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OPTIMAL FEEDBACK CONTROL OF HILFER FRACTIONAL EVOLUTION INCLUSIONS INVOLVING HISTORY-DEPENDENT OPERATORS

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Abstract. The main purpose of this paper is to study the feedback control systems governed by Hilfer fractional evolution inclusions involving history-dependent operators. We first show a priori estimates of the solutions to the fractional feedback control system. Then, by using the well-known Bohnenblust-Karlin fixed point theorem, we prove an existence theorem for the fractional feedback control system. Finally, we consider an optimal control problem driven by the fractional feedback control system, and establish its solvability.

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1. INTRODUCTION

In recent years, with the development of computer technology and scientific computing methods, the theory and application of fractional differential evolution equations have developed rapidly. The fractional differential evolution equations in infinite dimensional spaces have been applied in many fields such as economy, mechanics, physics and so on. The fractional differential evolution equations with Caputo type and Riemann-Liouville type have also been widely studied by a large number of scholars(see [9, 11, 17, 19]). Hilfer type fractional derivatives include both Caputo type and Riemann-Liouville type fractional derivatives, which have been widely studied by a large number of scholars (cf. [2]). For example, Gu-Trujillo [2] studied the existence of mild solutions of Hilfer fractional differential equations and the references therein.

Feedback control is a very important concept in control theory. Feedback control refers to the process of sending the output information of the system back to the input end, comparing with the input information, and using the deviation of the

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two to control. Its characteristic is that the control function affects the state function, and the state function will affect the control function in turn, so as to achieve the effect of feedback. In recent years, feedback control systems have been widely used in many fields, such as spacecraft, robot operation and greenhouse regulation (cf. [5, 10, 20–22]. In [20], Wang et al. considered optimal feedback control problems of fractional evolution equations with Caputo fractional derivatives. In [21, 22], the authors studied the feedback control problems of impulsive fractional evolution equations with Riemann-Liouville type. However, there are few studies on optimal feedback control problems of fractional evolution equations with Hilfer type fractional derivatives. This is one of the motivations of this paper. Based on the above considerations, we first study the existence of feasible pairs of Hilfer fractional evolution inclusion with feedback control of the following form:

$$\begin{cases} D_t^{\nu,\mu}x(t) \in Ax(t) + (\mathcal{R}x)(t) + F(t,x(t)) + B(t,x(t))u(t), & t \in (0,b], \\ u(t) \in U(t,t^{1-\alpha}x(t)), & \text{a.e. } t \in (0,b), \\ I_{0^+}^{(1-\nu)(1-\mu)}x(t)\mid_{t=0} = x_0, \end{cases} \tag{1.1}$$

where $D_t^{\nu,\mu}$ denotes the Hilfer fractional derivative, $\nu \in [0,1]$, $\mu \in (0,1)$.

Let $\alpha = \nu + \mu - \nu\mu$, then $1-\alpha = (1-\nu)(1-\mu) \geq 0$. A: $D(A) \subseteq X \to X$ is the infinitesimal generator of a uniformly bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a reflexive Banach space X. $\mathcal{R}$ is a history-dependent operator, $F\colon [0,b]\times X \to \mathcal{P}(X)$, $B\colon [0,b]\times X \to \mathcal{L}(V,X)$, where $\mathcal{L}(V,X)$ represents the space of all bounded linear operators from V to X with the standard norm $\|\cdot\|_{\mathcal{L}(V,X)}$, $U : [0,b]\times X \rightrightarrows V$ is a feedback multifunction.

An outline of this paper is organized as follows. In Section 2, we will present some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we present a priori estimates of the solutions to the fractional feedback control system and by using the well-known Bohnenblust-Karlin fixed point theorem, we prove an existence result of feasible pairs of the system (1.1). In Section 4, we will study an optimal control problem driven by the feedback control system and establish its solvability.

2. PRELIMINARIES

In order to study the feedback control systems of Hilfer fractional evolution inclusions involving history-dependent operators, we introduce the following basic definitions and preparatory knowledge. The norm of a Banach space X will be denoted by $\|\cdot\|_X$. In the sequel, we assume that V is a separable reflexive Banach space. For a uniformly bounded C_0-semigroup $T(t) (t\geq 0)$, we set $M := \sup_{t\in[0,\infty)}\|T(t)\|_{\mathcal{L}(X,X)}$. $C_{1-\alpha}([0,b],X) = \{x\colon\ y(t) = t^{1-\alpha}x(t),\ y \in C([0,b],X)\}$ with the norm $\|x\|_{C_{1-\alpha}} = \sup\{t^{1-\alpha}\|x(t)\|_X\colon t\in[0,b]\}$, where $0<\alpha\leq 1$.

In the sequel, we denote by $\mathcal{P}(Y)$ [$\mathcal{P}_f(Y)$, $\mathcal{P}_{fc}(Y)$, $\mathcal{P}_{fbc}(Y)$,$\mathcal{P}_{(w)cp}(Y)$] the collections of all nonempty [respectively, nonempty closed, nonempty closed and convex, nonempty closed,bounded and convex, nonempty (weakly) compact] subsets of a Banach space Y.

Now, let us recall the following basic definitions and properties related to fractional calculus that will be used in the sequel.

Definition 1 ([6, 19]). For a function $x(t)$ given in the interval $[0,\infty)$, the integral

$$I_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds, \qquad \alpha > 0,$$

is called the Riemann-Liouville fractional integral of order α, where Γ is the Gamma function.

Definition 2 ([6, 19]). For a function $x(t)$ given in the interval $[0,\infty)$, the integral

$${}^{L}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}x(s)ds, \qquad \alpha > 0,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

Definition 3 ([6, 19]). The Caputo fractional derivative of order $\alpha > 0$ is defined as

$${}^{C}D_t^{\alpha}x(t) = {}^{L}D_t^{\alpha}\Big[x(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}x^{(k)}(0)\Big], \qquad n-1 < \alpha < n.$$

If $x \in C^n[0,+\infty)$, then

$${}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds, \quad n-1 < \alpha < n.$$

Definition 4 ([17]). The Hilfer fractional derivative $D_t^{\nu,\mu}x(t)$ of order $\nu \in [0,1]$ and $\mu \in (0,1)$ is defined as

$$D_t^{\nu,\mu}x(t) = I_t^{\nu(1-\mu)}\frac{d}{dt}I_t^{(1-\nu)(1-\mu)}x(t),$$

provided the right side is point-wise defined on $[0,\infty)$.

Remark 1 ([17]). The properties of Hilfer fractional derivatives are as follows.

(i) For $\nu = 0$, $\mu \in (0,1)$, we have

$$D_t^{0,\mu}x(t) = \frac{d}{dt}I_t^{1-\mu}x(t),$$

which means the Hilfer fractional derivative $D_t^{0,\mu}x(t)$ is the Riemann-Liouville fractional derivative.

(ii) For $\nu = 1$, $\mu \in (0,1)$, we have

$$D_t^{1,\mu}x(t) = I_t^{1-\mu}\frac{d}{dt}x(t) = {}^{C}D_t^{\mu}x(t),$$

which means the Hilfer fractional derivative $D_t^{1,\mu}x(t)$ is the Caputo fractional derivative.

Lemma 1 ([2]). *Let $\nu \in [0,1]$ and $\mu \in (0,1)$, $\alpha = \nu+\mu-\nu\mu$, $h \in L^p(J,X)(p > \frac{1}{\mu})$. If $x \in C_{1-\alpha}(J,X)$ and x is a solution of the following problem*

$$\begin{cases} D_t^{\nu,\mu}x(t) = Ax(t)+h(t), \qquad t \in (0,b], \\ I_{0^+}^{(1-\nu)(1-\mu)}x(t)\mid_{t=0} = x_0 \in X, \end{cases} \tag{2.1}$$

then, x satisfies the following equation:

$$x(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)h(s)ds, \ \ t \in (0,b],$$

$$S_{\nu,\mu}(t) = I_t^{\nu(1-\mu)}T_\mu(t), \ T_\mu(t) = t^{\mu-1}P_\mu(t), \ P_\mu(t) = \mu\int_0^\infty \theta M_\mu(\theta)T(t^\mu\theta)d\theta,$$

where $M_\mu(\theta) = \sum\limits_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)}$ $(0<\mu<1)$ is the Wright function and satisfies

$$\int_0^\infty \theta^\delta M_\mu(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\mu\delta)}, \ \delta \geq 0.$$

According to Lemma 1, we give the following definition.

Definition 5. A function $x \in C_{1-\alpha}([0,b],X)$ is called a mild solution of system (1.1), if there exists $u(t) \in U(t,t^{1-\alpha}x(t))$ and $\xi(t) \in F(t,x(t))$ a.e. $t \in [0,b]$ such that x satisfies the following fractional integral equation

$$x(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)\big[(\mathcal{R}x)(s)+\xi(s)+B(s,x(s))u(s)ds, \ t \in [0,b]. \tag{2.2}$$

Obviously, from [2, 18], we have

Lemma 2. *Assume that $T(t)$ is strongly continuous and there exists $M > 1$ such that $\sup_{t\in[0,\infty)} \|T(t)\| \leq M$, we have the following properties.*

(i) *$P_\mu(t)$, $T_\mu(t)$ and $S_{\nu,\mu}(t)$ are linear and bounded operators, i.e. for all $t > 0$, $x \in X$ we can obtain:*

$$\|P_\mu(t)x\| \ \leq \ \frac{M\|x\|}{\Gamma(\mu)}, \qquad \|T_\mu(t)x\| \ \leq \frac{Mt^{\mu-1}\|x\|}{\Gamma(\mu)},$$

and

$$\|S_{\nu,\mu}(t)x\| \leq \frac{Mt^{\alpha-1}\|x\|}{\Gamma(\alpha)}, \quad \alpha = \nu+\mu-\nu\mu.$$

(ii) *Operators $P_\mu(t)$, $T_\mu(t)$ and $S_{\nu,\mu}(t)$ are strongly continuous.*

(iii) *For each $t > 0, S_{\nu,\mu}(t)$ and $P_\mu(t)$ are compact operators if $T(t)$ is compact.*

At the end of this section, we state the following Bohnenblust-Karlin fixed point theorem, which will play an important role to obtain the existence of solutions of feedback control system (1.1).

Theorem 1. *Let $\mathfrak{D}$ be a nonempty subset of Banach space X, which is bounded, closed and convex. Suppose $G : \mathfrak{D} \to \mathcal{P}(X)$ is u.s.c. with closed, convex values, and such that $G(\mathfrak{D}) \subseteq \mathfrak{D}$ and $\overline{G(\mathfrak{D})}$ is compact (i.e., $G(\mathfrak{D})$ is relatively compact). Then G has a fixed point.*

3. THE EXISTENCE OF FEASIBLE PAIRS

In this section, we first study the existence of feasible pairs for feedback control system (1.1). At first, we need the following assumptions on the data of our problems.

$H(T)$: $T(t)$ is compact for every $t > 0$.

$H(R)$ $\mathcal{R} : L^p([0,b],X) \to L^p([0,b],X)$ is a history-dependent operator, i.e., there exists a constant $c_{\mathcal{R}} > 0$ such that

$$\|(\mathcal{R}x_1)(t) - (\mathcal{R}x_2)(t)\|_X \leq c_{\mathcal{R}} \int_0^t \|x_1(s) - x_2(s)\|_X ds,$$
$$\text{for a.e. } t \in [0,b], \text{ all } x_1, x_2 \in L^p([0,b],X).$$

Remark 2. Obviously, if we denote $(\mathcal{R}0)(t)$ by $\varphi(t)$, then, one has from $H(R)$

$$\varphi(t) \in L^p([0,b]) \text{ and } \|(\mathcal{R}x)(t)\|_X \leq \varphi(t) + c_{\mathcal{R}} \int_0^t \|x(s)\|_X ds, \quad (3.1)$$
$$\text{for a.e. } t \in [0,b], \text{ all } x \in L^p([0,b],X).$$

$H(F)$: $[0,b] \times X \to \mathcal{P}_{fc}(X)$ is such that

(i) $t \mapsto F(t,x)$ is measurable on $[0,b]$ for all $x \in X$;

(ii) $F(t,\cdot)$ admits a strongly-weakly closed graph for a.e. $t \in [0,b]$;

(iii) there are a function $\varphi_F \in L^p([0,b])$ for some $p > \frac{1}{\mu}$ and a constant $c_F > 0$ such that

$$\|F(t,x)\|_X = \sup_{z \in F(t,x)} \|z\|_X \leq \varphi_F(t) + c_F t^{1-\alpha} \|x\|_X \quad \text{for all } x \in X \text{ and a.e. } t \in [0,b].$$

Remark 3. If we assume that $J : [0,b] \times X \to \mathbb{R}$ is such that

(i) $t \mapsto J(t,x)$ is measurable on $[0,b]$ for all $x \in X$;

(ii) $x \mapsto J(t,x)$ is locally Lipschitz on X for a.e. $t \in [0,b]$;

(iii) there are a function $\varphi_J \in L^p([0,b])$ for some $p > \frac{1}{\mu}$ and a constant $c_J > 0$ such that

$$\|\partial J(t,x)\|_{X^*} \leq \varphi_J(t) + c_J t^{1-\alpha} \|x\|_X \quad \text{for all } x \in X \text{ and a.e. } t \in [0,b],$$

where $\partial J(t,\cdot)$ stands for the generalized Clarke subdifferential of the local Lipschitz functional $J(t,\cdot)$ on X (cf.[1,15]). Then, ∂J satisfies all the conditions of $H(F)$. Thus, if we replace F by the generalized Clarke subdifferential ∂J, some recent problems involving hemivariational inequalities are special cases of our problems studied here [7,12–14,16,23–25].

$H(B)$: B: $[0,b]\times X\to\mathcal{L}(V,X)$ is such that

(i) $t\mapsto B(t,x)u$ is measurable on $[0,b]$ for any $(x,u)\in X\times V$;

(ii) $x\mapsto B^*(t,x)y$ is continuous for all $y\in X^*$ and a.e. $t\in[0,b]$;

(iii) there exists a constant $c_B>0$ such that for any $x\in X$ and a.e. $t\in[0,b]$

$$\|B(t,x)\|_{\mathcal{L}(U,X)}\le c_B.$$

$H(U)$: the feedback multifunction U: $[0,b]\times X\to\mathcal{P}_{fc}(V)$ is such that

(i) the map $t\to U(t,x)$ is measurable for all $x\in X$;

(ii) there exist a function $\varphi_U(\cdot)\in L^p([0,b],\mathbb{R}^+)$ $(p>\frac{1}{\mu})$ and a constant $c_U>0$ such that

$$\|U(t,x)\|=\sup_{z\in U(t,x)}\|z\|_V\le\varphi_U(t)+c_Ut^{1-\alpha}\|x\|_X,\quad \text{for all } (t,x)\in[0,b]\times X.$$

(iii) for a.e. $t\in[0,b]$, the function $x\to U(t,x)$ is upper semicontinuous.

Using the same ideas of the proof of Lemma 3.2 in [8, p.104], one easily obtain

Lemma 3. *If $H(T)$ holds, then the operator π: $L^p([0,b],X)\to C([0,b],X)$ for some $p>\frac{1}{\mu}$, given by*

$$(\pi h)(\cdot)=\int_0^{\cdot}(\cdot-s)^{\mu-1}P_\mu(\cdot-s)h(s)ds,\quad \forall h\in L^p(J,X)$$

is compact.

Definition 6. A pair $(x,u)\in C_{1-\alpha}([0,b],X)\times L^p([0,b],V)$ is said to be feasible if (x,u) satisfies (1.1).

For the sake of conveniences, we denote by $\mathcal{S}$ the collection of all the feasible pairs of (1.1). Now we will start to prove an existence result of mild solutions for feedback control system (1.1).

Proposition 1. *If $H(T)$, $H(R)$, $H(F)$, $H(B)$ and $H(U)$ hold and $(x,u)\in\mathcal{S}$, then there exist two constants $R_1,R_2>0$ such that the following inequalities hold*

$$\|x\|_{C_{1-\alpha}([0,b],X)}\le R_1,\quad \|u\|_{L^p([0,b],V)}\le R_2. \tag{3.2}$$

Proof. For the sake of convenience, in the sequel, we introduce an equivalent weighted norm on the Banach space $C_{1-\alpha}([0,b],X)$ as follows

$$\|x\|_r=\sup_{t\in[0,b]}t^{1-\mu}\|x(t)\|_Xe^{-rt}, \tag{3.3}$$

where r will be specified later. Let $(x,u)\in\mathcal{S}$. Then there exists $f(s)\in F(t,x(t))$ such that

$$\begin{cases} x(t)=S_{\nu,\mu}(t)x_0+\int_0^t(t-s)^{\mu-1}P_\mu(t-s)\big[(\mathcal{R}x)(s)+f(s)+B(s,x(s))u(s))\big]\,ds,\\ u(t)\in U(t,t^{1-\alpha}x(t)),\quad \text{a.e. } t\in[0,b]. \end{cases} \tag{3.4}$$

Notice that

$$\int_0^t(t-s)^{\alpha-1}e^{rs}ds=r^{-\alpha}e^{rt}\int_0^{rt}z^{\alpha-1}e^{-z}dz\le r^{-\alpha}e^{rt}\Gamma(\alpha),\quad \forall r>0,0<\alpha\le 1. \tag{3.5}$$

From the assumption $H(R)$, $H(F)$, $H(B)$, the formula (3.4) and the Hölder inequality, we have

$$\begin{aligned} t^{1-\alpha}\|x(t)\|_X &\le t^{1-\alpha}\|S_{\nu,\mu}(t)x_0\|_X\\ &\quad+t^{1-\alpha}\|\int_0^t(t-s)^{\mu-1}P_\mu(t-s)\big[(\mathcal{R}x)(s)+f(s)+B(s,x(s))u(s))\big]\,ds\|_X\\ &\le\frac{M}{\Gamma(\alpha)}\|x_0\|_X+\frac{Mb^{1-\alpha}}{\Gamma(\mu)}\int_0^t(t-s)^{\mu-1}\big[\varphi(s)+\varphi_F(s)+c_B\varphi_U(s)\big]\,ds\\ &\quad+\frac{Mt^{1-\alpha}[c_F+c_Bc_U]}{\Gamma(\mu)}\int_0^t(t-s)^{\mu-1}s^{1-\alpha}e^{rs}e^{-rs}\|x(s)\|_Xds\\ &\quad+\frac{Mt^{1-\alpha}c_{\mathcal{R}}}{\Gamma(\mu)}\int_0^t(t-s)^{\mu-1}\int_0^s\tau^{1-\alpha}\tau^{\alpha-1}e^{r\tau}e^{-r\tau}\|x(\tau)\|_X\,d\tau\big]\,ds\\ &\le\frac{M}{\Gamma(\alpha)}\|x_0\|_X\\ &\quad+\frac{Mb^{1-\alpha+\mu-\frac{1}{p}}}{\Gamma(\mu)}(\frac{p-1}{\mu p-1})^{\frac{p-1}{p}}[\|\varphi\|_{L^p}+\|\varphi_F\|\|_{L^p}+c_B\|\varphi_U\|\|_{L^p}]\\ &\quad+\big[Mt^{1-\alpha}(c_F+c_Bc_U)+\frac{Mtc_R}{\alpha}\big]r^{-\mu}e^{rt}\|x\|_r\\ &\le w+\big[Mb^{1-\alpha}(c_F+c_Bc_U)+\frac{Mbc_R}{\alpha}\big]r^{-\mu}e^{rt}\|x\|_r, \end{aligned}\tag{3.6}$$

where

$$w=\frac{M}{\Gamma(\alpha)}\|x_0\|_X+\frac{Mb^{1-\alpha+\mu-\frac{1}{p}}}{\Gamma(\mu)}(\frac{p-1}{\mu p-1})^{\frac{p-1}{p}}[\|\varphi\|_{L^p}+\|\varphi_F\|\|_{L^p}+c_B\|\varphi_U\|\|_{L^p}].$$

Let us choose

$$r\ge\big[2(Mb^{1-\alpha}(c_F+c_Bc_U)+\frac{Mbc_R}{\alpha})\big]^{\frac{1}{\mu}}. \tag{3.7}$$

Then from the above inequality, one has

$$\|x\|_r=\sup_{t\in[0,b]}t^{1-\alpha}\|x(t)\|_Xe^{-rt}\le R_1(:=2w).$$

By $H(U)$, there exists a constant $R_2 > 0$ such that

$$\|u\|_{L^p([0,b],V)} \le R_2 (:= \|\varphi_U\|_{L^p([0,b],\mathbb{R}^+)} + c_U b R_1),$$

which completes the proof. □

From Proposition 1, if $(x,u) \in \mathcal{S}$, then we have from $H(B)$, $H(F)$ and $H(U)$

$$\|F(t,x(t))\|_X + \|B(t,x(t))u(t)\|_X \le \varphi_F(t) + c_B\varphi_U(t) + (c_F + c_B c_U)t^{1-\alpha}\|x(t)\|_X,$$

a.e. $t \in [0,b]$, and set

$$\mathcal{G} := \{g \in L^p([0,b],X) \,|\, \|f\|_X \le \varphi_F(t) + c_B\varphi_U(t) + (c_F + c_B c_U)R_1, \text{ a.e. } t \in [0,b]\}. \tag{3.8}$$

We now consider the following Cauchy problem:

$$\begin{cases} D_t^{\nu,\mu} x(t) \in Ax(t) + (\mathcal{R}x)(t) + g(t), & t \in (0,b], \\ I_{0^+}^{(1-\nu)(1-\mu)} x(t)\mid_{t=0} = x_0. \end{cases} \tag{3.9}$$

Next we begin to prove the existence and uniqueness of mild solutions for (3.9).

Theorem 2. *If $H(T)$ and $H(R)$ hold, then for any $g \in \mathcal{G}$, the system (3.9) has a unique mild solution on $C_{1-\alpha}([0,b],X)$. Furthermore, the map $S\colon \mathcal{G} \to C_{1-\alpha}([0,b],X)$ defined by*

$$S(g) = x(g) \quad \forall g \in K,$$

where $x(g)$ is the unique solution of problem (3.9) corresponding to $g \in \mathcal{G}$, is continuous from $w - L^p([0,b],X)$ into $C_{1-\alpha}([0,b],X)$.

Proof. For any $g \in \mathcal{G}$, define the operator G: $C_{1-\alpha}(J,X) \to C_{1-\alpha}([0,b],X)$ by

$$(Gx)(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x)(s) + g(s)]ds, \quad t \in [0,b].$$

Here we also use the same equivalent weighted norm

$$\|x\|_r = \sup_{t\in(0,T]} t^{1-\alpha}\|x(t)\|_X e^{-rt}$$

defined by (3.3) in the Banach space $C_{1-\alpha}([0,b],X)$ and r also satisfies the inequality (3.7).

We show that the operator G is a contraction operator on $C_{1-\alpha}([0,b],X)$.

For any $x, y \in C_{1-\alpha}(J,X)$, if $t \in [0,b]$, we have

$$(Gx)(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x)(s) + g(s)]ds,$$

$$(Gy)(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}y)(s) + g(s)]ds.$$

So, we obtain from $H(R)$

$$t^{1-\alpha}\|(Gx)(t) - (Gy)(t)\|_X \le \frac{Mc_{\mathcal{R}}b}{\alpha} e^{rt} r^{-\mu}\|x-y\|_r,$$

which implies that from (3.7)

$$\|(Gx)-(Gy)\|_r = \sup_{t\in[0,t_1]} t^{1-\alpha}\|(Gx)(t)-(Gy)(t)\|_X e^{-rt} \leq \frac{Mc_{\mathcal{R}}b}{\alpha r^{\mu}}\|x-y\|_r \leq \frac{1}{2}\|x-y\|_r.$$

Therefore, G is a contraction operator on $C_{1-\alpha}([0,b],X)$. According to the Banach's fixed theorem, G has a unique fixed point on $C_{1-\alpha}([0,b],X)$ and this fixed point is the mild solution of system (3.9).

In the sequel, we show that the map $S\colon \mathcal{G} \to C_{1-\alpha}([0,b],X)$ is continuous from $w-L^p([0,b],X)$ into $C_{1-\alpha}([0,b],X)$.

Let $\{g_n\}\subset \mathcal{G}$ and $\{x_n\}\subset C_{1-\alpha}([0,b],X)$ be such that $x_n = S(g_n)$ and $g_n \rightharpoonup g$ in $L^p([0,b],X)$.

For each $n\in\mathbb{N}$, one has

$$x_n(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x_n)(s)+g_n(s)]ds, \quad t\in[0,b], \tag{3.10}$$

$$x(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x)(s)+g(s)]ds, \quad t\in[0,b], \tag{3.11}$$

for any $n\in\mathbb{N}$, which implies that

$$t^{1-\alpha}\|x_n(t)-x(t)\|_X \leq \frac{Mc_{\mathcal{R}}b}{\alpha}e^{rt}r^{-\mu}\|x_n-x\|_r + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[g_n(s)-g(s)]\,ds,$$

$$\|x_n-x\|_r \leq 2\sup_{t\in[0,b]}\int_0^t (t-s)^{\mu-1}P_\mu(t-s)[g_n(s)-g(s)]\,ds. \tag{3.12}$$

Notice that by Lemma 3, the following holds

$$\int_0^{\cdot} (\cdot-s)^{\mu-1}P_\mu(\cdot-s)[g_n(s)-g(s)]\,ds \to 0 \quad \text{in } C([0,b],X) \quad \text{as } n\to\infty. \tag{3.13}$$

Thus, we obtain

$$x_n \to x \text{ in } C_{1-\alpha}([0,b],X) \quad \text{as } n\to\infty.$$

The proof is complete. □

Theorem 3. *If $H(T)$, $H(R)$, $H(F)$, $H(B)$ and $H(U)$ hold, then the solution set $\mathcal{S}$ of feedback control system 1.1 is nonempty.*

Proof. We first consider the multifunction $S_U\colon C_{1-\alpha}([0,b],X)\to \mathcal{P}(L^p([0,b],V))$ defined by

$$S_U(x) := \{u\in L^p([0,b],V) \mid u(t)\in U(t,x(t)) \quad \text{for a.e. } t\in[0,b]\}.$$

From the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [3, p.158]) and hypotheses $H(U)$, we easily see that $S_U(x)\in \mathcal{P}_{fc}(L^p([0,b],V))$ for any $x\in C_{1-\alpha}([0,b],X)$.

We show that S_U is strongly-weakly u.s.c.,i.e., S_U is u.s.c. from $C_{1-\alpha}([0,b],X)$ to $w-L^p([0,b],V)$. To this end, we need to prove that for any weakly closed set $\mathcal{C}$ in $L^p([0,b],V)$ the set $S_U^-(\mathcal{C})$ is closed in $C_{1-\alpha}([0,b],X)$. Let $\{x_n\}\subset S_U^-(\mathcal{C})$ be such that

$$x_n \to x \text{ in } C_{1-\alpha}([0,b],X) \text{ for some } x\in C_{1-\alpha}([0,b],X). \tag{3.14}$$

So, there exists a sequence $\{u_n\}\subset L^p([0,b],V)$ with $u_n\in S_U(x_n)\cap\mathcal{C}$ for each $n\in\mathbb{N}$ such that

$$u_n(t)\in U(t,x_n(t)) \text{ for a.e. } t\in[0,b].$$

Condition $H(U)$(ii) implies that the sequence $\{u_n\}$ is bounded in $L^p([0,b],V)$. Passing to a subsequence if necessary, we may assume that

$$u_n\to u \text{ weakly in } L^p([0,b],V) \quad \text{for some } u\in L^p([0,b],V). \tag{3.15}$$

Applying Mazur Theorem, see e.g. [8, Chapter 2, Corollary 2.8], we have that there is a sequence $a_{il}\geq 0$ with $\sum_{i\geq 1}a_{il}=1$ such that

$$\overline{u}_l(\cdot):=\sum_{i\geq 1}a_{il}u_{i+l}(\cdot)\to u \text{ strongly in } L^p([0,b],V).$$

Hence, we may assume that

$$\overline{u}_l(t)\to u(t) \text{ in } V \text{ for a.e. } t\in[0,b]. \tag{3.16}$$

Notice that $x_n\to x$ in $C_{1-\alpha}([0,b],X)$, we have

$$t^{1-\alpha}x_n(t)\to t^{1-\alpha}x(t) \text{ in } X \text{ for all } t\in[0,b]. \tag{3.17}$$

Since $x\mapsto U(t,x)$ is u.s.c., then for any $\varepsilon>0$, there exists $k>0$ large enough such that

$$u_k(t)\in U(t,x_k(t))\subset U(t,x(t))+B_\varepsilon \text{ for a.e. } t\in[0,b],$$

where B_ε is an open ball with radius $\varepsilon>0$ centered at 0_V. So, if l is large enough, one also has

$$\overline{u}_l(t)\in U(t,x(t))+B_\varepsilon \text{ for a.e. } t\in[0,b],$$

due to the convexity of $U(t,x(t))+B_\varepsilon$. This combined with the convergence (3.16) deduces

$$u(t)\in\overline{U(t,x(t))+B_\varepsilon} \text{ for a.e. } t\in[0,b].$$

Letting $\varepsilon\to 0$, we have

$$u(t)\in\overline{U(t,x(t))} \text{ for a.e. } t\in[0,b].$$

Since U has closed values, so, we have $u(t)\in\overline{U(t,x(t))}=U(t,x(t))$ for a.e. $t\in[0,b]$. This means that $u\in S_U(x)$. But, the weak closedness of $\mathcal{C}$ infers that $u\in\mathcal{C}$, thus is, $x\in S_U^-(\mathcal{C})$. So, S_U is u.s.c. from $C_{1-\alpha}([0,b],X)$ to $w-L^p([0,b],V)$.

Let us consider the multifunction $\Lambda\colon C_{1-\alpha}([0,b],X)\to 2^{C_{1-\alpha}([0,b],X)}$ defined by

$$\Lambda(x)=S(\mathcal{F}_B(x)) \text{ for all } x\in C_{1-\alpha}([0,b],X), \tag{3.18}$$

where S is the solution map defined in Theorem 2 and

$$\mathcal{F}_B\colon C_{1-\alpha}([0,b],X) \to 2^{L^p([0,b],X)}$$

is given by

$$\mathcal{F}_B(x) := \{F(\cdot,x(\cdot)) + B(\cdot,x(\cdot))u(\cdot) \mid u \in S_U(x)\} \text{ for all } x \in C_{1-\alpha}([0,b],X). \quad (3.19)$$

It is easy to see that for each $g \in \mathcal{F}_B(x)$, there exist $f(t) \in F(t,x(t)$ and $u \in S_U(x)$ such that

$$g(t) = f(t) + B(t,x(t))u(t) \text{ for a.e. } t \in [0,b]$$

and

$$\begin{aligned}\|g(t)\|_X &= \|f(t)+B(t,x(t))u(t)\|_X \le \|f(t)\|_X + c_B\|u(t)\|_V \\ &\le \varphi_F(t) + c_F t^{1-\alpha}\|x(t)\|_X + c_B(a_U(t) + c_U t^{1-\alpha}\|x(t)\|_X) \text{ for a.e. } t \in [0,b].\end{aligned}$$

Obviously, for each $x \in C_{1-\alpha}([0,b],X)$, $\mathcal{F}_B(x)$ is a bounded, closed and convex subset of $L^p([0,b],X)$, due to the closedness and convexity of $F(\cdot,x(\cdot))$ and $S_U(x)$ in $L^p([0,b],X)$ and $L^p([0,b],V)$, respectively. Besides, we also say that $\mathcal{F}_B$ is u.s.c. from $C_{1-\alpha}([0,b],X)$ to $w-L^p([0,b],X)$. For any weakly closed set $\mathcal{D}$ in $L^p([0,b],X)$, let $\{x_n\} \subset \mathcal{F}_B^-(\mathcal{D})$ be such that

$$x_n \to x \text{ in } C_{1-\alpha}([0,b],X) \ \text{ for some } x \in C_{1-\alpha}([0,b],X).$$

So, for each $n \in \mathbb{N}$, there exist $f_n(\cdot) \in F(\cdot,x_n(\cdot))$ and $u_n \in S_U(x_n)$ such that

$$f_n(\cdot) + B(\cdot,x_n(\cdot))u_n(\cdot) \in \mathcal{D} \cap \mathcal{F}_B(x_n).$$

Conditions $H(F)$(iii) and $H(U)$(ii) point out that $\{F(\cdot,x_n(\cdot))\}$ and $\{u_n(\cdot)\}$ are bounded in $L^p([0,b],X)$ and $L^p([0,b],V)$, respectively. Thus we may assume that $f_n \rightharpoonup f$ and $u_n \rightharpoonup u$ in $L^p([0,b],X)$ and $L^p([0,b],V)$,respectively. By $H(F)$(ii), one has $f(\cdot) \in F(\cdot,x(\cdot))$. Using the same arguments as the proof of the upper semicontinuity of S_U, we obtain $u \in S_U(x)$. For any $y \in [L^p([0,b],X)]^* (= L^{p'}([0,b],X^*))$, it has

$$\int_0^b \langle B(t,x_n(t))u_n(t), y(t)\rangle\, dt = \int_0^b \langle u_n(t), B^*(t,x_n(t))y(t)\rangle\, dt.$$

The continuity of $x \mapsto B^*(t,x)$ and Lebesgue dominated convergence theorem entail

$$\begin{aligned}\lim_{n\to\infty}\int_0^b \langle B(t,x_n(t))u_n(t), y(t)\rangle\, dt &= \lim_{n\to\infty}\int_0^b \langle u_n(t), [B^*(t,x_n(t)) - B^*(t,x(t))]y(t)\rangle\, dt \\ &\quad + \lim_{n\to\infty}\int_0^b \langle u_n(t), B^*(t,x(t))y(t)\rangle\, dt \\ &= \int_0^b \langle u(t), B^*(t,x(t))y(t)\rangle\, dt \\ &= \int_0^b \langle B(t,x(t))u(t), y(t)\rangle\, dt,\end{aligned}$$

so, $B(\cdot,x_n(\cdot))u_n(\cdot) \to B(\cdot,x(\cdot))u(\cdot)$ weakly in $L^p([0,b],X)$. The fact $u \in S_U(x)$ turns out $f(\cdot)+B(\cdot,x(\cdot))u(\cdot) \in \mathcal{F}_B(x)$. This combined with the weak closedness of $\mathcal{D}$ implies

$$f(\cdot)+B(\cdot,x(\cdot))u(\cdot) \in \mathcal{F}_B(x) \cap \mathcal{D},$$

that is, $x \in \mathcal{F}_B^-(\mathcal{D})$. So, $\mathcal{F}_B$ is strongly-weakly u.s.c.

The continuity of S and $\mathcal{F}_B$ concludes that the multifunction $\Lambda\colon C_{1-\alpha}([0,b],X) \to 2^{C_{1-\alpha}([0,b],X)}$ is u.s.c. as well (see e.g. [4, Theorem 1.2.8]). Besides, it is easy to prove that Λ has closed and convex values. Set

$$\mathcal{B}(R_1) := \{x \in C_{1-\alpha}([0,b],X) \mid \|x(t)\|_{C_{1-\alpha}([0,b],X)} \le R_1 \text{ for all } t \in [0,b]\}. \tag{3.20}$$

$$\mathcal{G} := \{g \in L^p([0,b],X) \mid \|g(t)\|_X \le \varphi_F(t) + c_B a_U(t) + (c_F + c_B c_U)R_1 \text{ for a.e. } t \in [0,b]\} \tag{3.21}$$

It is easy to see that for each $g \in \mathcal{F}_B(x)$, there exist $f(t) \in F(t,x(t)$ and $u \in S_U(x)$ such that

$$g(t) = f(t) + B(t,x(t))u(t) \text{ for a.e. } t \in [0,b].$$

Now, we prove that Λ maps $\mathcal{B}(R_1)$ into itself. For any $x \in \mathcal{B}(R_1)$ and $y \in \Lambda(x)$, there exist $f(\cdot) \in F(\cdot,x(\cdot))$ and $u(\cdot) \in S_U(x)$ such that $y = S(f(\cdot)+B(\cdot,x(\cdot))u(\cdot))$, i.e.,

$$y(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1} P_\mu(t-s)[(\mathcal{R}y)(s) + f(s) + B(s,x(s)u(s)]\,ds.$$

Using the same arguments in the proof of Proposition 1, we can also show that $\|y(t)\|_{C_{1-\alpha}([0,b],X)} \le R_1$. So, Λ maps $\mathcal{B}(R_1)$ into itself. Moreover, condition $H(U)$(ii) can imply the inclusion $F(\cdot,\mathcal{B}(R_1)) + B(\cdot,\mathcal{B}(R_1))S_U(\mathcal{B}(R_1)) \subset \mathcal{G}$. But, Theorem 2 turns out that the set $\Lambda(\mathcal{B}(R_1))$ is relatively compact in $C_{1-\alpha}([0,b],X)$.

Since all the conditions of Theorem 1 are satisfied, we apply this theorem to conclude that Λ has a fixed point, i.e., there exists $x \in \mathcal{B}(R_1)$ such that $x \in \Lambda(x)$. Then, we can find $f(\cdot) \in F(\cdot,x(\cdot)), u \in S_U(x)$ such that $x = \Gamma(f(\cdot)+B(\cdot,x(\cdot))u(\cdot))$, namely,

$$\begin{cases} D_t^{\nu,\mu}x(t) \in Ax(t) + (\mathcal{R}x)(t) + F(t,x(t)) + B(t,x(t))u(t), & t \in (0,b], \\ u(t) \in U(t,t^{1-\alpha}x(t)), & \text{a.e. } t \in (0,b), \\ I_{0+}^{(1-\nu)(1-\mu)}x(t)\,|_{t=0} = x_0. \end{cases} \tag{3.22}$$

Consequently, $(x,u) \in C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$ is a solution of feedback control system 1.1. □

4. Existence of optimal feedback control pairs

In this section, we study the existence of solutions for the following optimal feedback control problem: find $(x^*,u^*) \in \mathcal{S}$ such that

$$I(x^*,u^*) = \inf_{(x,u)\in\mathcal{S}} I(x,u), \tag{4.1}$$

where $\mathcal{S}$ is the collection of all the feasible pairs of feedback control system (1.1) and I is defined by

$$I(x,u) = \int_0^b h(t,x(t),u(t))\,dt \text{ for all } (x,u) \in C_{1-\alpha}(J,X) \times L^p(J,V).$$

In the sequel, we need the following hypotheses $H(h)$: $h\colon [0,b] \times X \times V \to \mathbb{R}$ is such that

(i) for all $(x,u) \in X \times V$, the map $t \mapsto h(t,x,u)$ is measurable;

(ii) there exist $k_1, k_2 \in L^1([0,b],\mathbb{R})$ and $c_h \geq 0$ such that

$$|h(t,x,u)| \leq k_1(t) + k_2(t)t^{1-\alpha}\|x\|_X + c_h\|u\|_V \quad \text{for all } (x,u) \in X \times V \text{ and a.e. } t \in [0,b];$$

(iii) for each bounded set $\mathcal{D} \subset L^p([0,b],V)$, there exists a function $a_D \in L^p([0,b],\mathbb{R})$ such that

$$|h(t,x,u(t)) - h(t,y,u(t))| \leq a_D(t)t^{1-\alpha}\|x-y\|_X \quad \text{for all } x,y \in X,\ u \in \mathcal{D} \text{ and a.e. } t \in [0,b];$$

(iv) for a.e. $t \in [0,b]$ and $x \in X$, the function $u \mapsto h(t,x,u)$ is lower semicontinuous and convex.

Theorem 4. *If $H(T)$, $H(R)$, $H(F)$, $H(B)$, $H(U)$ and $H(h)$ hold, then the optimal feedback control problem (4.1) has at least one solution.*

Proof. For any $(x,u) \in C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$, using $H(h)$(ii) and Hölder inequality, one has

$$\begin{aligned} I(x,u) &= \int_0^b h(t,x(t),u(t))\,dt \\ &\geq -\int_0^b \left(k_1(t) + k_2(t)t^{1-\alpha}\|x(t)\|_X + c_h\|u(t)\|_V\right) dt \\ &\geq -(\|k_1\|_{L^1} + \|x\|_{C_{1-\alpha}([0,b],X)}\|k_2\|_{L^1} + c_h b^{1-\frac{1}{p}}\|u\|_{L^p([0,b],V)}), \end{aligned}$$

which implies that the functional I is bounded from below on $\mathcal{S}$, due to the boundedness of $\mathcal{S}$ in $C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$ (recall Proposition 1).

Let $\{(x_n,u_n)\} \subset \mathcal{S}$ be a minimizing sequence of optimal feedback control problem 4.1, i.e.,

$$\inf = \lim_{n\to\infty} I(x_n,u_n). \tag{4.2}$$

For each $n \in \mathbb{N}$, there exist $f_n(\cdot) \in F(\cdot,x_n(\cdot))$, $u_n(\cdot) \in S_U(x_n)$ such that $\forall t \in [0,b]$

$$x_n(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x_n)(s) + f_n(s) + B(s,x_n(s))u_n(s)]ds. \tag{4.3}$$

The boundedness of $\{u_n\}$, $H(F)$(iii) and $H(B)$(iii) allow us to assume that

$$u_n \to u \text{ weakly in } L^p([0,b],V) \text{ for some } u \in L^p([0,b],V) \tag{4.4}$$

$$f_n \to f \text{ weakly in } L^p([0,b],X) \text{ for some } f \in L^p([0,b],X) \tag{4.5}$$

$$B(\cdot,x_n(\cdot))u_n(\cdot) \to z \text{ weakly in } L^p([0,b],X) \text{ for some } z \in L^p([0,b],X). \tag{4.6}$$

However, the continuity of S (see Theorem 2) implies

$$x_n = S(f_n(\cdot)+B(\cdot,x_n(\cdot))u_n) \to S(f+z) := x \text{ in } C_{1-\alpha}([0,b],X).$$

Obviously, by $H(F)$(ii), we have

$$f(t) \in F(t,x(t)) \quad \text{a.e. } t \in [0,b]. \tag{4.7}$$

We shall show that $z = B(\cdot,x(\cdot))u(\cdot)$. To this end, for any $y \in \big(L^p([0,b],X)\big)^*$, we have from $H(B)$(ii)

$$\begin{aligned}\lim_{n\to\infty}\int_0^b \langle B(t,x_n(t))u_n(t),y(t)\rangle_X\,dt &= \int_0^b \langle u(t),B^*(t,S(f+z)(t))y(t)\rangle\,dt\\ &= \int_0^b \langle B(t,S(f+z)(t))u(t),y(t)\rangle\,dt.\end{aligned}$$

Hence, $B(\cdot,x_n(\cdot))u_n(\cdot) \to z = B(t,S(f+z)(\cdot))u(\cdot) = B(t,x(\cdot))u(\cdot)$ weakly in $L^p([0,b],X)$. From the definition of S and $x = S(f+z)$, we can see that $(x,u) \in C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$ is a solution of feedback control system (1.1), namely, $(x,u) \in \mathcal{S}$.

In what follows, we prove that $(x,u) \in \mathcal{S}$ is also a solution of optimal feedback control problem 4.1. Now, we show that $u \mapsto I(x,u)$ is weakly lower semicontinuous.

Firstly, we claim that for any $x \in C_{1-\alpha}([0,b],X)$, $u \mapsto I(x,u)$ is convex. In fact, for any $u,v \in L^p([0,b],V)$ and $s \in [0,1]$, we have

$$\begin{aligned}I(x,su+(1-s)v) &= \int_0^b h(t,x(t),su(t)+(1-s)v(t))\,dt\\ &\le s\int_0^b h(t,x(t),u(t))\,dt + (1-s)\int_T h(t,x(t),v(t))\,dt\\ &= sI(x,u)+(1-s)I(x,v),\end{aligned}$$

i.e., $u \mapsto I(x,u)$ is convex.

Let $\{u_n\} \to u$ in $L^p([0,b],V)$ for some $u \in L^p([0,b],V)$. Without any loss of generality, we may assume that $u_n(t) \to u(t)$ in V for a.e. $t \in [0,b]$. It follows from $H(h)$(iv) and Fatou lemma that

$$\begin{aligned}\liminf_{n\to\infty} I(x,u_n) &= \liminf_{n\to\infty}\int_0^b h(t,x(t),u_n(t))\,dt\\ &\ge \int_0^b \liminf_{n\to\infty} h(t,x(t),u_n(t))\,dt \ge \int_0^b h(t,x(t),u(t))\,dt.\end{aligned}$$

So, $u \mapsto I(x,u)$ is l.s.c for any $x \in C_{1-\alpha}([0,b],X)$. Moreover, it is weakly lower semicontinuous by the convexity of $u \mapsto I(x,u)$.

Let $\{(x_n,u_n)\} \subset C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$ be such that $x_n \to x$ in $C_{1-\alpha}([0,b],X)$ for some $x \in C_{1-\alpha}([0,b],X)$ and $u_n \rightharpoonup u$ in $L^p([0,b],V)$ for some $u \in L^p([0,b],V)$. It follows from $H(h)$(ii), (iii), (iv) and Fatou lemma that

$$\begin{aligned}\liminf_{n\to\infty} I(x_n,u_n) &\geq \liminf_{n\to\infty}[I(x_n,u_n) - I(x,u_n)] + \liminf_{n\to\infty} I(x,u_n)\\ &\geq -\limsup_{n\to\infty}\int_0^b a_D(t)t^{1-\alpha}\|x_n(t)-x(t)\|_X\,dt + \liminf_{n\to\infty} I(x,u_n) \geq I(x,u),\end{aligned}$$

where the weak lower semicontinuity of $u \mapsto I(x,u)$. The above estimates and (4.2) conclude that $\inf \leq I(x,u) \leq \liminf_{n\to\infty} I(x_n,u_n) = \inf$, due to $(x,u) \in \mathcal{S}$. So, $(x,u) \in \mathcal{S}$ is also a solution of optimal feedback control problem 4.1. □

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