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# NEW APPROACHES FOR $m$-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRAL OPERATORS WITH STRONG KERNELS

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*Abstract.* We have established this paper on $m-$convex functions, which can be expressed as a general form of the convex function concept. First of all, some inequalities of Hadamard type are proved with fairly simple conditions. Next, an integral identity containing Atangana-Baleanu fractional integral operators is obtained to prove new inequalities for differentiable $m$-convex functions. Using this identity, various properties of $m-$convex functions and classical inequalities, some new integral inequalities have been proved.

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## 1. INTRODUCTION

Although the definition of convex functions has an aesthetic form due to its algebraic and geometric structure, it has been an important place in mathematical analysis with its applications in many fields. Several modified versions and variants of convex functions have been established, which have become the center of consideration of researchers in many branches of applied sciences, especially in areas such as statistics, numerical analysis, convex programming and approximation theory.

The definition of $m-$convex function, which is one of these general forms, is given as follows.

**Definition 1.** [30] The function $\Upsilon : [0,b] \to \mathbb{R}$ is said to be $m$-convex $m \in [0,1]$, if for every $x_1, x_2 \in [0,b]$ and $\tau \in [0,1]$, we have

$$\Upsilon(\tau x_1 + m(1-\tau)x_2) \leq \tau\Upsilon(x_1) + m(1-\tau)\Upsilon(x_2).$$

Let us recall what important results were obtained in some of the basic studies available in the literature for $m$-convex functions. In the study presented in [12], firstly some basic properties of $m$-convex functions are given; then, results including Hermite-Hadamard inequality for $m$-convex functions are obtained by Dragomir. In

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[29], Set *et al.* gave place to two basic theorems for $m$-convex functions involving fractional integrals. While the first of these basic results is a Hermite-Hadamard type result obtained via fractional integrals for $m$-convex functions; the second theorem pays attention to an inequality obtained including fractional integrals for a new function defined with the help of an $m$-convex function. In [22], Mehreen and Anwar generalized the inequality obtained by Set *et al.* for a new function defined by $m$-convex function via Katugampola fractional integrals. In [21], addition to the results including Hermite-Hadamard inequality for $m$-convex and $(\alpha,m)$-convex functions, Klaričić Bakula *et al.* have also obtained inequalities for the product of $m$-convex functions and the product of $(\alpha,m)$ -convex functions. In [8], Chen generalized the results on the product of two $m$-convex functions and the product of two $(\alpha,m)$-convex functions of Klaričić Bakula *et al.* with the help of Riemann-Liouville fractional integrals. Different from the above definition, in [25] Pavić and Avcı Ardıç obtained interesting results for $m$-convex functions by using the intervals of real numbers that contain zero. For other results obtained on $m$-convex functions, see the references in [6], [14] and [24].

Mathematics is a language that explains nature and events and is as old and ancient as the history of humanity. The adventure, which started with simple calculations based on classical analysis, continued towards more advanced problems with the increasing needs of humanity. New orientations in mathematics were needed because it was insufficient to explain real world problems, physical phenomena and the infrastructure that would form the basis of engineering sciences with classical analysis. At this point, the orientation of mathematicians has been towards fractional analysis. Because it is known that fractional analysis produces more effective results for the solutions of differential equation systems and differs from classical analysis in terms of memory effect.

The development in fractional analysis has gained momentum with the definition of fractional derivative operators and associated integral operators. New operators are changing with the structural differences in their core structures and their effectiveness in the application areas, and each new fractional operator has brought a new approach method to the field. Especially the operators defined by strong kernels with non-local and non-singular properties have been preferred by mathematicians. To provide more information related to new fractional operators, integral inequalities and applications, see the papers [2–4, 7, 9–11, 13, 15–20, 23] and [26–29].

Now, recall the Atangana-Baleanu fractional integral operators associated with the fractional derivative described with a kernel structure containing the Mittag-Leffler function.

**Definition 2.** [5] The fractional integral associate to the new fractional derivative with non-local kernel of a function $\Upsilon \in H^1(x_1, x_2)$ as defined:

$$ {}_{x_1}^{AB}I_t^{\alpha}\{\Upsilon(t)\} = \frac{1-\alpha}{B(\alpha)}\Upsilon(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{x_1}^{t}\Upsilon(u)(t-u)^{\alpha-1}du $$

where $x_2 > x_1, \alpha \in [0,1]$, $B(\alpha)$ is normalization function.

In [1], the authors have given the right hand side of integral operator as following;

$$\left({}^{AB}I^{\alpha}_{x_2}\right)\{\Upsilon(t)\} = \frac{1-\alpha}{B(\alpha)}\Upsilon(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^{x_2}\Upsilon(u)(u-t)^{\alpha-1}du.$$

Here, $\Gamma(\alpha)$ is the Gamma function. Since the normalization function $B(\alpha)$ is positive, it immediately follows that the fractional $AB-$integral of a positive function is positive. It should be noted that, when the order $\alpha \longrightarrow 1$, we recover the classical integral. Also, the initial function is recovered whenever the fractional order $\alpha \longrightarrow 0$.

The paper is organized as follows: In section 2, some new integral inequalities have been derived via Atangana-Baleanu fractional integrals for $m-$convex functions. In section 3, a new integral identity containing Atangana-Baleanu fractional integrals as well as generalizations of some integral inequalities available in the literature for $m-$convex functions has been proved. Besides, some new Hadamard type integral inequalities based on this integral identity are given.

## 2. Inequalities for $m$-convex functions without using their derivatives

In this section, we will first give the inequalities that we have obtained without using the derivative of a function that is $m-$convex. Next, we have established some new inequalities for the product of two $m-$convex functions. Of course, let's state once again that we make use of the Atangana-Baleanu integral operator while obtaining these results.

**Theorem 1.** *Let $0 \leq x_1 < \frac{x_2}{m}$, $\xi, m \in (0,1]$ and $\Upsilon : [0,\infty) \longrightarrow \mathbb{R}$ be an m-convex function. If $\Upsilon \in L[x_1, \frac{x_2}{m}]$, we have the following*

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}_{x_1}I^{\xi}_{x_2}\{\Upsilon(x_2)\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_1)}{\xi+1} + \frac{m\Upsilon\left(\frac{x_2}{m}\right)}{\xi(\xi+1)}\right] + \frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2) \tag{2.1}$$

*and*

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}I^{\xi}_{x_2}\{\Upsilon(x_1)\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_2)}{\xi+1} + \frac{m\Upsilon\left(\frac{x_1}{m}\right)}{\xi(\xi+1)}\right] + \frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_1). \tag{2.2}$$

*Proof.* Let us consider the definition of the $m$-convex function, we can write for all $\tau \in [0,1]$

$$\Upsilon(\tau x_1 + (1-\tau)x_2) \leq \tau\Upsilon(x_1) + m(1-\tau)\Upsilon\left(\frac{x_2}{m}\right) \tag{2.3}$$

and

$$\Upsilon(\tau x_2 + (1-\tau)x_1) \leq \tau\Upsilon(x_2) + m(1-\tau)\Upsilon\left(\frac{x_1}{m}\right). \tag{2.4}$$

If we multiply both sides of (2.3) by $\tau^{\xi-1}$, and integrate the resulting inequality over $[0,1]$ with respect to $\tau$, we get

$$\int_0^1 \tau^{\xi-1}\Upsilon(\tau x_1+(1-\tau)x_2)\,d\tau \tag{2.5}$$
$$\leq \int_0^1 \tau^{\xi-1}\left(\tau\Upsilon(x_1)+m(1-\tau)\Upsilon\left(\frac{x_2}{m}\right)\right)d\tau = \frac{\Upsilon(x_1)}{\xi+1}+\frac{m\Upsilon\left(\frac{x_2}{m}\right)}{\xi(\xi+1)}.$$

If appropriate variable changing is made for the left hand side of the above inequality, we obtain

$$\int_0^1 \tau^{\xi-1}\Upsilon(\tau x_1+(1-\tau)x_2)\,d\tau = \frac{1}{(x_2-x_1)^{\xi}}\int_{x_1}^{x_2}(x_2-y)^{\xi-1}\Upsilon(y)\,dy. \tag{2.6}$$

Substituting the equation in (2.6) for the inequality in (2.5), then multiplying the two sides of the inequality by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and adding by $\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)$ we get the inequality in (2.1).

Similar steps are followed to obtain the inequality in (2.2). First, we multiply both sides of (2.4) by $\tau^{\xi-1}$, and integrate the resulting inequality over $[0,1]$. After these operations, if we multiply the two sides of the last inequality by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and add by $\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_1)$, we obtain the inequality in (2.2). □

*Remark* 1. In inequalities (2.1) and (2.2), if we choose $\xi = 1$ and evaluate the obtained inequalities together, we achieve Theorem 2 of Dragomir's paper as in [12].

**Theorem 2.** *Let $0 \leq x_1 < x_2$, $\xi, m \in (0,1]$ and $\Upsilon : [0,\infty) \longrightarrow \mathbb{R}$ be an m-convex function. If $\Upsilon \in L[mx_1, x_2]$, we have the following inequality*

$$\frac{1}{(mx_2-x_1)^{\xi}}\left[{}^{AB}_{x_1}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{mx_2}\{\Upsilon(x_1)\}\right] \tag{2.7}$$
$$+\frac{1}{(x_2-mx_1)^{\xi}}\left[{}^{AB}I^{\xi}_{x_2}\{\Upsilon(mx_1)\}+{}^{AB}_{mx_1}I^{\xi}_{x_2}\{\Upsilon(x_2)\}\right]$$
$$\leq \frac{m+1}{B(\xi)\Gamma(\xi)}(\Upsilon(x_1)+\Upsilon(x_2))+\frac{1-\xi}{(mx_2-x_1)^{\xi}B(\xi)}(\Upsilon(x_1)+\Upsilon(mx_2))$$
$$+\frac{1-\xi}{(x_2-mx_1)^{\xi}B(\xi)}(\Upsilon(mx_1)+\Upsilon(x_2)).$$

*Proof.* For the proof, we will basically make use of the $m-$convex function definition and the definition of Atangana-Baleanu fractional integral oprator. If we use the definition of $m$-convex function, we can write for all $\tau \in [0,1]$ and $x_1, x_2 \in [0,\infty)$

$$\Upsilon(\tau x_1+m(1-\tau)x_2) \leq \tau\Upsilon(x_1)+m(1-\tau)\Upsilon(x_2),$$
$$\Upsilon((1-\tau)x_1+m\tau x_2) \leq (1-\tau)\Upsilon(x_1)+m\tau\Upsilon(x_2),$$
$$\Upsilon(\tau x_2+m(1-\tau)x_1) \leq \tau\Upsilon(x_2)+m(1-\tau)\Upsilon(x_1)$$

and

$$\Upsilon((1-\tau)x_2+m\tau x_1) \leq (1-\tau)\Upsilon(x_2)+m\tau\Upsilon(x_1).$$

If we add these four inequalities above, multiply both sides of the new inequality by $\tau^{\xi-1}$ and integrate the last inequality over $[0,1]$ with respect to $\tau$, we obtain

$$\begin{aligned}
&\int_0^1 \tau^{\xi-1}\Upsilon(\tau x_1+m(1-\tau)x_2)\,d\tau+\int_0^1 \tau^{\xi-1}\Upsilon((1-\tau)x_1+m\tau x_2)\,d\tau \\
&\quad+\int_0^1 \tau^{\xi-1}\Upsilon(\tau x_2+m(1-\tau)x_1)\,d\tau+\int_0^1 \tau^{\xi-1}\Upsilon((1-\tau)x_2+m\tau x_1)\,d\tau \\
&\leq \frac{(m+1)}{\xi}(\Upsilon(x_1)+\Upsilon(x_2)).
\end{aligned} \tag{2.8}$$

Here, following equalities hold:

$$\begin{aligned}
\int_0^1 \tau^{\xi-1}\Upsilon(\tau x_1+m(1-\tau)x_2)\,d\tau &= \frac{1}{(mx_2-x_1)^{\xi}}\int_{x_1}^{mx_2}(mx_2-y)^{\xi-1}\Upsilon(y)dy,\\
\int_0^1 \tau^{\xi-1}\Upsilon((1-\tau)x_1+m\tau x_2)\,d\tau &= \frac{1}{(mx_2-x_1)^{\xi}}\int_{x_1}^{mx_2}(y-x_1)^{\xi-1}\Upsilon(y)dy,\\
\int_0^1 \tau^{\xi-1}\Upsilon(\tau x_2+m(1-\tau)x_1)\,d\tau &= \frac{1}{(x_2-mx_1)^{\xi}}\int_{mx_1}^{x_2}(y-mx_1)^{\xi-1}\Upsilon(y)dy
\end{aligned}$$

and

$$\int_0^1 \tau^{\xi-1}\Upsilon((1-\tau)x_2+m\tau x_1)\,d\tau = \frac{1}{(x_2-mx_1)^{\xi}}\int_{mx_1}^{x_2}(x_2-y)^{\xi-1}\Upsilon(y)dy.$$

Here, if we take these equations into account in (2.8) and firstly multiply both sides of the inequality by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and secondly add the terms $\frac{1-\xi}{(mx_2-x_1)^{\xi}B(\xi)}(\Upsilon(x_1)+\Upsilon(mx_2))$, $\frac{1-\xi}{(x_2-mx_1)^{\xi}B(\xi)}(\Upsilon(mx_1)+\Upsilon(x_2))$ to both sides of the resulting inequality, we obtain the inequality in (2.7). $\square$

*Remark* 2. In (2.7), if we choose $\xi=1$, we achieve Theorem 5 in [12].

**Theorem 3.** *Assume that the assumptions given in the Theorem 1 are valid. Then, we have the following inequalities:*

$$\begin{aligned}
&\frac{\Upsilon\left(\frac{x_1+x_2}{2}\right)}{B(\xi)\Gamma(\xi)}+\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)+\frac{m^{\xi+1}(1-\xi)}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon\left(\frac{x_1}{m}\right) \\
&\leq \frac{1}{2(x_2-x_1)^{\xi}}\left[{}^{AB}_{x_1}I^{\xi}_{x_2}\{\Upsilon(x_2)\}+m^{(\xi+1)AB}I^{\xi}_{\frac{x_2}{m}}\left\{\Upsilon\left(\frac{x_1}{m}\right)\right\}\right] \\
&\leq \frac{\xi}{2B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_1)+m^2\Upsilon\left(\frac{x_2}{m^2}\right)}{\xi+1}+\frac{m\left(\Upsilon\left(\frac{x_2}{m}\right)+\Upsilon\left(\frac{x_1}{m}\right)\right)}{\xi(\xi+1)}\right]
\end{aligned} \tag{2.9}$$

$$+\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)+\frac{m^{\xi+1}(1-\xi)\Upsilon\left(\frac{x_1}{m}\right)}{2(x_2-x_1)^{\xi}B(\xi)}.$$

*Proof.* Since $\Upsilon$ is $m$-convex function, we can write $w_1=\tau x_1+(1-\tau)x_2$ and $w_2=(1-\tau)x_1+\tau x_2$ in the inequality

$$\Upsilon\left(\frac{w_1+w_2}{2}\right)\leq\frac{1}{2}\left(\Upsilon(w_1)+m\Upsilon\left(\frac{w_2}{m}\right)\right)$$

where all $w_1$ and $w_2$ belongs to $[0,\infty)$, we get

$$\Upsilon\left(\frac{x_1+x_2}{2}\right)\leq\frac{1}{2}\left[\Upsilon(\tau x_1+(1-\tau)x_2)+m\Upsilon\left(\frac{(1-\tau)x_1+\tau x_2}{m}\right)\right]. \tag{2.10}$$

If we multiply both sides of (2.10) by $\tau^{\xi-1}$, and integrate the resulting inequality over $[0,1]$ with respect to $\tau$, we get

$$\begin{aligned}&\frac{1}{\xi}\Upsilon\left(\frac{x_1+x_2}{2}\right) \\ &\leq\frac{1}{2(x_2-x_1)^{\xi}}\left[\int_{x_1}^{x_2}(x_2-y)^{\xi-1}\Upsilon(y)dy+m^{\xi+1}\int_{\frac{x_1}{m}}^{\frac{x_2}{m}}\left(y-\frac{x_1}{m}\right)^{\xi-1}\Upsilon(y)dy\right].\end{aligned} \tag{2.11}$$

If we multiply both sides of the inequality in (2.11) by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and add the terms $\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)$ and $\frac{m^{\xi+1}(1-\xi)}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(\frac{x_1}{m})$ to both sides of the resulting inequality, we obtain the first inequality in (2.9).

We will follow a similar method to prove the second inequality in (2.9). Since $\Upsilon$ is $m$-convex function, we can write

$$\begin{aligned}&\frac{1}{2}\left[\Upsilon(\tau x_1+(1-\tau)x_2)+m\Upsilon\left(\frac{(1-\tau)x_1+\tau x_2}{m}\right)\right] \\ &\quad\leq\frac{1}{2}\left[\tau\Upsilon(x_1)+m(1-\tau)\Upsilon\left(\frac{x_2}{m}\right)+m\left((1-\tau)\Upsilon\left(\frac{x_1}{m}\right)+m\tau\Upsilon\left(\frac{x_2}{m^2}\right)\right)\right].\end{aligned}$$

After multiplying both sides of the above inequality by $\tau^{\xi-1}$, and integrating the resulting inequality over $[0,1]$ with respect to $\tau$, if we multiply by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ both sides of the last inequality and add the terms $\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)$ and $\frac{m^{\xi+1}(1-\xi)}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(\frac{x_1}{m})$, we deduce the second inequality in (2.9). □

*Remark* 3. In (2.9), if we choose $\xi=1$, we achieve a result similar to the result in Theorem 4 in [12].

**Theorem 4.** *Under the assumptions of Theorem 1, let us define the mapping* $\Psi(w_1,w_2)_{(\tau)}:[0,1]\to\mathbb{R}$,

$$\Psi(w_1,w_2)_{(\tau)}=\frac{1}{2}\left[\Upsilon(\tau w_1+m(1-\tau)w_2)+\Upsilon((1-\tau)w_1+m\tau w_2)\right].$$

*We have for all* $\tau \in [0,1]$

$$\frac{\xi}{B(\xi)\Gamma(\xi)(x_2-x_1)^{\xi}} \int_{x_1}^{x_2} (x_2-y)^{\xi-1} \Psi\left(y, \frac{x_1+x_2}{2}\right)_{\left(\frac{x_2-y}{x_2-x_1}\right)} dy \tag{2.12}$$
$$\leq \frac{1}{2(x_2-x_1)^{\xi}} \left({}^{AB}_{x_1}I^{\xi}_{x_2} \{\Upsilon(x_2)\}\right) + \frac{m}{2B(\xi)\Gamma(\xi)} \Upsilon\left(\frac{x_1+x_2}{2}\right) - \frac{1-\xi}{2(x_2-x_1)^{\xi} B(\xi)} \Upsilon(x_2).$$

*Proof.* Considering that $\Upsilon$ is $m$-convex function, we can write

$$\Psi(w_1,w_2)_{(\tau)} \leq \frac{1}{2}[\tau\Upsilon(w_1) + m(1-\tau)\Upsilon(w_2) + (1-\tau)\Upsilon(w_1) + m\tau\Upsilon(w_2)]$$
$$= \frac{1}{2}[\Upsilon(w_1) + m\Upsilon(w_2)]$$

and

$$\Psi\left(w_1, \frac{x_1+x_2}{2}\right)_{(\tau)} \leq \frac{1}{2}\left[\Upsilon(w_1) + m\Upsilon\left(\frac{x_1+x_2}{2}\right)\right]. \tag{2.13}$$

If we write $w_1 = \tau x_1 + (1-\tau)x_2$ in the equation (2.13), and after that if we multiply the both sides of resulting inequality with $\tau^{\xi-1}$ and integrate the last inequality over $[0,1]$, we have

$$\frac{1}{(x_2-x_1)^{\xi}} \int_{x_1}^{x_2} (x_2-y)^{\xi-1} \Psi\left(y, \frac{x_1+x_2}{2}\right)_{\left(\frac{x_2-y}{x_2-x_1}\right)} dy \tag{2.14}$$
$$\leq \frac{1}{2(x_2-x_1)^{\xi}} \int_{x_1}^{x_2} (x_2-y)^{\xi-1} \Upsilon(y) dy + \frac{m}{2\xi} \Upsilon\left(\frac{x_1+x_2}{2}\right).$$

To complete the proof, we must multiply the inequality in (2.14) by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and we must add $\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)$. So we obtain the requested result. □

*Remark* 4. In (2.12), if we choose $\xi = 1$, we achieve the $\alpha = 1$ special case of the inequality in Theorem 6 given by Set *et al.* in [29].

**Theorem 5.** *Let* $0 \leq x_1 < x_2$, *and* $\Upsilon, \Omega : [0,\infty) \to [0,\infty)$ *be functions such that* $\Upsilon\Omega \in L[x_1,x_2]$. *If* $\Upsilon$ *and* $\Omega$ *are* $m_1$*-convex and* $m_2$*-convex on* $[x_1,x_2]$ *respectively with* $m_1, m_2 \in (0,1]$ *we obtain*

$$\frac{1}{(x_2-x_1)^{\xi}} \left({}^{AB}_{x_1}I^{\xi}_{x_2} \{(\Upsilon\Omega)(x_2)\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)} \left[\frac{\Upsilon(x_1)\Omega(x_1)}{\xi+2}\right. \tag{2.15}$$
$$\left. + \left(m_2\Upsilon(x_1)\Omega\left(\frac{x_2}{m_2}\right) + m_1\Omega(x_1)\Upsilon\left(\frac{x_2}{m_1}\right)\right) \frac{1}{(\xi+1)(\xi+2)}\right]$$
$$+ \frac{2m_1m_2}{B(\xi)\Gamma(\xi)(\xi+1)(\xi+2)} \Upsilon\left(\frac{x_2}{m_1}\right)\Omega\left(\frac{x_2}{m_2}\right) + \frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_2)$$

*and*

$$
\begin{aligned}
&\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}I^{\xi}_{x_2}\left\{(\Upsilon\Omega)(x_1)\right\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_2)\Omega(x_2)}{\xi+2}\right. \\
&\quad\left.+\left(m_2\Upsilon(x_2)\Omega\left(\frac{x_1}{m_2}\right)+m_1\Omega(x_2)\Upsilon\left(\frac{x_1}{m_1}\right)\right)\frac{1}{(\xi+1)(\xi+2)}\right] \\
&\quad+\frac{2m_1m_2}{B(\xi)\Gamma(\xi)(\xi+1)(\xi+2)}\Upsilon\left(\frac{x_1}{m_1}\right)\Omega\left(\frac{x_1}{m_2}\right)+\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_1)
\end{aligned}
\tag{2.16}
$$

*where* $\xi \in (0,1]$.

*Proof.* We will start by proving the inequality in the equation (2.15). In the hypothesis of the Theorem 5, it is given that the functions $\Upsilon$ and $\Omega$ are $m_1$-convex and $m_2$-convex on $[x_1,x_2]$ respectively. So, we can write

$$\Upsilon(\tau x_1+(1-\tau)x_2) \leq \tau\Upsilon(x_1)+m_1(1-\tau)\Upsilon\left(\frac{x_2}{m_1}\right) \tag{2.17}$$

and

$$\Omega(\tau x_1+(1-\tau)x_2) \leq \tau\Omega(x_1)+m_2(1-\tau)\Omega\left(\frac{x_2}{m_2}\right). \tag{2.18}$$

If we multiply (2.17) and (2.18), we obtain

$$
\begin{aligned}
\Upsilon(\tau x_1+(1-\tau)x_2)\,\Omega(\tau x_1+(1-\tau)x_2) &\leq \tau^2\Upsilon(x_1)\Omega(x_1) \\
&+\tau(1-\tau)m_2\Upsilon(x_1)\Omega\left(\frac{x_2}{m_2}\right)+\tau(1-\tau)m_1\Omega(x_1)\Upsilon\left(\frac{x_2}{m_1}\right) \\
&+(1-\tau)^2m_1m_2\Upsilon\left(\frac{x_2}{m_1}\right)\Omega\left(\frac{x_2}{m_2}\right).
\end{aligned}
\tag{2.19}
$$

After multiplying both sides of (2.19) by $\tau^{\xi-1}$, and integrating the resulting inequality over $[0,1]$ with respect to $\tau$, if we multiply both sides of the last inequality by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and add the term $\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_2)$ we complete the proof of the inequality in the equation (2.15).

Secondly, to prove the inequality in the equation (2.16), one can reach the result by using the method similar to the proof of the first inequality for the product of the following inequalities:

$$\Upsilon(\tau x_2+(1-\tau)x_1) \leq \tau\Upsilon(x_2)+m_1(1-\tau)\Upsilon\left(\frac{x_1}{m_1}\right)$$

and

$$\Omega(\tau x_2+(1-\tau)x_1) \leq \tau\Omega(x_2)+m_2(1-\tau)\Omega\left(\frac{x_1}{m_2}\right).$$

Based on the above estimates, we can easily obtain the desired results. □

**Corollary 1.** *If we choose $m_1 = m_2 = 1$ in Theorem 5, we have the following inequalities*

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}_{x_1}I^{\xi}_{x_2}\{(\Upsilon\Omega)(x_2)\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_1)\Omega(x_1)}{\xi+2}\right.$$
$$\left.+(\Upsilon(x_1)\Omega(x_2)+\Omega(x_1)\Upsilon(x_2))\frac{1}{(\xi+1)(\xi+2)}\right]$$
$$+\frac{2}{B(\xi)\Gamma(\xi)(\xi+1)(\xi+2)}\Upsilon(x_2)\Omega(x_2)+\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_2)$$

*and*

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}I^{\xi}_{x_2}\{(\Upsilon\Omega)(x_1)\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_2)\Omega(x_2)}{\xi+2}\right.$$
$$\left.+(\Upsilon(x_2)\Omega(x_1)+\Omega(x_2)\Upsilon(x_1))\frac{1}{(\xi+1)(\xi+2)}\right]$$
$$+\frac{2}{B(\xi)\Gamma(\xi)(\xi+1)(\xi+2)}\Upsilon(x_1)\Omega(x_1)+\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_1)$$

*where $\Upsilon$ and $\Omega$ are convex functions on $[0,\infty)$.*

*Remark* 5. In inequalities (2.15) and (2.16), if we choose $\xi = 1$ and evaluate the obtained inequalities together, we achieve Theorem 2.4 in [21].

## 3. Hermite-Hadamard type inequalities for $m$-convex functions via Atangana-Baleanu integral operators

The lemma containing the Atangana-Baleanu integral operator below incorporates the left side of the Hermite-Hadamard inequality.

**Lemma 1.** *Let $x_1 < x_2$, $x_1, x_2 \in J^{\circ}$ and $\Upsilon : J \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on $J^{\circ}$. If $\Upsilon' \in L[x_1, mx_2]$, identity for Atangana-Baleanu integral operators in equation (3.1) is valid for all $\tau, \xi \in [0,1]$ and $m \in (0,1]$*

$$\begin{aligned}
&{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\} + {}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}[\Upsilon(x_1)+\Upsilon(mx_2)] \\
&= \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\int_0^{\frac{1}{2}}\tau^{\xi}\Upsilon'(\tau x_1+m(1-\tau)x_2)d\tau\right. \\
&\qquad \left.-\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\Upsilon'(\tau x_1+m(1-\tau)x_2)d\tau\right\}.
\end{aligned} \tag{3.1}$$

*Proof.* By making use of integration by parts, we can write

$$\begin{aligned}
&\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\int_0^{\frac{1}{2}}\tau^{\xi}\Upsilon'(\tau x_1+m(1-\tau)x_2)d\tau \qquad (3.2)\\
&=\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left[\tau^{\xi}\frac{\Upsilon(\tau x_1+m(1-\tau)x_2)}{x_1-mx_2}\bigg|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\xi\tau^{\xi-1}\frac{\Upsilon(\tau x_1+m(1-\tau)x_2)}{x_1-mx_2}d\tau\right]\\
&=-\frac{(mx_2-x_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)+\frac{\xi}{B(\xi)\Gamma(\xi)}\int_{\frac{x_1+mx_2}{2}}^{mx_2}(mx_2-y)^{\xi-1}\Upsilon(y)\,dy
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\Upsilon'(\tau x_1+m(1-\tau)x_2)d\tau \qquad (3.3)\\
&=-\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left[(1-\tau)^{\xi}\frac{\Upsilon(\tau x_1+m(1-\tau)x_2)}{x_1-mx_2}\bigg|_{\frac{1}{2}}^{1}\right.\\
&\qquad\left.+\int_{\frac{1}{2}}^{1}\xi(1-\tau)^{\xi-1}\frac{\Upsilon(\tau x_1+m(1-\tau)x_2)}{x_1-mx_2}d\tau\right]\\
&=-\frac{(mx_2-x_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)+\frac{\xi}{B(\xi)\Gamma(\xi)}\int_{x_1}^{\frac{x_1+mx_2}{2}}(y-x_1)^{\xi-1}\Upsilon(y)\,dy.
\end{aligned}$$

If we add (3.2) and (3.3), and then by adding $\frac{1-\xi}{B(\xi)}\Upsilon(x_1)+\frac{1-\xi}{B(\xi)}\Upsilon(mx_2)$ to two sides of resulting equality, we complete the proof of Lemma 1. □

**Theorem 6.** *Let $x_1<x_2$, $x_1,x_2\in J^{\circ}$and $\Upsilon:J\subset[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on $J^{\circ}$ and $\Upsilon'\in L[x_1,mx_2]$. If $|\Upsilon'|$ is an m-convex function on $[x_1,x_2]$, following inequality is achieved*

$$\begin{aligned}
&\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}\\
&\qquad-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}[\Upsilon(x_1)+\Upsilon(mx_2)]\Bigg|\\
&\le\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left(\frac{1}{2^{\xi+1}(\xi+1)}\right)\left(|\Upsilon'(x_1)|+m|\Upsilon'(x_2)|\right)
\end{aligned}$$

*for all $\xi\in[0,1]$ and $m\in(0,1]$.*

*Proof.* By using the equality in (3.1), property of modulus, we have

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}$$

$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg|$$

$$\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\int_0^{\frac{1}{2}}\tau^{\xi}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|d\tau\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|d\tau\right\}.$$

By applying $m$-convexity of $|\Upsilon'|$, we obtain

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\left\{\Upsilon(mx_2)\right\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\left\{\Upsilon(x_1)\right\}$$

$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg|$$

$$\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\int_0^{\frac{1}{2}}\tau^{\xi}\left[\tau\left|\Upsilon'(x_1)\right|+m(1-\tau)\left|\Upsilon'(x_2)\right|\right]d\tau\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\left[\tau\left|\Upsilon'(x_1)\right|+m(1-\tau)\left|\Upsilon'(x_2)\right|\right]d\tau\right\}.$$

We complete the proof by making the necessary calculations in above. □

**Theorem 7.** *Let $x_1 < x_2$, $x_1,x_2 \in J^{\circ}$and $\Upsilon : J \subset [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on $J^{\circ}$ and $\Upsilon' \in L[x_1,mx_2]$. If $|\Upsilon'|^q$ is an $m$-convex function on $[x_1,x_2]$, following inequality is achieved for all $\xi \in [0,1]$ and $m \in (0,1]$*

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\left\{\Upsilon(mx_2)\right\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\left\{\Upsilon(x_1)\right\}$$

$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg|$$

$$\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left(\frac{1}{2^{\xi+1}(\xi+1)}\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\frac{1}{2^{\xi+2}(\xi+2)}\left|\Upsilon'(x_1)\right|^q+\frac{m(\xi+3)}{2^{\xi+2}(\xi+1)(\xi+2)}\left|\Upsilon'(x_2)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\xi+3)}{2^{\xi+2}(\xi+1)(\xi+2)}\left|\Upsilon'(x_1)\right|^q+\frac{m}{2^{\xi+2}(\xi+2)}\left|\Upsilon'(x_2)\right|^q\right)^{\frac{1}{q}}\right\}$$

*where $q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* If we use Hölder inequality, we have

$$\begin{aligned}
&\Big| {}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\left\{\Upsilon(mx_2)\right\} + {}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\left\{\Upsilon(x_1)\right\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Big| \\
&\quad \le \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\left(\int_0^{\frac{1}{2}}\tau^{\xi}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\tau^{\xi}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\right. \\
&\qquad \left.+\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By using $m$-convexity of $|\Upsilon'|^q$, we get

$$\begin{aligned}
&\Big| {}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\left\{\Upsilon(mx_2)\right\} + {}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\left\{\Upsilon(x_1)\right\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Big| \\
&\le \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\left(\int_0^{\frac{1}{2}}\tau^{\xi}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\tau^{\xi}\left[\tau\left|\Upsilon'(x_1)\right|^q+m(1-\tau)\left|\Upsilon'(x_2)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right. \\
&\qquad \left.+\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\left[\tau\left|\Upsilon'(x_1)\right|^q+m(1-\tau)\left|\Upsilon'(x_2)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

If we calculate the integrals above, we have the desired result. □

**Theorem 8.** *Suppose that the assumptions given in the Theorem 7 are valid. Then, we have the following inequality:*

$$\begin{aligned}
&\Big| {}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\left\{\Upsilon(mx_2)\right\} + {}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\left\{\Upsilon(x_1)\right\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Big| \\
&\quad \le \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left(\frac{1}{2^{\xi p+1}(\xi p+1)}\right)^{\frac{1}{p}} \\
&\qquad \times\left[\left(\frac{|\Upsilon'(x_1)|^q+3m|\Upsilon'(x_2)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\Upsilon'(x_1)|^q+m|\Upsilon'(x_2)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

*Proof.* By Hölder's inequality in a different variant, we can write

$$\begin{aligned}
&\Bigg| {}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\} + {}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg| \\
&\quad\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\Bigg\{\left(\int_0^{\frac{1}{2}}\tau^{\xi p}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}} \\
&\qquad +\left(\int_{\frac{1}{2}}^1(1-\tau)^{\xi p}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\Bigg\}
\end{aligned}$$

By taking into account the $m$-convexity of $|\Upsilon'|^q$, we have

$$\begin{aligned}
&\Bigg| {}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\} + {}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg| \\
&\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\Bigg\{\left(\int_0^{\frac{1}{2}}\tau^{\xi p}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\tau\left|\Upsilon'(x_1)\right|^q+m(1-\tau)\left|\Upsilon'(x_2)\right|^q\right]d\tau\right)^{\frac{1}{q}} \\
&\qquad +\left(\int_{\frac{1}{2}}^1(1-\tau)^{\xi p}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[\tau\left|\Upsilon'(x_1)\right|^q+m(1-\tau)\left|\Upsilon'(x_2)\right|^q\right]d\tau\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

We complete the proof by calculating the integrals. □

**Theorem 9.** *Assume that the assumptions given in the Theorem 7 are valid. Then, we have the following inequality:*

$$\begin{aligned}
&\Bigg| {}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\} + {}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg| \\
&\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left(\frac{q-1}{2^{\xi\left(\frac{q-p}{q-1}\right)+1}\left(\xi(q-p)+q-1\right)}\right)^{\frac{1}{p}} \\
&\qquad \times\Bigg\{\left(\frac{1}{2^{\xi p+2}(\xi p+2)}\left|\Upsilon'(x_1)\right|^q+\frac{m(\xi p+3)}{2^{\xi p+2}(\xi p+1)(\xi p+2)}\left|\Upsilon'(x_2)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{(\xi p+3)}{2^{\xi p+2}(\xi p+1)(\xi p+2)}\left|\Upsilon'(x_1)\right|^q+\frac{m}{2^{\xi p+2}(\xi p+2)}\left|\Upsilon'(x_2)\right|^q\right)^{\frac{1}{q}}\Bigg\}$$

*where* $q \geq p > 1$.

*Proof.* Applying by different way of Hölder inequality, we have

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}$$

$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg|$$

$$\leq\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\left(\int_0^{\frac{1}{2}}\tau^{\xi\left(\frac{q-p}{q-1}\right)}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\tau^{\xi p}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi\left(\frac{q-p}{q-1}\right)}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi p}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\right\}.$$

If we use $m$-convexity of $|\Upsilon'|^q$ above and calculate the integrals, we complete the proof. □

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