A FIXED POINT THEOREM FOR GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we obtain a generalization of a fixed point theorem given by Popescu [O. Popescu, Comput. Math. Appl., vol. 62, no. 10, pp. 3912–3919, 2011]. An example is also given to support our main result.

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1. INTRODUCTION AND PRELIMINARIES

Let \((X, d)\) be a metric space and let \(T : X \to X\) be a mapping. We say that \(x \in X\) is a fixed point for \(T\), if \(Tx = x\). The sequence \((x_n) \subseteq X\) is said to be an approximate fixed point sequence (a.f.p.s., for short) for \(T\), if \(d(x_n, Tx_n) \to 0\).

In 2011, Popescu [6] proved the following result.

**Theorem 1.** Let \((X, d)\) be a complete metric space and let \(T : X \to X\) be a mapping satisfying

\[
\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq ad(x, y) + b(d(x, Tx) + d(y, Ty)) + c(d(x, Ty) + d(y, Tx)),
\]

for all \(x, y \in X\), with \(a \geq 0, b > 0, c > 0\) and \(a + 2b + 2c = 1\). Then, \(T\) has a unique fixed point.

**Definition 1.** Let \((X, d)\) be a metric space, \(\lambda \in [0, 1)\) and let \(a, b, c \in \mathbb{R}\) with \(a + 2b + 2c = 1\). We say that a mapping \(T : X \to X\) is a \((\lambda, a, b, c)\)-generalized nonexpansive mapping if for all \(x, y \in X\),

\[
\lambda d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq ad(x, y) + b(d(x, Tx) + d(y, Ty)) + c(d(x, Ty) + d(y, Tx)).
\]
In the following example, we see that many previously known classes of generalized nonexpansive mappings are actually \((\lambda, a, b, c)\)-generalized nonexpansive, for some \(\lambda \in [0, 1]\) and \(a, b, c \in \mathbb{R}\).

**Example 1.**

(i) Every nonexpansive mapping is \((0, 1, 0, 0)\)-generalized nonexpansive.

(ii) Every mapping which satisfies condition \((C)\) [2, 4] is \((\frac{1}{2}, 1, 0, 0)\)-generalized nonexpansive.

(iii) Every Kannan nonexpansive mapping [5] is \((0, 0, 0, 1)\)-generalized nonexpansive.

(iv) Every Reich nonexpansive mapping [5] is \((0, a, \alpha, 0)\)-generalized nonexpansive, where \(a \geq 0\) and \(\alpha \geq 0\).

(v) Every generalized \(\alpha\)-nonexpansive mapping [5] is \((\frac{1}{2}, 1 - 2\alpha, 0, 0)\)-generalized nonexpansive, where \(\alpha \in [0, 1]\).

(vi) Every Reich-Suzuki nonexpansive mapping [5] is \((\frac{1}{2}, 1 - 2\alpha, \alpha, 0)\)-generalized nonexpansive, where \(\alpha \in [0, 1]\).

(vii) Every Gobel-Kirk-Shimi generalized nonexpansive mapping [1] is \((0, a, b, c)\)-generalized nonexpansive, where \(a \geq 0, b > 0\) and \(c > 0\).

(viii) The class of \((0, a, b, 0)\)-generalized nonexpansive mappings, where \(a > 0\) and \(b > 0\), was defined and studied by Greguš [3].

(ix) The class of \((\frac{1}{2}, a, b, c)\)-generalized nonexpansive mappings, where \(a \geq 0, b > 0\) and \(c > 0\), was defined and studied by Popescu [6].

In connection with Theorem 1, the following problem arises:

**Problem 1.** Determine all \((\lambda, a, b, c) \in [0, 1) \times \mathbb{R}^3\) with \(a + 2b + 2c = 1\), such that every \((\lambda, a, b, c)\)-generalized nonexpansive mapping \(T : X \to X\) has a fixed point, where \((X, d)\) is a complete metric space.

Fixed point theory for generalized nonexpansive mappings, that recall in example 1, have been studied by some authors [1–6]. In this paper, we give a partial answer to the problem 1, which is also a generalization of Theorem 1.

2. Main result

García-Falset et al. [2], studied a class of mappings satisfying the following condition.

**Definition 2.** Let \(C\) be a nonempty subset of a metric space \(X\). For \(\mu \geq 1\), we say that a mapping \(T : C \to X\) satisfies condition \((E_\mu)\) on \(C\) if for each \(x, y \in C\),

\[
d(x, Ty) \leq \mu d(x, Tx) + d(x, y).
\]

We say that \(T\) satisfies condition \((E)\) on \(C\) whenever \(T\) satisfies \((E_\mu)\) on \(C\) for some \(\mu \geq 1\).
Proposition 1. Let $T: X \to X$ be $(\frac{1}{2}, a, b, c)$-generalized nonexpansive mapping with $b \geq 0$, $c \geq 0$ and $b + c < 1$. Then, $T$ satisfies condition (E) with $\mu = \frac{b + c}{1 - b - c}$.

Proof. Since $ \frac{1}{2}d(x, Tx) \leq d(x, Ty)$, for each $x \in X$, then
\[
d(Tx, T^2x) \leq ad(x, Tx) + b(d(x, Tx) + d(Tx, T^2x)) + cd(x, T^2x) \leq ad(x, Tx) + b(d(x, Tx) + d(Tx, T^2x)) + c(d(x, Tx) + d(Tx, T^2x)).
\]
Using the assumptions $b + c < 1$ and $a + 2b + 2c = 1$, from the above we conclude that (note that the above inequality holds for any $(\lambda, a, b, c)$-generalized nonexpansive mapping with $\lambda \in [0, 1]$).
\[
d(Tx, T^2x) \leq d(x, Tx) \quad \text{for each } x \in X. \quad (2.1)
\]
Now, we prove that for each $x, y \in X$,
\[
either \quad \frac{1}{2}d(x, Tx) \leq d(x, y) \quad \text{or} \quad \frac{1}{2}d(Tx, T^2x) \leq d(Tx, y). \quad (2.2)
\]

On the contrary, we assume that there exist $x, y \in X$ such that
\[
\frac{1}{2}d(x, Tx) > d(x, y) \quad \text{and} \quad \frac{1}{2}d(Tx, T^2x) > d(Tx, y).
\]
Hence by (2.1), we have
\[
d(x, Tx) \leq d(x, y) + d(y, Tx)
\]
\[
\leq \frac{1}{2}d(x, Tx) + \frac{1}{2}d(Tx, T^2x)
\]
\[
\leq \frac{1}{2}d(x, Tx) + \frac{1}{2}d(x, Tx) = d(x, Tx),
\]
a contradiction. So, (2.2) holds for each $x, y \in X$.

Now, from (2.2) we get that for each $x, y \in X$, either
\[
d(Tx, Ty) \leq ad(x, y) + b(d(x, Tx) + d(y, Ty)) + c(d(x, Ty) + d(y, Tx)), \quad (2.3)
\]
or
\[
d(T^2x, Ty) \leq ad(Tx, y) + b(d(Tx, T^2x) + d(y, Ty)) + c(d(Tx, Ty) + d(y, T^2x)). \quad (2.4)
\]
Let $x, y \in X$. We first assume that (2.3) holds. Then
\[
d(x, Ty) \leq d(x, Tx) + d(Tx, Ty)
\]
\[
\leq d(x, Tx) + ad(x, y)
\]
\[
+ b(d(x, Tx) + d(y, Ty)) + c(d(x, Ty) + d(y, Tx))
\]
\[
\leq (1 + b + c)d(x, Tx) + (b + c)d(x, Ty) + (1 - b - c)d(x, y).
\]
Since $b + c < 1$, we obtain
\[
\begin{align*}
  d(x, Ty) &\leq (1 + b + c)d(x, Tx) + d(x, y) \\
  &\leq (3 + b + c)d(x, Tx) + d(x, y).
\end{align*}
\]

Now assume that (2.4) holds. Then by (2.1), we have
\[
\begin{align*}
  d(x, Ty) &\leq d(x, Tx) + d(Tx, T^2x) + d(T^2x, Ty) \\
  &\leq 2d(x, Tx) + ad(Tx, y) \\
  &\quad + b(d(Tx, T^2x) + d(y, Ty)) + c(d(Tx, Ty) + d(y, T^2x)) \\
  &\leq 2d(x, Tx) + ad(Tx, y) \\
  &\quad + b(d(Tx, T^2x) + d(y, Tx) + d(x, Tx) + d(x, Ty)) \\
  &\quad + c(d(x, Tx) + d(x, Ty) + d(Tx, y) + d(Tx, T^2x)) \\
  &\leq 2d(x, Tx) + ad(Tx, y) \\
  &\quad + b(d(y, Tx) + 2d(x, Tx) + d(x, Ty)) \\
  &\quad + c(d(x, Ty) + d(Tx, y) + 2d(x, Tx)).
\end{align*}
\]

Since $b + c < 1$, we obtain
\[
(1 - b - c)d(x, Ty) \leq (2 + 2b + 2c)d(x, Tx) + (1 - b - c)d(Tx, y) \\
\leq (2 + 2b + 2c)d(x, Tx) + (1 - b - c)(d(Tx, Tx) + d(x, x)).
\]

Hence
\[
d(x, Ty) \leq \left( \frac{3 + b + c}{1 - b - c} \right)d(x, Tx) + d(x, y).
\]

Now, we are ready to state our first main result.

**Theorem 2.** Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a $(\lambda, a, b, c)$-generalized nonexpansive mapping with $\lambda \in [0, 1)$, $b > 0$, $c > 0$ and $b + c < 1$. Then, $T$ has an a.f.p.s.

**Proof.** Let $m = \inf_{x \in X} d(x, Tx)$. On the contrary, assume that $m > 0$. Let $(x_n)$ be a sequence in $X$ such that
\[
m = \lim_{n \to \infty} d(x_n, Tx_n). \tag{2.5}
\]

By the diagonal argument, we may assume all the following limits exist:
\[
\lim_{n \to \infty} d(T^i x_n, T^j x_n), \quad \forall i, j \in \mathbb{N} \cup \{0\}. \tag{2.6}
\]

Replacing $x$ by $T x_n$ and $y$ by $T^2 x_n$ in Definition 1, for each $n \in \mathbb{N}$ we obtain
\[
d(T^2 x_n, T^3 x_n) \leq ad(T x_n, T^2 x_n) + bd(T x_n, T^2 x_n)
\]
Thus
\[ m = \lim_{n \to \infty} d(T^2 x_n, T^3 x_n) \]
\[ \leq a \lim_{n \to \infty} d(T x_n, T^2 x_n) + b \lim_{n \to \infty} d(T x_n, T^2 x_n) + b \lim_{n \to \infty} d(T x_n, T^3 x_n) + c \lim_{n \to \infty} d(T x_n, T^3 x_n). \]

From (2.1) and (2.5), we obtain
\[ m = \lim_{n \to \infty} d(T^k x_n, T^{k+1} x_n), \text{ for each } k \in \mathbb{N}. \]  
(2.8)

Hence by using (2.8) and (2.7), we get
\[ m \leq \frac{1}{2} \lim_{n \to \infty} d(T x_n, T^3 x_n), \]
and so (note the conditions (2.5) and (2.6) are also satisfied by \((T x_n)\))
\[ m \leq \frac{1}{2} \lim_{n \to \infty} d(T^2 x_n, T^4 x_n). \]

Now we prove by induction that
\[ m \leq \frac{1}{k} \lim_{n \to \infty} d(T x_n, T^{k+1} x_n) \text{ for each } k \geq 2. \]  
(2.9)

Assume that (2.9) holds for some \( k \geq 2 \), then
\[ m \leq \frac{1}{k} \lim_{n \to \infty} d(T^2 x_n, T^{k+2} x_n). \]

Therefore from (2.9) and for sufficiently large \( n \), we have
\[ \lambda d(T x_n, T^2 x_n) \leq \lim_{n \to \infty} d(T x_n, T^2 x_n) = m \]
\[ \leq \frac{1}{k} \lim_{n \to \infty} d(T x_n, T^{k+1} x_n) \]
\[ \leq d(T x_n, T^{k+1} x_n), \]
and so, we may assume that
\[ \lambda d(T x_n, T^2 x_n) \leq d(T x_n, T^{k+1} x_n) \text{ for each } n \in \mathbb{N}. \]

Replacing \( x \) by \( T x_n \) and \( y \) by \( T^{k+1} x_n \) in Definition (1), for each \( n \in \mathbb{N} \) we obtain
\[ d(T^2 x_n, T^{k+2} x_n) \leq a d(T x_n, T^{k+1} x_n) \]
\[ + b d(T x_n, T^2 x_n) + b d(T^{k+1} x_n, T^{k+2} x_n) + c d(T x_n, T^{k+2} x_n) + c d(T^2 x_n, T^{k+1} x_n). \]

Now we claim that
\[ a \lim_{n \to \infty} d(T x_n, T^{k+1} x_n) \leq a m. \]  
(2.10)
In the case $a < 0$, the inequality follows from (2.9). In the case $a \geq 0$, (2.10) follows from the inequality
\[ d(Tx_n, T^{k+1}x_n) \leq d(Tx_n, T^2x_n) + d(T^2x_n, T^3x_n) + \ldots + d(T^kx_n, T^{k+1}x_n). \]
Thus
\[ km \leq \lim_{n \to \infty} d(T^2x_n, T^{k+2}x_n) \]
\[ \leq a \lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) + b \lim_{n \to \infty} d(Tx_n, T^2x_n) + b \lim_{n \to \infty} d(T^{k+1}x_n, T^{k+2}x_n) + c \lim_{n \to \infty} d(Tx_n, T^{k+2}x_n) + c \lim_{n \to \infty} d(T^2x_n, T^{k+1}x_n) \]
\[ \leq akm + 2bm + c \lim_{n \to \infty} d(Tx_n, T^{k+2}x_n) + c(k-1)m, \]
and so
\[ m \leq \frac{c}{(c+2b)(k+1) - 4b} \lim_{n \to \infty} d(Tx_n, T^{k+2}x_n) \]
\[ \leq \frac{1}{k+1} \lim_{n \to \infty} d(Tx_n, T^{k+2}x_n), \]
and so (2.9) holds.
Let
\[ M = \max \left\{ \left( \frac{a+b+c}{b} \right) m, \left( \frac{2(b+c)}{1-c} \right) m, \left( \frac{b+c}{b} \right) m \right\}. \]
We show that for each $k \geq 2$,
\[ \lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \leq M. \tag{2.11} \]
To prove the claim, note that by (2.1) and (2.9) and for each $k \geq 2$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,
\[ \lambda d(x_n, Tx_n) \leq \lim_{n \to \infty} d(x_n, Tx_n) = m \leq \frac{1}{k} \lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \]
\[ \leq \frac{1}{k} \lim_{n \to \infty} d(x_n, T^{k}x_n) \leq d(x_n, T^kx_n). \]
Thus for sufficiently large $n$, we obtain
\[ d(Tx_n, T^{k+1}x_n) \leq ad(x_n, T^kx_n) + b(d(x_n, Tx_n) + d(T^kx_n, T^{k+1}x_n)) \]
\[ + c(d(x_n, T^{k+1}x_n) + d(Tx_n, T^{k+1}x_n)). \tag{2.12} \]
Assume first that $a \geq 0$. Then by (2.12),
\[ d(Tx_n, T^{k+1}x_n) \leq a(d(x_n, Tx_n) + d(Tx_n, T^{k+1}x_n) + d(Tx_n, T^{k+1}x_n)) \]
\[ + b(d(x_n, Tx_n) + d(T^kx_n, T^{k+1}x_n)) \]
\[ + c(d(x_n, Tx_n) + d(Tx_n, T^{k+1}x_n)) \]
and so by letting \( n \to \infty \), we get
\[
(1 - a - 2c) \lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \leq 2(a + b + c)m.
\]
Hence for each \( k \geq 2 \)
\[
\lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \leq \left( \frac{a + b + c}{b} \right) m \leq M.
\]
If \( a < 0 \) and \( a + c \leq 0 \), then by (2.12),
\[
d(Tx_n, T^{k+1}x_n) \leq ad(x_n, T^k x_n) + b(d(x_n, Tx_n) + d(T^k x_n, T^{k+1}x_n))
\]
\[
+ c(d(x_n, Tx_n) + d(Tx_n, T^{k+1}x_n)) + c(d(x_n, Tx_n) + d(x_n + T^k x_n)).
\]
Thus by letting \( n \to \infty \), we obtain
\[
(1 - c) \lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \leq (a + c) \lim_{n \to \infty} d(x_n, T^k x_n) + 2(b + c)m,
\]
and so (note that \( c < 1 \) and \( a + c \leq 0 \))
\[
\lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \leq \frac{2(b + c)}{1 - c} m \leq M.
\]
Now if \( a < 0 \) and \( a + c > 0 \), then thanks to (2.12),
\[
d(Tx_n, T^{k+1}x_n) \leq ad(x_n, T^k x_n) + b(d(x_n, Tx_n) + d(T^k x_n, T^{k+1}x_n))
\]
\[
+ (a + c)d(x_n, T^{k+1}x_n) - ad(x_n, T^{k+1}x_n) + cd(Tx_n, T^k x_n)
\]
\[
\leq (-a)(d(x_n, T^{k+1}x_n) - d(x_n, T^k x_n))
\]
\[
+ b(d(x_n, Tx_n) + d(T^k x_n, T^{k+1}x_n))
\]
\[
+ (a + c)(d(x_n, Tx_n) + d(Tx_n, T^{k+1}x_n))
\]
\[
+ c(d(Tx_n, T^{k+1}x_n) + d(T^k x_n, T^{k+1}x_n))
\]
\[
\leq (-a)d(T^k x_n, T^{k+1}x_n)
\]
\[
+ b(d(x_n, Tx_n) + d(T^k x_n, T^{k+1}x_n))
\]
\[
+ (a + c)(d(x_n, Tx_n) + d(Tx_n, T^{k+1}x_n))
\]
\[
+ c(d(Tx_n, T^{k+1}x_n) + d(T^k x_n, T^{k+1}x_n)).
\]
Thus by letting \( n \to \infty \),
\[
(1 - a - 2c) \lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \leq 2(b + c)m.
\]
Hence
\[
\lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \leq \left( \frac{b + c}{b} \right) m \leq M.
\]
Therefore (2.11) holds. Now from (2.9) and (2.11), we obtain
\[ m \leq \frac{1}{k} \lim_{n \to \infty} d(Tx_n, T^{k+1}x_n) \leq \frac{1}{k} M, \text{ for each } k \geq 2, \]
and so by letting \( k \to \infty \), we get \( m = 0 \), a contradiction. So, \( T \) has an a.f.p.s. \qed

Now we state our second main result.

**Theorem 3.** Let \( (X, d) \) be complete metric space and let \( T : X \to X \) be \((\frac{1}{2}, a, b, c)\)-generalized nonexpansive mapping with \( b > 0 \), \( c > 0 \) and \( b + c < 1 \). Then \( T \) has a unique fixed point.

**Proof.** For each \( n \in \mathbb{N} \) we set \( K_n := \{ x \in X : d(x, Tx) \leq \frac{1}{n} \} \). By Theorem 2, \( \inf_{x \in X} d(x, Tx) = 0 \), therefore for each \( n \in \mathbb{N}, K_n \neq \emptyset \). We show that \( \text{diam } (K_n) \to 0 \) as \( n \to \infty \), where for each \( n \in \mathbb{N} \), \( \text{diam } (K_n) := \sup_{x, y \in K_n} d(x, y) \). Let \( n \in \mathbb{N} \) and \( x, y \in K_n \). We have either \( d(x, y) \leq \frac{1}{n} \) or \( d(x, y) > \frac{1}{n} \). If \( d(x, y) > \frac{1}{n} \), then \( \frac{1}{2}d(x, Tx) \leq \frac{1}{n} < d(x, y) \). Hence
\[ d(Tx, Ty) \leq ad(x, y) + b(d(x, Tx) + d(y, Ty)) + c(d(x, Ty) + d(y, Tx)). \]
Then
\[
\begin{align*}
  d(x, y) &\leq d(x, Tx) + d(Tx, Ty) + d(y, Ty) \\
          &\leq \frac{2}{n} + ad(x, y) + b(d(x, Tx) + d(y, Ty)) + c(d(x, Ty) + d(y, Tx)) \\
          &\leq \frac{2}{n} + \frac{2b}{n} + ad(x, y) + c(d(x, y) + d(y, Ty) + d(x, Ty)) \\
          &\leq 2 + \frac{2b + 2c}{n} + (a + 2c)d(x, y) \\
          &\leq 2 + \frac{2b + 2c}{n} + (1 - 2b)d(x, y).
\end{align*}
\]
So
\[ d(x, y) \leq \frac{1 + b + c}{nb}. \]
Then for each \( x, y \in K_n \), we have
\[ d(x, y) \leq \max \left\{ \frac{1}{n}, \frac{1 + b + c}{nb} \right\}. \]
Thus \( \text{diam } (K_n) \to 0 \) when \( n \to \infty \). Since for each \( n \in \mathbb{N} \), \( \text{diam } (K_n) = \text{diam } (K_n) \) and \( (K_n) \) is a decreasing sequence of nonempty sets, so \( (K_n) \) is a decreasing sequence of nonempty closed sets such that \( \text{diam } (K_n) \to 0 \) as \( n \to \infty \). Therefore \( \bigcap_{n \in \mathbb{N}} K_n \) is singleton, say, \( \bigcap_{n \in \mathbb{N}} K_n = \{ x_0 \} \). Then there exists a sequence \( (x_n) \subseteq X \) with \( x_n \in K_n \) for each \( n \in \mathbb{N} \), such that \( x_n \to x_0 \). Now by Proposition 1 there exists \( \mu \geq 1 \), such that for each \( n \in \mathbb{N} \),
\[ d(x_n, Tx_0) \leq \mu d(x_n, Tx_n) + d(x_n, x_0). \]
Hence as \( n \to \infty \) we obtain \( d(x_0, T x_0) = 0 \), that is, \( x_0 \) is a fixed point for \( T \).

To prove the uniqueness, suppose that \( x, y \in X \) are two fixed point of \( T \). Since
\[
\frac{1}{4}d(x, T x) = 0 \leq d(x, y)
\]
then (note that \( b > 0 \))
\[
d(x, y) = d(T x, T y) \leq ad(x, y) + b(d(x, T x) + d(y, T y)) + c(d(x, T y) + d(y, T x))
\]
\[
\leq ad(x, y) + c(d(x, y) + d(y, T y) + d(x, y) + d(x, T x))
\]
\[
= (a + 2c)d(x, y) \leq (1 - 2b)d(x, y),
\]
and so \( x = y \). \( \square \)

In the following example, we define a metric space \((X, d)\) and a mapping \( T : X \to X \) which satisfies the assumptions of Theorem 3, but it is easy to see that \( T \) is not in any classes of generalized nonexpansive mappings which are listed in Example 1. This example also shows that Theorem 3 is a real generalization of Theorem 2.1 of [6].

**Example 2.** Let \((X, d)\) be a metric space such that \( X := \{1, 2, 3, 4, 5\} \) and \( d(1, 2) = d(4, 5) = d(3, 4) = 1, d(1, 5) = d(2, 3) = d(1, 3) = d(2, 5) = 1.7, d(1, 4) = d(2, 4) = 2.2 \) and \( d(3, 5) = 1.8 \). Let \( T : X \to X \) define by \( T 1 = 5, T 2 = 3 \) and \( T 3 = T 4 = T 5 = 4 \). It is straightforward to show that \( T \) is \((\frac{1}{2}, -0.2, 0.3, 0.3)\)-generalized nonexpansive mapping, and so by Theorem 3 has a unique fixed point. But we show that \( T \) does not satisfy the assumptions of Theorem 2.1 of [6]. On the contrary, assume that \( T \) is \((\frac{1}{2}, a, b, c)\)-generalized nonexpansive mapping with \( a \geq 0, b > 0 \) and \( c > 0 \). Then, for \( x = 1 \) and \( y = 2 \) we have \( \frac{1}{2}d(1, T 1) \leq d(1, 2) \). Then (note that \( a \geq 0 \))
\[
1.8 = d(5, 3) = d(T 1, T 2)
\]
\[
\leq ad(1, 2) + b(d(1, T 1) + d(2, T 2)) + c(d(1, T 2) + d(2, T 1))
\]
\[
= a + (2b + 2c)1.7
\]
\[
\leq (a + 2b + 2c)1.7 = 1.7,
\]
a contradiction.

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