s$ and φ is increasing, $\varphi(0) \leq \varphi(s)$, it follows that $s \leq \varphi(s) < s$, which is contradiction, hence $\varphi(0) = 0$. $\square$

Proposition 5. *Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ and let (T,g) be a pair of self-mappings defined on X and also let $\varphi \in \Phi$. Then the following conditions are equivalent:*

(I) $d(Tx,Ty) \leq \varphi(d(gx,gy))$ *with* $(gx,gy) \in \mathcal{R}$;
(II) $d(Tx,Ty) \leq \varphi(d(gx,gy))$ *with* $[gx,gy] \in \mathcal{R}$.

Proof. Obviously $(II) \Rightarrow (I)$. We claim that $(I) \Rightarrow (II)$, choose $x,y \in X$ such that $[gx,gy] \in \mathcal{R}$. If $(gx,gy) \in \mathcal{R}$, then (II) immediately follows from (I). Otherwise, if $(gy,gx) \in \mathcal{R}$, then by (I) and owing to the symmetry of d (metric), we conclude the claim. □

For the sake of completeness, we state the main results of Arif et al., [6] and Alam Imdad [5] respectively, which are given under:

Theorem 1 ([6])**.** *Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined on X. Suppose that the following conditions hold:*

(i) *(X,d) is $\mathcal{R}$-complete;*
(ii) *$\mathcal{R}$ is T-closed and locally T-transitive;*
(iii) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed;*
(iv) *$X(T,\mathcal{R})$ is non-empty;*
(v) *there exists a comparison function φ such that*

$$d(Tx,Ty) \leq \varphi(d(x,y)) \ \forall\, x,y \in X with (x,y) \in \mathcal{R}.$$

Then $F(T) \neq \emptyset$. Moreover, if $T(X)$ is $\mathcal{R}^s$-connected, then $F(T)$ is singleton.

Theorem 2 ([4], Theorem 2)**.** *Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$, let (T,g) be a pair of self-mappings defined on X and Z be an $\mathcal{R}$-complete subspace of X. Assume that the following conditions hold:*

(a) *$T(X) \subseteq Z \cap g(X)$;*
(b) *$X(T,g,\mathcal{R})$ is non-empty:*
(c) *$\mathcal{R}$ is (T,g)-closed;*
(d) *there exists $\alpha \in [0,1)$ such that $d(Tx,Ty) \leq \alpha d(gx,gy)$ for all $x,y \in X$ with $(gx,gy) \in \mathcal{R}$;*
(e) (e_1) *(T,g) is $\mathcal{R}$-compatible;*
(e_2) *g is $\mathcal{R}$-continuous;*
(e_3) *T is $\mathcal{R}$-continuous or $\mathcal{R}$ is (g,d)-self-closed;*

or, alternatively

(e') (e'_1) $Z \subseteq g(X)$;
(e'_2) *either T is $(g,\mathcal{R})$-continuous or T and g are continuous or $\mathcal{R}|_Z$ is d-self-closed.*

Then $C(T,g) \neq \emptyset$.

Indeed, the main results of this paper are based on the following points:

- Theorem 1 is extended for a pair of self-mappings (T,g) defined on a non-empty set X,
- Theorem 2 is improved by replacing relatively weaker contraction condition,
- Theorem 1 is generalized by replacing comparatively weaker notions namely $\mathcal{R}$-completeness of any subspace $Z \subseteq X$, with $T(X) \subseteq Z$ rather than $\mathcal{R}$ completeness of whole space X,
- some examples are constructed to demonstrate the realized improvement in the proved results in this article.

3. Main Results

Now, we are equipped to prove our main result as follows:

Theorem 3. *Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$, let (T,g) be a pair of self-mappings defined on X and Z be an $\mathcal{R}$-complete subspace of X. Assume that hypotheses (a), (b), (e), or $\big[(e')$ in which modified (e_2') : either T is $(g,\mathcal{R})$-continuous or T is continuous and g is bijective and bi-continuous or $\mathcal{R}|_Z$ is d-self-closed.)$\big]$ of Theorem 2 along with the following conditions hold:*

(k) $\mathcal{R}$ is (T,g)-closed and locally T-transitive;

(l) there exists a comparison function φ such that

$$d(Tx,Ty) \leq \varphi(d(gx,gy)) \ for\ all\ x,y \in X\ with\ (gx,gy) \in \mathcal{R}.$$

Then $C(T,g) \neq \varnothing$.

Proof. Let $x_0 \in X$ such that $(gx_0, Tx_0) \in \mathcal{R}$ (due to hypothesis (b)). Construct a Picard Jungck sequence $\{gx_n\}$ based at the point x_0, i.e.,

$$g(x_{n+1}) = T(x_n) \quad \forall\, n \in \mathbb{N}_0. \tag{3.1}$$

Since $(gx_0, Tx_0) \in \mathcal{R}$, $\mathcal{R}$ is (T,g)-closed and using (3.1), inductively we have

$$(gx_n, gx_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0. \tag{3.2}$$

In view of (3.1) and (3.2), we have $\{Tx_n\}$ is $\mathcal{R}$-preserving in Z (due to hypothesis (a)), i.e.,

$$(Tx_n, Tx_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0. \tag{3.3}$$

Now, if $d(gx_{n_0+1}, gx_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then in view of (3.1), we have $T(x_{n_0}) = g(x_{n_0})$ so that x_{n_0} is a coincidence point of T and g and hence we are done. On the other hand, if $d(gx_{n+1}, gx_n) > 0\ \forall\, n \in \mathbb{N}_0$, then applying the contractivity condition (l) to (3.2), we deduce, for all $n \in \mathbb{N}$ that

$$d(gx_{n+1}, gx_n) \leq \varphi(d(gx_n, gx_{n-1})),$$

which on using (3.2), contractive condition (l) and increasing property of φ, reduces to

$$d(gx_{n+1}, gx_n) \leq \varphi^n(d(gx_1, gx_0)). \tag{3.4}$$

Making $n \to \infty$, in (3.4) and using the definition of comparison function, we get

$$\lim_{n\to\infty} d(gx_{n+1}, gx_n) = 0. \tag{3.5}$$

Fix $\varepsilon > 0$. Then in view of (3.5), we can choose fix $n \in \mathbb{N}$ (corresponding to given $\varepsilon > 0$) such that

$$d(gx_{n+1}, gx_n) < \varepsilon - \varphi(\varepsilon). \tag{3.6}$$

Now, we claim that $\{gx_n\}$ is a Cauchy sequence. To substantiate the claim, using increasing property of φ, (3.1), (3.2) and (3.6), we obtain

$$\begin{aligned} d(gx_{n+2}, gx_n) &\leq d(gx_{n+2}, gx_{n+1}) + d(gx_{n+1}, gx_n) \\ &< d(Tx_{n+1}, Tx_n) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(d(gx_{n+1}, gx_n)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(\varepsilon - \varphi(\varepsilon)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(\varepsilon) + \varepsilon - \varphi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

Now, again using the increasing property of φ, (3.2) and locally T-transitivity of $\mathcal{R}$, we obtain

$$\begin{aligned} d(gx_{n+3}, gx_n) &\leq d(gx_{n+3}, gx_{n+1}) + d(gx_{n+1}, gx_n) \\ &< (d(Tx_{n+2}, Tx_n)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(d(gx_{n+2}, gx_n)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(\varepsilon - \varphi(\varepsilon)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(\varepsilon) + \varepsilon - \varphi(\varepsilon) \\ &= \varepsilon, \end{aligned}$$

so that inductively yields,

$$d(gx_{n+k}, gx_n) < \varepsilon \text{ for all } k \in \mathbb{N}. \tag{3.7}$$

Set $m_0 := n$ and $m := n+k$ for all $k \in \mathbb{N}$ in (3.7), then the inequality (3.7) yields that $d(gx_m, gx_n) < \varepsilon \ \forall\ m,n \geq m_0$ with $m > n$, which shows that the sequence $\{gx_n\}$ is a Cauchy and $\{gx_n\}$ is $\mathcal{R}$-preserving. By $\mathcal{R}$-completeness of (Z,d), there exists $z \in Z$ such that

$$\lim_{n\to\infty} g(x_n) = z. \tag{3.8}$$

On using (3.1), (3.2) and (3.8), we obtain

$$\lim_{n\to\infty} T(x_n) = z. \tag{3.9}$$

Now, to complete the proof by using (e) or (e'). Assume that (e) holds. Using (3.2), (3.8) and assumption (e_2) (i.e., $\mathcal{R}$-continuity of g), we have

$$\lim_{n\to\infty} g(gx_n) = g(\lim_{n\to\infty} gx_n) = g(z). \tag{3.10}$$

Now, utilizing (3.1), (3.2) and (3.9) and assumption (e_2) (i.e., $\mathcal{R}$-continuity of g), we have

$$\lim_{n\to\infty} g(Tx_n) = g(\lim_{n\to\infty} Tx_n) = g(z). \tag{3.11}$$

Since $\{Tx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$-preserving (owing to (3.2) and (3.3)) and $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} T(x_n) = z$ (owing to (3.8) and (3.9)), using assumption (e_1) (i.e., $\mathcal{R}$-compatibility of T and g), give rise

$$\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0. \tag{3.12}$$

We need to prove that z is a coincidence point of T and g. To do this, we use assumption (e_3). Assume that T is $\mathcal{R}$-continuous. On using (3.2), (3.9) and $\mathcal{R}$-continuity of T, we obtain

$$\lim_{n\to\infty} T(gx_n) = T(\lim_{n\to\infty} gx_n) = T(z). \tag{3.13}$$

Applying (3.11), (3.12), (3.13) and the continuity of d, we obtain

$$d(gz, Tz) = d(\lim_{n\to\infty} gTx_n, \lim_{n\to\infty} Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0,$$

so that

$$g(z) = T(z).$$

Thus $z \in C(T, g)$.

Alternatively, assume that $\mathcal{R}$ is (g,d)-self-closed. As $\{gx_n\}$ is $\mathcal{R}$-preserving (due to (3.2)) and $g(x_n) \xrightarrow{d} z$ (in view of (3.8)), due to (g,d)-self-closedness of $\mathcal{R}$, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$[ggx_{n_k}, gz] \in \mathcal{R} \quad \forall\, k \in \mathbb{N}_0. \tag{3.14}$$

Since $g(x_{n_k}) \xrightarrow{d} z$, so equations (3.10)-(3.13) also hold for $\{x_{n_k}\}$ (instead of $\{x_n\}$). On using (3.14), assumption (l) and Propositions 3 and 4 (either $d(ggx_{n_k}, gz)$ is zero or non-zero), we have

$$d(Tgx_{n_k}, Tz) \le \varphi(d(ggx_{n_k}, gz)) \le d(ggx_{n_k}, gz) \;\; \forall\, k \in \mathbb{N}_0. \tag{3.15}$$

Employing the triangular inequality, continuity of ‘d’, (3.10), (3.11), (3.12) and (3.15), we get

$$d(gz, Tz) \le d(gz, gTx_{n_k}) + d(gTx_{n_k}, Tgx_{n_k}) + d(Tgx_{n_k}, Tz)$$

$$\leq d(gz, gTx_{n_k}) + d(gTx_{n_k}, Tgx_{n_k}) + d(ggx_{n_k}, gz)$$
$$\to 0 \text{ as } k \to \infty$$

so that

$$g(z) = T(z).$$

Thus $z \in C(T,g)$ and hence again we are done.
Now, we assume that (e') holds. Owing to assumption (e'_1) (*i.e.*, $Z \subseteq g(X)$), we can find some $u \in X$ such that $z = g(u)$. Hence, (3.8) and (3.9) respectively reduce to

$$\lim_{n\to\infty} g(x_n) = g(u). \tag{3.16}$$

$$\lim_{n\to\infty} T(x_n) = g(u). \tag{3.17}$$

Now, we need to prove that u is a coincidence point of T and g. To accomplish this, we use assumption (e'_2). Firstly, assume that T is $(g,\mathcal{R})$-continuous, then using (3.2) and (3.16), we get

$$\lim_{n\to\infty} T(x_n) = T(u). \tag{3.18}$$

On using (3.17) and (3.18), we get $g(u) = T(u)$. Hence, $u \in C(T,g)$.
Secondly, assume that T is continuous and g is bijective and bi-continuous. Then on using (3.16) and (3.17), we get

$$T(u) = Tg^{-1}(gu) = Tg^{-1}(\lim_{n\to\infty} gx_n) = \lim_{n\to\infty} Tg^{-1}(gx_n) = \lim_{n\to\infty} T(x_n) = g(u).$$

Thus $u \in C(T,g)$ and hence again we are through.

Finally, assume that $\mathcal{R}|_Z$ is d-self-closed. Since $\{gx_n\}$ is $\mathcal{R}|_Z$-preserving (due to (3.2)) and $g(x_n) \xrightarrow{d} g(u) \in Z$ (due to (3.8)), using d-self-closeness of $\mathcal{R}|_Z$, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$[gx_{n_k}, gu] \in \mathcal{R}|_Z \quad \forall\, k \in \mathbb{N}_0. \tag{3.19}$$

Applying (3.1), (3.19), assumption (l), Propositions 3 and 4, (whether $d(gx_{n_k}, gu)$ is zero or non-zero), we obtain

$$d(gx_{n_k+1}, Tu) = d(Tx_{n_k}, Tu) \leq \varphi(d(gx_{n_k}, gu)) \leq d(gx_{n_k}, gu) \quad \forall\, k \in \mathbb{N}_0 \tag{3.20}$$

Applying (3.16), (3.20) and continuity of d, we get

$$d(gu, Tu) = d(\lim_{k\to\infty} gx_{n_k+1}, Tu) = \lim_{k\to\infty} d(gx_{n_k+1}, Tu) \leq \lim_{k\to\infty} d(gx_{n_k}, gu),$$

so that

$$g(u) = T(u).$$

Hence $u \in C(T,g)$. This completes the proof. □

Now, we present a consequence of Theorem 3 by assuming the map $g = I$ (identity mapping on X), remains a sharpened form of Theorem 1 (due to Arif et al. [6]) in the context of $\mathcal{R}$-complete subspace of X, which runs as:

Corollary 1. *Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$, let T be a self-mapping on X and Z be an $\mathcal{R}$-complete subspace of X with $T(X) \subseteq Z$. Suppose that the following conditions hold:*

(i) *$\mathcal{R}$ is T-closed and locally T-transitive;*
(ii) *either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed;*
(iii) *$X(T,\mathcal{R})$ is non-empty;*
(iv) *there exists a comparison function φ such that*

$$d(Tx,Ty) \leq \varphi(d(x,y)) \ \forall\, x,y \in X \text{ with } (x,y) \in \mathcal{R}.$$

Then $F(T) \neq \varnothing$.

Corollary 2. *The conclusion of Theorem 3 (also, Corollary 1) holds, if locally T-transitivity of $\mathcal{R}$ is replaced by any one of the following conditions:*

(i) *$\mathcal{R}$ is transitive;*
(ii) *$\mathcal{R}$ is T-transitive;*
(iii) *$\mathcal{R}$ is g-transitive;*
(iv) *$\mathcal{R}$ is locally transitive.*

4. Uniqueness Results

In this section, we present the corresponding uniqueness results, as follows:

Theorem 4. *If in the hypotheses of Theorem 3, the assumption $\mathcal{R}|_{g(X)}^{s}$-connectedness of $T(X)$ is added, then $\overline{C}(T,g)$ is singleton.*

Proof. Owing to Theorem 3, let $\bar{x}$ and $\bar{y}$ be the two elemtents of $\overline{C}(T,g)$, then there exist $x,y \in X$ such that

$$\bar{x} = g(x) = T(x) \ \text{ and } \ \bar{y} = g(y) = T(y). \tag{4.1}$$

We claim that $\bar{x} = \bar{y}$. Since $T(x),T(y) \in T(X) \subseteq g(X)$, by assumption the $\mathcal{R}|_{g(X)}^{s}$-connectedness of $T(X)$, there exists a path (say $\{gw_0,gw_1,gw_2,\ldots,gw_k\}$) of some finite length k in $\mathcal{R}|_{g(X)}^{s}$ from $T(x)$ to $T(y)$ (where $w_0,w_1,w_2,\ldots,w_k \in X$). Owing to (4.1), without loss of generality, we may set $w_0 = x$ and $w_k = y$. Thus, we obtain

$$[gw_i,gw_{i+1}] \in \mathcal{R}|_{g(X)} \text{ for each } i \ (0 \leq i \leq k-1). \tag{4.2}$$

Define the constant sequences $w_n^0 = x$ and $w_n^k = y$. On using (4.1), we have $g(w_{n+1}^0) = T(w_n^0) = \bar{x}$ and $g(w_{n+1}^k) = T(w_n^k) = \bar{y} \quad \forall\, n \in \mathbb{N}_0$. Set $w_0^1 = w_1, w_0^2 = w_2, \ldots, w_0^{k-1} = w_{k-1}$. As $T(X) \subseteq g(X)$, on the lines similar to proof of Theorem 3, we can furnish sequences $\{w_n^1\},\{w_n^2\},\ldots,\{w_n^{k-1}\}$ in X such that $g(w_{n+1}^1) = T(w_n^1), g(w_{n+1}^2) = T(w_n^2),\ldots,g(w_{n+1}^{k-1}) = T(w_n^{k-1}) \ \forall\, n \in \mathbb{N}_0$. Hence, we have

$$g(w_{n+1}^i) = T(w_n^i) \quad \forall\, n \in \mathbb{N}_0 \text{ and for each } i \ (0 \leq i \leq k). \tag{4.3}$$

Now, we assert that

$$[gw_n^i, gw_n^{i+1}] \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0 \text{ and for each } i\ (0 \le i \le k-1). \tag{4.4}$$

We prove this assertion by an inductive method with respect to index n. It follows from (4.3) that (4.4) holds for $n = 0$. Assume that (4.4) holds for $n = r > 0$, i.e.,

$$[gw_r^i, gw_r^{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \le i \le k-1).$$

Since $\mathcal{R}$ is (T, g)-closed and owing to Proposition 2, we obtain

$$[Tw_r^i, Tw_r^{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \le i \le k-1),$$

which on utilizing (4.3), give rise

$$[gw_{r+1}^i, gw_{r+1}^{i+1}] \in \mathcal{R} \text{ for each } i\ (0 \le i \le k-1).$$

It follows that (4.4) holds for $n = r+1$. Therefore, by induction, (4.4) holds for all $n \in \mathbb{N}_0$. Now, for all $n \in \mathbb{N}_0$ and for each i $(0 \le i \le k-1)$, define $\eta_n^i := d(gw_n^i, gw_n^{i+1})$. We claim that

$$\lim_{n\to\infty} \eta_n^i = 0, \text{ for each } i\ (0 \le i \le k-1). \tag{4.5}$$

With fix i, two cases arise. Firstly, assume that $\eta_{n_0}^i := d(gw_{n_0}^i, gw_{n_0}^{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, then by the assumption (l) and Proposition 4, we obtain $d(Tw_{n_0}^i, Tw_{n_0}^{i+1}) = 0$. Consequently on using (4.3), we get $\eta_{n_0+1}^i = d(gw_{n_0+1}^i, gw_{n_0+1}^{i+1}) = d(Tw_{n_0}^i, Tw_{n_0}^{i+1}) = 0$. Thus by induction, we get $\eta_n^i = 0\ \forall\, n \ge n_0$, yielding thereby $\lim\limits_{n\to\infty} \eta_n^i = 0$. Secondly, assume that $\eta_n^i > 0\ \forall\, n \in \mathbb{N}_0$, then on using (4.3), (4.4), the assumption (l), increasing property of φ and Proposition 3, we obtain

$$\begin{aligned}
\eta_n^i = d(gw_n^i, gw_n^{i+1}) = d(Tw_{n-1}^i, Tw_{n-1}^{i+1}) \le \varphi(d(gw_{n-1}^i, w_{n-1}^{i+1})) &= \varphi(\eta_{n-1}^i) \\
&\le \varphi^2(\eta_{n-2}^i) \\
&\ \ \vdots \\
&\le \varphi^n(\eta_0^i).
\end{aligned}$$

Tending with $n \to \infty$, we get $\lim\limits_{n\to\infty} \eta_n^i = 0$. Therefore, in both cases, claim (4.5) is proved for each i $(0 \le i \le k-1)$. On using triangular inequality and (4.5), we obtain

$$d(\bar{x}, \bar{y}) \le \eta_n^0 + \eta_n^1 + \cdots + \eta_n^{k-1} \to 0 \text{ as } n \to \infty$$

Thus $\bar{x} = \bar{y}$, which ends the proof. □

Corollary 3. *If in the hypotheses of Theorem 4, we replace the assumption $\mathcal{R}|_{g(X)}^s$-connectedness of $T(X)$ by either completeness of $\mathcal{R}|_{T(X)}$ or $\mathcal{R}|_{g(X)}^s$-directedness of $T(X)$, then the conclusion of Theorem 4 also holds.*

Theorem 5. *If in the hypotheses of Theorem 4, the assumption of injectivity of any one of the mappings T and g is added, then $\overline{C}(T, g)$ is singleton.*

Theorem 6. *If in the hypotheses of Theorem 4, the assumption of weakly compatibility of T and g (embodied in conditions (e')) is added, then $F(T,g)$ is singleton.*

The proofs of Corollary 3, Theorems 5 and 6 can be obtained on the lines similar to Corollary 4.6, Theorems 4.7, and 4.8 respectively, contained in [4].

Remark 4. If in addition to hypotheses of Corollary 1, $\mathcal{R}^s$-connectedness of $T(X)$ is added, then $F(T)$ is singleton.

5. Illustrative Examples

Finally, we furnish some examples to demonstrate the realized improvement of our proved results.

Example 1. Let $(X=[0,\infty),d)$ (where $d(x,y):=|x-y|$, for all $x,y\in X$) be a metric space endowed with a binary relation $\mathcal{R}$, where $\mathcal{R}:=\{(x,y)\in X^2 : x-y>0,\ x,y\in [0,\frac{1}{2})\}\cup\{(\frac{1}{n},\frac{1}{n+1})_{n=2}^{\infty}\}$. On X, define a pair of self-mappings (T,g) by $T(x)=\frac{x}{1+2x}$ and $g(x)=\frac{x}{1+x}$ for all $x\in X$. Clearly, neither $\mathcal{R}$ is transitive nor g-transitive but it is T-transitive, hence it is locally T-tranisitive and also $\mathcal{R}$ is (T,g)-closed. Define a comparison function φ by $\varphi(s)=\frac{s}{s+1}\ \forall\, s\in[0,\infty)$. On choosing $\mathcal{R}$-complete subspace $Z=[0,1)$ of X with $T(X)\subseteq Z\cap g(X)$ (as $T(X)=[0,\frac{1}{2})$ and $g(X)=[0,1)$). Now, for all $(gx,gy)\in\mathcal{R}$, we have

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{x}{1+2x}-\frac{y}{1+2y}\right| = \left|\frac{(x-y)}{1+2x+2y+4xy}\right| \\ &\leq \left(\frac{(x-y)}{1+x+y+2xy+(x-y)}\right) \\ &= \left(\frac{(x-y)/(1+x+y+2xy)}{(1+x+y+2xy+(x-y))/(1+x+y+2xy)}\right) \\ &= \frac{\left(\frac{x}{1+x}-\frac{y}{1+y}\right)}{1+\left(\frac{x}{1+x}-\frac{y}{1+y}\right)} \\ &= \frac{(gx-gy)}{1+(gx-gy)} = \frac{d(gx,gy)}{1+d(gx,gy)} = \varphi(d(gx,gy)). \end{aligned}$$

Hence the pair (T,g) and φ satisfy the contractive condition (l) of Theorem 3. Further, it can be easily seen that $X(T,g,\mathcal{R})\neq\varnothing$ and continuity of the pair (T,g). Observe that in view of Theorem 3, (T,g) has a coincidence point (namely: $x=0$).

Further, the conditions $\mathcal{R}|_{g(X)}^{s}$-connectedness of $T(X)$ and weakly compatibility of (T,g) of Theorem 6 can be easily verified. Notice that $x=0$, is the unique common fixed point of (T,g).

But contraction condition (d) of Theorem 2 (due to Alam and Imdad [4]) is not satisfied as on choosing very small positive number ε with $(g\varepsilon, g0) \in \mathcal{R}$, we have

$$\frac{\varepsilon}{1+2\varepsilon} = d(T\varepsilon, T0) \leq \alpha d(g\varepsilon, g0) = \alpha \frac{\varepsilon}{1+\varepsilon},$$

wich implies that $\alpha \geq \frac{1+\varepsilon}{1+2\varepsilon}$. However, $\alpha \geq 1$, as ε tends to zero, which is a contradiction, this shows that genuineness of our newly proved results.

Example 2. Let $(X = [0,4), d)$ be a metric space endowed with a binary relation $\mathcal{R} = \{(0,0),(0,1),(1,0),(1,1),(3,0)\} \cup \{(x_n, x_{n+1}) : x_n = 4 - \frac{1}{n}\}$, where d is a usual metric on X. It is easy to see that $\mathcal{R}$ is neither transitive nor g-transitive but it is locally T-transitive. On X, define a pair of self-mappings (T,g) by

$$T(x) = \begin{cases} 0, & 0 \leq x \leq 1; \\ 1, & 1 < x < 4, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} [x], & 0 \leq x \leq 1; \\ 3, & 1 < x < 4, \end{cases}$$

where $[\cdot]$ is a greatest integer function. Clearly, $\mathcal{R}$ is (T,g)-closed. Let $Z = \{0,1\}$, then Z is $\mathcal{R}$-complete and $T(X) = \{0,1\} \subseteq Z \subseteq g(X) = \{0,1,3\}$. Define a comparison function φ by $\varphi(s) = \frac{1}{2}s$, for all $s \in [0,\infty)$.

Let $\{x_n\}$ be any $\mathcal{R}|_Z$-preserving sequence in Z such that $x_n \xrightarrow{d} x$. As $(x_n, x_{n+1}) \in \mathcal{R}|_Z$, for all $n \in \mathbb{N}$, there exists positive integer $N \in \mathbb{N}$ such that $x_n = x \in \{0,1\}$ for all $n \geq N$. Hence, we can choose a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} = x$, for all $k \in \mathbb{N}$, which amounts to saying that $[x_{n_k}, x] \in \mathcal{R}|_Z$, for all $k \in \mathbb{N}$. Hence, $\mathcal{R}|_Z$ is d-self-closed. It can be easily seen that contraction condition (l) and remaining hypotheses of Theorem 3 are also satisfied.

Observe that, in view of Theorem 3, (T,g) has a coincidence point (namely $x = 0$). Furthermore, the remaining hypotheses of Theorem 6 also holds. Thus, all the hypotheses of Theorem 6 are satisfied and hence (T,g) has a unique common fixed point (namely $x = 0$).

On setting $g = I$, identity mapping on X in the present example and further all the conditions of Corollary 1 can be easily verified. Notice that, in view of Corollary 1 and Remark 4, $x = 0$ is the unique fixed point of T. But (X,d) is not $\mathcal{R}$-complete, therefore this example cannot be covered by Theorem 1, which substantiate the utility of Corollary 1 over Theorem 1 (due to Arif et al. [6]).

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