SOME ALGEBRAIC PROPERTIES OF GENERALIZED $q$–CESÀRO MATRIX $C_{g}(q)$

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Abstract. In this paper, we define the generalized $q$-Cesàro matrix by using generalized Cesàro matrix and $q$-Cesàro matrix. We investigate normality, self-adjointedness and Hilbert-Schmidt properties of generalized $q$-Cesàro matrices.

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1. INTRODUCTION

In [1], Bustoz and Gordillo defined the $q$-Cesàro matrix. For $0 < q < 1$, $q$-Cesàro matrix $C_{1}(q)$ is defined by

$$c_{nk} = \begin{cases} \frac{q^{n-k}}{1+q+\cdots+q^{n-k}}, & 0 < n \leq k \\ 0, & n < k \end{cases}.$$  

Since $c_{nk}(q) \to \frac{1}{n+1}$, for $q \to 1$, the $q$-analog of $C_{1} = (c_{nk})$ is matrix $C_{1}(q)$ where the entries of Cesàro matrix are

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & 0 < k \leq n \\ 0, & n < k \end{cases}$$

and $q$–analog of the integer $n$ is

$$[n]_{q} = \frac{1-q^{n}}{1-q}, \quad (q \neq 1).$$

In [3] Durna and Türkay obtained the spectrum of $q$-Cesàro matrix $C_{1}(q)$ on the space of convergent sequences $c$. Also, they calculated spectral decomposition in the sense of Goldberg. In [6] El Shabrawy investigated boundedness, compactness and spectra of $q$-Cesàro matrix $C_{1}(q)$.

Let us denote the space $H(D) = \{ f : D \to \mathbb{C} : f \text{ is an analytic function} \}$, on $D$ the standard Hardy space $H^{p}$ ($1 \leq p < \infty$) where $D$ is a unit disk, and $p$-summable
complex-valued sequences on the set of non-negative integers \( \ell^p \). You can see [7] for details.

Previously, some authors have generalized various matrix transformations. For example, Young generalized the Cesàro matrix in his PhD thesis ([10]). For all \( f \in H^2 \) the generalized Cesàro operator with the symbol \( g \) is defined as

\[
C_g(f)(z) = \frac{1}{z} \int_0^z f(t) g(t) \, dt.
\]

Here, \( g \) is analytic in \( \mathbb{D} \) and has Taylor series representation

\[
g(z) = \sum_{j=0}^{\infty} a_j z^j,
\]

where \( \{z^{n-1}\}_{n=1}^{\infty} \) is the standard basis of \( H^2 \). The matrix form of \( C_g \) in the standard basis is

\[
C_g = \begin{pmatrix}
a_0 & 0 & 0 & 0 & \cdots \\
a_1 & a_0 & 0 & 0 & \cdots \\
\frac{a_2}{3} & \frac{a_1}{3} & a_0 & 0 & \cdots \\
\frac{a_3}{4} & \frac{a_2}{4} & \frac{a_1}{4} & a_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

And so, one can express entries of matrix \( C_g \) as follows:

\[
(c_g)_{nj} = \begin{cases}
a_n - j \frac{1}{n}, & n \geq j \\
0, & n < j
\end{cases}
\]

For all \( 0 < q < 1 \) we can define the generalized \( q \)-Cesàro matrix \( C_g(q) \) by

\[
C_g(q) = \begin{pmatrix}
a_0 & 0 & 0 & 0 & \cdots \\
q \frac{a_1}{1+q} & \frac{1}{1+q} & a_0 & 0 & \cdots \\
\frac{q^2 a_2}{1+q^2} & \frac{q}{1+q^2} & \frac{a_1}{1+q^2} & a_0 & \cdots \\
\frac{q^3 a_3}{1+q^3} & \frac{q^2}{1+q^3} & \frac{q}{1+q^3} & \frac{a_1}{1+q^3} & a_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

and we have

\[
(c_g(q))_{nk} = \begin{cases}
q^{n-k} \frac{a_n - k}{n}, & 0 \leq k \leq n \\
0, & n < k
\end{cases}
\] (1.1)

Since \( \lim_{q \to 1^-} (c_g(q))_{nk} = \frac{a_n - k}{n} \), matrix \( C_g(q) \) is a \( q \)-analogue of generalized Cesàro matrix \( C_g \). If we take \( g(z) = \frac{1}{1-z} \) and limit for \( q \to 1 \), then we obtain \( C_g(q) = C(q) \). Thus generalized \( q \)-Cesàro matrix contains both of generalized Cesàro matrix and ordinary Cesàro matrix.
The conjugate transpose of the matrix $C_g(q)$ is

$$
C_g^*(q) = \begin{pmatrix}
\alpha_0 & q & q^2 & \cdots \\
1 & 1 + q & 1 + q + q^2 & \cdots \\
0 & 1 + q & 1 + q + q^2 & \cdots \\
0 & 0 & 1 + q + q^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

and so,

$$
(c^*_g(q))_{nk} = \begin{cases} 
\frac{q^{n-k}}{1 + q + \ldots + q^{k-1}} \alpha_{n-k}, & 0 \leq n \leq k \\
0, & k < n
\end{cases}
$$

(1.2)

Inspired by the relationship between Rhaly and Cesàro matrices, Durna and Yıldırım in [4] defined the generalized Rhaly matrix using the generalized Cesàro matrix. They also investigated the topological properties of the generalized Rhaly matrix. In [5], Durna and Yıldırım show that if an infinite matrix $B$ is commutative with every generalized terraced matrix, then $B = kI$, where $I$ is a unit matrix and $k \in \mathbb{C}$. Also, they investigated normality and self-adjointedness of generalized terraced matrix. Now, let us examine normality and self-adjointedness of generalized $q$-Cesàro matrix $C_g(q)$.

2. Results

**Theorem 1.** $C_g(q)$ is a normal matrix iff for $c \in \mathbb{C}$, $g(z) = c$.

**Proof.** We consider $(1, 1)$-entries of $C_g(q)C_g^*(q)$ and $C_g^*(q)C_g(q)$:

$$
[C_g(q)C_g^*(q)]_{11} = \sum_{k=1}^{\infty} (c_g(q))_{1k} (c_g^*(q))_{k1}
$$

$$
= (c_g(q))_{11} (c_g^*(q))_{11}
$$

$$
= a_0 \alpha_0 = |a_0|^2
$$

and

$$
[C_g^*(q)C_g(q)]_{11} = \sum_{k=1}^{\infty} (c_g^*(q))_{k1} (c_g(q))_{1k}
$$

$$
= a_0 \alpha_0 + \sum_{k=2}^{\infty} \frac{q^{k-1}}{1 + q + \ldots + q^{k-1}} a_{k-1} \frac{q^{k-1}}{1 + q + \ldots + q^{k-1}} a_{k-1}
$$

$$
= |a_0|^2 + \sum_{k=2}^{\infty} \left( \frac{q^{k-1}}{1 + q + \ldots + q^{k-1}} \right)^2 |a_{k-1}|^2.
$$

Since $C_g(q)$ is a normal matrix, we get $\sum_{k=2}^{\infty} \left( \frac{q^{k-1}}{1 + q + \ldots + q^{k-1}} \right)^2 |a_{k-1}|^2 = 0$. From here, we obtain that for all $i > 1$, $a_i = 0$. And so, $g(z) = a_0$, $a_0 \in \mathbb{C}$.
Conversely, if $g(z) = a_0$, $a_0 \in \mathbb{C}$ then we have

$$C_g(q) = \begin{pmatrix}
a_0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{1+q}a_0 & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{1+q+q^2}a_0 & 0 & \cdots \\
& \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

hence $C_g(q)$ is normal. \hfill \Box

**Corollary 1.** $C_1$ is not normal.

**Corollary 2.** $C_1(q)$ is not normal.

**Theorem 2.** $C_g(q)$ is a self-adjoint matrix iff for $c \in \mathbb{R}$, $g(z) = c$.

**Proof.** From (1.1) and (1.2), $a_0 = a_0$ and

$$\frac{q}{1+q}a_1 = \frac{q^2}{1+q+q^2}a_2 = \frac{q^3}{1+q+q^2+q^3}a_3 = \cdots = 0.$$

Since, for all $n \in \mathbb{N}$, $\frac{q^n}{1+q+\cdots+q^n} \neq 0$, then

$$a_0 = a_0, a_1 = a_2 = a_3 = \cdots = 0.$$

Hence $a_0 \in \mathbb{R}$ and $g(z) = a_0 \in \mathbb{R}$.

The other direction is obvious. \hfill \Box

**Corollary 3.** $C_1$ is not self-adjoint.

**Corollary 4.** $C_1(q)$ is not self-adjoint.

Note that if $n \geq 0$ and if $T_{ij} = 0$ for $j < i + n$, we say that $T$ is an $n$-triangular operator matrix ([9]).

**Lemma 1** ([9, Theorem 2]). Let $T = (T_{ij}) \in \mathcal{L} \left( \bigoplus_{n=0}^{\infty} \mathcal{H} \right)$ be a $1$–triangular operator matrix. Suppose that $T_{i,i+1}$ has dense range for all $i$. Then $T$ is cyclic.

**Lemma 2** ([2, Proposition 3.6]). Let $T$ be a triangular operator whose diagonal entries with respect to some orthonormal basis for $\mathcal{H}$ are distinct. Then $T$ is cyclic.

**Theorem 3.** $C_g^*(q)$ is cyclic for all $g \in \mathcal{B}$, where

$$\mathcal{B} = \left\{ g : \mathbb{D} \to \mathbb{C} : \int_0^z g(t) \, dt \text{ is a bounded mean oscillation function} \right\}.$$

**Proof.** Suppose that $g(0) = 0$. From Lemma 1, we obtain the state of the theorem. Let $g(0) \neq 0$. Therefore the diagonal entries of matrix $C_g^*(q)$ are distinct. And so, from Lemma 2, $C_g^*(q)$ is cyclic. \hfill \Box
Theorem 4. If \( g_\beta (z) = g (\beta z) \) with \( |\beta| = 1 \), then \( C_{g_\beta} (q) \) is unitarily equivalent to \( C_g (q) \). We denote this by \( C_{g_\beta} (q) \cong C_g (q) \).

Proof. Define the map \( U_\beta : H^2 \to H^2 \) by \( U_\beta (f) (z) = f (\beta z) \). It is easy to see that \( U_\beta \) is unitary with \( U_\beta^* = U_\beta^{-1} \).

Now, to show the unitary equivalence, we must prove that \( U_\beta^* C_{g_\beta} (q) U_\beta = C_g (q) \).

The matrix representation of \( U_\beta \) in the basis \( \{ z^n \}_{n=1}^\infty \) is the diagonal matrix \( \text{diag} \{ \beta^n \} \). Moreover, we know that \( (U_\beta)^* = U_\beta^{-1} \). Using these matrix representations we have \( U_\beta^* C_{g_\beta} (q) U_\beta = C_g (q) \) and consequently \( C_{g_\beta} (q) \cong C_g (q) \). \( \square \)

Corollary 5. If \( \beta \in \partial (\mathbb{D}) \), then the spectrum of \( C_{\frac{1}{1-\beta}} \) is

\[
\sigma \left( C_{\frac{1}{1-\beta}} \right) = \sigma (C(q)) = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{1+q} \right| \leq \frac{q}{1+q} \right\}.
\]

Proof. This is immediate from the unitary equivalence and [6], Theorem 2.2. \( \square \)

In [8] Mursaleen et al. studied the spectrum and Hilbert Schmidt properties of generalized Rhaly matrices. Now, let us show that \( C_g (q) \) is a Hilbert-Schmidt operator. For this, we shall give the following lemma.

Lemma 3 ([8]). If \( \alpha \in \mathbb{Z} \), \( B_{j-1} = 0 \), \( B_k = \sum_{j=1}^{n} \beta^j \) for \( k = j, \ldots, n \) and \( (a_k) \) is a positive decreasing sequence, then

\[
\left| \sum_{k=j}^{n} \frac{\beta^k}{a_k} \right| \leq \frac{2}{|1-\beta|} \left( \frac{1}{a_j} + \frac{1}{a_n} - \frac{1}{a_1} \right).
\]

The set of all Hilbert Schmidt operators on \( H \) is denoted by \( B_2 (H) \).

Theorem 5. If \( \beta_1 \neq \beta_2 \), then \( C_{\beta_1} (q) C_{\beta_2} (q) \) is a Hilbert-Schmidt operator.

Proof. Notice that

\[
C_{\beta_1} (q) C_{\beta_2} (q) \in B_2 (H^2) \iff U_{\beta_2} C_{\beta_1} (q) U_{\beta_2} U_{\beta_2} C_{\beta_2} (q) U_{\beta_2} \in B_2 (H^2).
\]

Since \( U_{\beta_2} C_{\beta_2} (q) U_{\beta_2} = C(q) \), then we can assume \( \beta_2 = 1 \). Rewrite \( \beta_1 \) as \( \beta \). Since \( C_{\beta} (q) = C_{\frac{1}{1-\beta}} (q) \), we have

\[
[C_{\beta} (q)]_{n,k} = \begin{cases} 
\frac{\beta^{n-k}}{1+q^{n-k}}, & n \geq k, \ n,j = 1,2,\ldots \\
0, & n < j
\end{cases}
\]

Since \( [C(q)]_{n,k} = \begin{cases} 
\frac{q^{n-k}}{1+q^{n-k}}, & n \geq k, \ n,k = 1,2,\ldots \\
0, & n < k
\end{cases} \), we get that

\[
[C_{\beta} (q) C_{\beta} (q)]_{n,k} = \begin{cases} 
\frac{q^{n-k}}{1+q^{n-k}} \sum_{j=0}^{n} \beta^{n-j} q^{j-k}, & n \geq j, \ n,j = 1,2,\ldots \\
0, & n < k
\end{cases}
\]
\[
C_{\beta}(q)C(q) = \begin{cases}
\beta_1 q^i n - j \sum_{k=j}^{n} \frac{\beta_k}{1 + q + \cdots + q^{k-1}}, & n \geq j, n, j = 1, 2, \ldots \\
0, & n < j
\end{cases}
\]

From here,

\[
\|C_{\beta}(q)C(q)\|_{H.S}^2 = \sum_{n,j} \left| \left[ C_{\beta}(q)C(q) \right]_{n,j} \right|^2
\]

\[
= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \left| \left[ C_{\beta}(q)C(q) \right]_{n,j} \right|^2
\]

\[
= \sum_{n=1}^{\infty} \frac{\beta_2 n 2n}{(1 + q + \cdots + q^{n-1})} \sum_{j=1}^{n} q^{-2j} \left| \frac{\beta_k}{1 + q + \cdots + q^{k-1}} \right|^2.
\]

From Lemma 3,

\[
\left| \sum_{k=j}^{n} \frac{\beta_k}{1 + \cdots + q^{k-1}} \right|
\leq \frac{2}{1 - \beta} \left( \frac{1}{1 + q + \cdots + q^{n-1}} \right)
\]

\[
\leq \frac{2}{1 - \beta} \frac{1}{1 + q + \cdots + q^{j-1}}.
\]

We use this in the last equation,

\[
\|C_{\beta}(q)C(q)\|_{H.S}^2 = \sum_{n=1}^{\infty} \frac{\beta_2 n 2n}{(1 + q + \cdots + q^{n-1})} \sum_{j=1}^{n} q^{-2j} \left| \frac{\beta_k}{1 + q + \cdots + q^{k-1}} \right|^2
\]

\[
\leq \frac{4}{1 - \beta} \sum_{n=1}^{\infty} \frac{\beta_2 n 2n}{(1 + q + \cdots + q^{n-1})} \sum_{j=1}^{n} q^{-2j} \left| \frac{\beta_k}{1 + q + \cdots + q^{k-1}} \right|^2
\]

\[
\leq \frac{4}{1 - \beta} \sum_{n=1}^{\infty} \frac{\beta_2 n 2n}{(1 + q + \cdots + q^{n-1})} < \infty.
\]

Thus \( C_{\beta}(q)C(q) \) is a Hilbert-Schmidt operator.

\[\square\]

**Theorem 6.** If \( \beta_1 \neq \beta_2 \), then \((C_{\beta_1}(q))^* C_{\beta_2}(q)\) and \(C_{\beta_1}(q)(C_{\beta_2}(q))^*\) are Hilbert-Schmidt operators.
Proof. Without loss of generality, let us take \( \beta_1 = \beta \). Firstly, we will show that \((C_{\beta_1}(q))^*C_{\beta_2}(q) \in B_2(H^2)\). 

\[
[(C_{\beta}(q))^* C(q)]_{nj} = \begin{cases} 
0, & n > j \\
\frac{\beta^n q^n}{1 + q + \cdots + q^{n-1}}, & n \leq j, n, j = 1, 2, \ldots
\end{cases}
\]

Therefore,

\[
[(C_{\beta}(q))^* C(q)]_{nj} = \frac{\beta^n q^n q^{-j}}{1 + q + \cdots + q^{n-1}} \sum_{k=\max(n,j)}^{\infty} \frac{\beta^k q^k}{1 + q + \cdots + q^{k-1}}, n, j = 1, 2, \ldots
\]

Since

\[
\sum_{k=\max(n,j)}^{\infty} \frac{\beta^k q^k}{1 + q + \cdots + q^{k-1}} \leq \frac{2}{|1-q\beta|} \left( \frac{q^{n-1}}{1 + q + \cdots + q^{n-1}} + \frac{q^{l-1}}{1 + q + \cdots + q^{l-1}} - \frac{q^{n-1}}{1 + q + \cdots + q^{n-1}} \right)
\]

\[
\leq \frac{2}{|1-q\beta|} \left( \frac{q^{n-1}}{1 + q + \cdots + q^{n-1}} \right) \leq \frac{2}{|1-q\beta|} \left( \frac{q^{n-1}}{1 + q + \cdots + q^{n-1}} \right),
\]

we have

\[
[(C_{\beta}(q))^* C(q)]_{nj} = \sum_{n,j=1}^{\infty} \frac{\beta^n q^n q^{-j}}{1 + q + \cdots + q^{n-1}} \leq \sum_{n,j=1}^{\infty} \frac{4 q^{2j-2}}{|1-q\beta|^2 (1 + q + \cdots + q^{k-1})^2} = \frac{4 q^{2j-2}}{|1-q\beta|^2} \sum_{n,j=1}^{\infty} \frac{1}{(1 + q + \cdots + q^{k-1})^2} < \infty.
\]

Thus \((C_{\beta_1}(q))^*C_{\beta_2}(q)\) is a Hilbert-Schmidt operator.

Now, let us show that \(C_{\beta}(q) (C(q))^* \in B_2(H^2)\).

\[
[C_{\beta}(q) (C(q))^*]_{nj} = \min\{n,j\} \frac{\beta^n q^n q^{-j}}{1 + q + \cdots + q^{n-1}} \sum_{k=\min(n,j)}^{\infty} \frac{\beta^k}{1 + q + \cdots + q^{k-1}}
\]

and so,

\[
\frac{\beta^n q^n q^{-j}}{1 + q + \cdots + q^{n-1}} \sum_{k=\min(n,j)}^{\infty} \frac{\beta^k}{1 + q + \cdots + q^{k-1}} \leq \frac{q^n q^{-j}}{1 + q + \cdots + q^{n-1}} \sum_{k=\min(n,j)}^{\infty} \frac{\beta^k}{1 + q + \cdots + q^{k-1}}
\]
\[
\|C_\beta (q) (C(q))^*\|_{H,S} = \sum_{n,j=1}^{\infty} \left| \frac{\beta^n q^{n-j}}{1 + q + \cdots + q^{n-1}} \right| \leq \frac{4 q^{n-j}}{1 - \beta^2} \sum_{n,j=1}^{\infty} \frac{q^{2n} q^{-2j}}{(1 + \cdots + q^{n-1})^2} < \infty.
\]

Thus \(C_\beta (q) (C(q))^* \in B_2 (H^2)\) is a Hilbert-Shmidt operator. \(\square\)

**REFERENCES**


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