Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1117–1126

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4103

# A FIXED POINT THEOREM FOR GENERALIZED NONEXPANSIVE MAPPINGS

ALIREZA AMINI-HARANDI, MAHDI GOLI, AND HAMID REZA HAJISHARIFI

*Received 25 January, 2022*

*Abstract.* In this paper, we obtain a generalization of a fixed point theorem given by Popescu [O. Popescu, Comput. Math. Appl., vol. 62, no. 10, pp. 3912–3919, 2011]. An example is also given to support our main result.

2010 *Mathematics Subject Classification:* 47H04; 47H10; 54H25

*Keywords:* fixed point, approximate fixed point sequence, metric space, generalized nonexpansive mapping

## 1. Introduction and Preliminares

Let $(X,d)$ be a metric space and let $T: X \to X$ be a mapping. We say that $x \in X$ is a fixed point for $T$, if $Tx = x$. The sequence $(x_n) \subseteq X$ is said to be an approximate fixed point sequence (a.f.p.s., for short) for $T$, if $d(x_n, Tx_n) \to 0$.

In 2011, Popescu [6] proved the following result.

**Theorem 1.** *Let $(X,d)$ be a complete metric space and let $T: X \to X$ be a mapping satisfying*

$$\frac{1}{2}d(x,Tx) \le d(x,y) \;\Rightarrow$$
$$d(Tx,Ty) \le ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)),$$

*for all $x,y \in X$, with $a \ge 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then, $T$ has a unique fixed point.*

**Definition 1.** Let $(X,d)$ be a metric space, $\lambda \in [0,1)$ and let $a,b,c \in \mathbb{R}$ with $a + 2b + 2c = 1$. We say that a mapping $T: X \to X$ is a $(\lambda,a,b,c)$-generalized nonexpansive mapping if for all $x,y \in X$,

$$\lambda d(x,Tx) \le d(x,y) \;\Rightarrow$$
$$d(Tx,Ty) \le ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)).$$

The third author was in part supported by a grant from IPM (No.1402460315).

In the following example, we see that many previously known classes of generalized nonexpansive mappings are actually $(\lambda,a,b,c)$-generalized nonexpansive, for some $\lambda\in[0,1)$ and $a,b,c\in\mathbb{R}$.

*Example* 1.

(i) Every nonexpansive mapping is $(0,1,0,0)$-generalized nonexpansive.
(ii) Every mapping which satisfies condition $(C)$ [2,4] is $(\frac{1}{2},1,0,0)$-generalized nonexpansive.
(iii) Every Kannan nonexpansive mapping [5] is $(0,0,0,1)$-generalized nonexpansive.
(iv) Every Reich nonexpansive mapping [5] is $(0,a,\alpha,0)$-generalized nonexpansive, where $a\geq 0$ and $\alpha\geq 0$.
(v) Every generalized $\alpha$-nonexpansive mapping [5] is $(\frac{1}{2},1-2\alpha,0,\alpha)$-generalized nonexpansive, where $\alpha\in[0,1)$.
(vi) Every Reich-Suzuki nonexpansive mapping [5] is $(\frac{1}{2},1-2\alpha,\alpha,0)$-generalized nonexpansive, where $\alpha\in[0,1)$.
(vii) Every Gobel- Kirk-Shimi generalized nonexpansive mapping [1] is $(0,a,b,c)$-generalized nonexpansive, where $a\geq 0$, $b>0$ and $c>0$.
(viii) The class of $(0,a,b,0)$-generalized nonexpansive mappings, where $a>0$ and $b>0$, was defined and studied by Greguš [3].
(ix) The class of $(\frac{1}{2},a,b,c)$-generalized nonexpansive mappings, where $a\geq 0$, $b>0$ and $c>0$, was defined and studied by Popescu [6].

In connection with Theorem 1, the following problem arises:

**Problem 1.** *Determine all $(\lambda,a,b,c)\in[0,1)\times\mathbb{R}^3$ with $a+2b+2c=1$, such that every $(\lambda,a,b,c)$-generalized nonexpansive mapping $T:X\to X$ has a fixed point, where $(X,d)$ is a complete metric space.*

Fixed point theory for generalized nonexpansive mappings, that recall in example 1, have been studied by some authors [1–6]. In this paper, we give a partial answer to the problem 1, which is also a generalization of Theorem 1.

## 2. Main result

García-Falset et al. [2], studied a class of mappings satisfying the following condition.

**Definition 2.** Let $C$ be a nonempty subset of a metric space $X$. For $\mu\geq 1$, we say that a mapping $T:C\to X$ satisfies condition $(E_\mu)$ on $C$ if for each $x,y\in C$,

$$d(x,Ty)\leq \mu d(x,Tx)+d(x,y).$$

We say that $T$ satisfies condition $(E)$ on $C$ whenever $T$ satisfies $(E_\mu)$ on $C$ for some $\mu\geq 1$.

**Proposition 1.** *Let $T: X \to X$ be $(\frac{1}{2}, a, b, c)$-generalized nonexpansive mapping with $b \geq 0$, $c \geq 0$ and $b + c < 1$. Then, $T$ satisfies condition $(E)$ with $\mu = \frac{3+b+c}{1-b-c}$.*

*Proof.* Since $\frac{1}{2}d(x,Tx) \leq d(x,Tx)$, for each $x \in X$, then

$$\begin{aligned} d(Tx,T^2x) &\leq ad(x,Tx) + b(d(x,Tx) + d(Tx,T^2x)) + cd(x,T^2x) \\ &\leq ad(x,Tx) + b(d(x,Tx) + d(Tx,T^2x)) + c(d(x,Tx) + d(Tx,T^2x)). \end{aligned}$$

Using the assumptions $b + c < 1$ and $a + 2b + 2c = 1$, from the above we conclude that (note that the above inequality holds for any $(\lambda, a, b, c)$-generalized nonexpansive mapping with $\lambda \in [0,1)$),

$$d(Tx,T^2x) \leq d(x,Tx) \quad \text{for each } x \in X. \tag{2.1}$$

Now, we prove that for each $x, y \in X$,

$$\text{either} \quad \frac{1}{2}d(x,Tx) \leq d(x,y) \quad \text{or} \quad \frac{1}{2}d(Tx,T^2x) \leq d(Tx,y). \tag{2.2}$$

On the contrary, we assume that there exist $x, y \in X$ such that

$$\frac{1}{2}d(x,Tx) > d(x,y) \quad \text{and} \quad \frac{1}{2}d(Tx,T^2x) > d(Tx,y).$$

Hence by (2.1), we have

$$\begin{aligned} d(x,Tx) &\leq d(x,y) + d(y,Tx) \\ &< \frac{1}{2}d(x,Tx) + \frac{1}{2}d(Tx,T^2x) \\ &\leq \frac{1}{2}d(x,Tx) + \frac{1}{2}d(x,Tx) = d(x,Tx), \end{aligned}$$

a contradiction. So, (2.2) holds for each $x, y \in X$.
Now, from (2.2) we get that for each $x, y \in X$, either

$$d(Tx,Ty) \leq ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)), \tag{2.3}$$

or

$$d(T^2x,Ty) \leq ad(Tx,y) + b(d(Tx,T^2x) + d(y,Ty)) + c(d(Tx,Ty) + d(y,T^2x)). \tag{2.4}$$

Let $x, y \in X$. We first assume that (2.3) holds. Then

$$\begin{aligned} d(x,Ty) &\leq d(x,Tx) + d(Tx,Ty) \\ &\leq d(x,Tx) + ad(x,y) \\ &\quad + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)) \\ &\leq (1+b+c)d(x,Tx) + (b+c)d(x,Ty) + (1-b-c)d(x,y). \end{aligned}$$

Since $b+c<1$, we obtain

$$\begin{aligned} d(x,Ty) &\le (\frac{1+b+c}{1-b-c})d(x,Tx)+d(x,y) \\ &\le (\frac{3+b+c}{1-b-c})d(x,Tx)+d(x,y). \end{aligned}$$

Now assume that (2.4) holds. Then by (2.1), we have

$$\begin{aligned} d(x,Ty) &\le d(x,Tx)+d(Tx,T^2x)+d(T^2x,Ty) \\ &\le 2d(x,Tx)+ad(Tx,y) \\ &\quad +b(d(Tx,T^2x)+d(y,Ty))+c(d(Tx,Ty)+d(y,T^2x)) \\ &\le 2d(x,Tx)+ad(Tx,y) \\ &\quad +b(d(Tx,T^2x)+d(y,Tx)+d(x,Tx)+d(x,Ty)) \\ &\quad +c(d(x,Tx)+d(x,Ty)+d(Tx,y)+d(Tx,T^2x)) \\ &\le 2d(x,Tx)+ad(Tx,y) \\ &\quad +b(d(y,Tx)+2d(x,Tx)+d(x,Ty)) \\ &\quad +c(d(x,Ty)+d(Tx,y)+2d(x,Tx)). \end{aligned}$$

Since $b+c<1$, we obtain

$$\begin{aligned} (1-b-c)d(x,Ty) &\le (2+2b+2c)d(x,Tx)+(1-b-c)d(Tx,y) \\ &\le (2+2b+2c)d(x,Tx)+(1-b-c)(d(x,Tx)+d(x,y)). \end{aligned}$$

Hence

$$d(x,Ty) \le (\frac{3+b+c}{1-b-c})d(x,Tx)+d(x,y).$$

□

Now, we are ready to state our first main result.

**Theorem 2.** *Let $(X,d)$ be a complete metric space and let $T: X\to X$ be a $(\lambda,a,b,c)$-generalized nonexpansive mapping with $\lambda\in[0,1)$, $b>0$, $c>0$ and $b+c<1$. Then, $T$ has an a.f.p.s.*

*Proof.* Let $m=\inf_{x\in X} d(x,Tx)$. On the contrary, assume that $m>0$. Let $(x_n)$ be a sequence in $X$ such that

$$m=\lim_{n\to\infty} d(x_n,Tx_n). \tag{2.5}$$

By the diagonal argument, we may assume all the following limits exist:

$$\lim_{n\to\infty} d(T^ix_n,T^jx_n), \quad \forall\, i,j\in\mathbb{N}\cup\{0\}. \tag{2.6}$$

Replacing $x$ by $Tx_n$ and $y$ by $T^2x_n$ in Definition 1, for each $n\in\mathbb{N}$ we obtain

$$d(T^2x_n,T^3x_n)\le ad(Tx_n,T^2x_n)+bd(Tx_n,T^2x_n)$$

$$+bd(T^2x_n,T^3x_n)+cd(Tx_n,T^3x_n).$$

Thus

$$\begin{aligned} m &= \lim_{n\to\infty} d(T^2x_n,T^3x_n) \\ &\le a\lim_{n\to\infty} d(Tx_n,T^2x_n)+b\lim_{n\to\infty} d(Tx_n,T^2x_n) \\ &\quad +b\lim_{n\to\infty} d(T^2x_n,T^3x_n)+c\lim_{n\to\infty} d(Tx_n,T^3x_n). \end{aligned} \tag{2.7}$$

From (2.1) and (2.5), we obtain

$$m=\lim_{n\to\infty} d(T^kx_n,T^{k+1}x_n), \quad \text{for each } k\in\mathbb{N}. \tag{2.8}$$

Hence by using (2.8) and (2.7), we get

$$m\le \frac{1}{2}\lim_{n\to\infty} d(Tx_n,T^3x_n),$$

and so (note the conditions (2.5) and (2.6) are also satisfied by $(Tx_n)$)

$$m\le \frac{1}{2}\lim_{n\to\infty} d(T^2x_n,T^4x_n).$$

Now we prove by induction that

$$m\le \frac{1}{k}\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n) \quad \text{for each } k\ge 2. \tag{2.9}$$

Assume that (2.9) holds for some $k\ge 2$, then

$$m\le \frac{1}{k}\lim_{n\to\infty} d(T^2x_n,T^{k+2}x_n).$$

Therefore from (2.9) and for sufficiently large $n$, we have

$$\begin{aligned} \lambda d(Tx_n,T^2x_n) &\le \lim_{n\to\infty} d(Tx_n,T^2x_n)=m \\ &\le \frac{1}{k}\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n) \\ &\le d(Tx_n,T^{k+1}x_n), \end{aligned}$$

and so, we may assume that

$$\lambda d(Tx_n,T^2x_n)\le d(Tx_n,T^{k+1}x_n) \ \text{ for each } n\in\mathbb{N}.$$

Replacing $x$ by $Tx_n$ and $y$ by $T^{k+1}x_n$ in Definition (1), for each $n\in\mathbb{N}$ we obtain

$$\begin{aligned} d(T^2x_n,T^{k+2}x_n) &\le ad(Tx_n,T^{k+1}x_n) \\ &\quad +bd(Tx_n,T^2x_n)+bd(T^{k+1}x_n,T^{k+2}x_n) \\ &\quad +cd(Tx_n,T^{k+2}x_n)+cd(T^2x_n,T^{k+1}x_n). \end{aligned}$$

Now we claim that

$$a\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n)\le akm. \tag{2.10}$$

In the case $a<0$, the inequality follows from (2.9). In the case $a\geq 0$, (2.10) follows from the inequality

$$d(Tx_n,T^{k+1}x_n)\leq d(Tx_n,T^2x_n)+d(T^2x_n,T^3x_n)+\ldots+d(T^kx_n,T^{k+1}x_n).$$

Thus

$$\begin{aligned}
km &\leq \lim_{n\to\infty} d(T^2x_n,T^{k+2}x_n)\\
&\leq a\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n)\\
&\quad + b\lim_{n\to\infty} d(Tx_n,T^2x_n)+b\lim_{n\to\infty} d(T^{k+1}x_n,T^{k+2}x_n)\\
&\quad + c\lim_{n\to\infty} d(Tx_n,T^{k+2}x_n)+c\lim_{n\to\infty} d(T^2x_n,T^{k+1}x_n)\\
&\leq akm+2bm+c\lim_{n\to\infty} d(Tx_n,T^{k+2}x_n)+c(k-1)m,
\end{aligned}$$

and so

$$\begin{aligned}
m &\leq \frac{c}{(c+2b)(k+1)-4b}\lim_{n\to\infty} d(Tx_n,T^{k+2}x_n)\\
&\leq \frac{1}{k+1}\lim_{n\to\infty} d(Tx_n,T^{k+2}x_n),
\end{aligned}$$

and so (2.9) holds.
Let

$$M=\max\left\{\left(\frac{a+b+c}{b}\right)m,\left(\frac{2(b+c)}{1-c}\right)m,\left(\frac{b+c}{b}\right)m\right\}.$$

We show that for each $k\geq 2$,

$$\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n)\leq M. \tag{2.11}$$

To prove the claim, note that by (2.1) and (2.9) and for each $k\geq 2$, there exists $N\in\mathbb{N}$ such that for $n\geq N$,

$$\begin{aligned}
\lambda d(x_n,Tx_n) &\leq \lim_{n\to\infty} d(x_n,Tx_n)=m\leq \frac{1}{k}\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n)\\
&\leq \frac{1}{k}\lim_{n\to\infty} d(x_n,T^kx_n)\leq d(x_n,T^kx_n).
\end{aligned}$$

Thus for sufficiently large $n$, we obtain

$$\begin{aligned}
d(Tx_n,T^{k+1}x_n) &\leq ad(x_n,T^kx_n)+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\
&\quad +c(d(x_n,T^{k+1}x_n)+d(Tx_n,T^kx_n)).
\end{aligned}\tag{2.12}$$

Assume first that $a\geq 0$. Then by (2.12),

$$\begin{aligned}
d(Tx_n,T^{k+1}x_n) &\leq a(d(x_n,Tx_n)+d(Tx_n,T^{k+1}x_n)+d(T^kx_n,T^{k+1}x_n))\\
&\quad +b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\
&\quad +c(d(x_n,Tx_n)+d(Tx_n,T^{k+1}x_n))
\end{aligned}$$

$$+c(d(Tx_n,T^{k+1}x_n)+d(T^kx_n+T^{k+1}x_n)),$$

and so by letting $n\to\infty$, we get

$$(1-a-2c)\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\leq 2(a+b+c)m.$$

Hence for each $k\geq 2$

$$\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\leq\left(\frac{a+b+c}{b}\right)m\leq M.$$

If $a<0$ and $a+c\leq 0$, then by (2.12),

$$\begin{aligned}d(Tx_n,T^{k+1}x_n)&\leq ad(x_n,T^kx_n)+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\&\quad+c(d(x_n,Tx_n)+d(Tx_n,T^{k+1}x_n))+c(d(x_n,Tx_n)+d(x_n+T^kx_n)).\end{aligned}$$

Thus by letting $n\to\infty$, we obtain

$$(1-c)\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\leq(a+c)\lim_{n\to\infty}d(x_n,T^kx_n)+2(b+c)m,$$

and so (note that $c<1$ and $a+c\leq 0$ )

$$\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\leq\frac{2(b+c)}{1-c}m\leq M.$$

Now if $a<0$ and $a+c>0$, then thanks to (2.12),

$$\begin{aligned}d(Tx_n,T^{k+1}x_n)&\leq ad(x_n,T^kx_n)+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\&\quad+(a+c)d(x_n,T^{k+1}x_n)-ad(x_n,T^{k+1}x_n)+cd(Tx_n,T^kx_n)\\&\leq(-a)(d(x_n,T^{k+1}x_n)-d(x_n,T^kx_n))\\&\quad+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\&\quad+(a+c)(d(x_n,Tx_n)+d(Tx_n,T^{k+1}x_n))\\&\quad+c(d(Tx_n,T^{k+1}x_n)+d(T^kx_n,T^{k+1}x_n))\\&\leq(-a)d(T^kx_n,T^{k+1}x_n)\\&\quad+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\&\quad+(a+c)(d(x_n,Tx_n)+d(Tx_n,T^{k+1}x_n))\\&\quad+c(d(Tx_n,T^{k+1}x_n)+d(T^kx_n,T^{k+1}x_n)).\end{aligned}$$

Thus by letting $n\to\infty$,

$$(1-a-2c)\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\leq 2(b+c)m.$$

Hence

$$\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\leq(\frac{b+c}{b})m\leq M.$$

Therefore (2.11) holds. Now from (2.9) and (2.11), we obtain

$$m \leq \frac{1}{k} \lim_{n\to\infty} d(Tx_n, T^{k+1}x_n) \leq \frac{1}{k}M, \quad \text{for each } k \geq 2,$$

and so by letting $k \to \infty$, we get $m = 0$, a contradiction. So, $T$ has an a.f.p.s. □

Now we state our second main result.

**Theorem 3.** *Let $(X,d)$ be complete metric space and let $T : X \to X$ be $(\frac{1}{2}, a, b, c)$-generalized nonexpansive mapping with $b > 0$, $c > 0$ and $b + c < 1$. Then, $T$ has a unique fixed point.*

*Proof.* For each $n \in \mathbb{N}$ we set $K_n := \{x \in X : d(x,Tx) \leq \frac{1}{n}\}$. By Theorem 2, $\inf_{x\in X} d(x,Tx) = 0$, therefore for each $n \in \mathbb{N}$, $K_n \neq \varnothing$. We show that diam $(K_n) \to 0$ as $n \to \infty$, where for each $n \in \mathbb{N}$, diam $(K_n) := \sup_{x,y\in K_n} d(x,y)$. Let $n \in \mathbb{N}$ and $x, y \in K_n$. We have either $d(x,y) \leq \frac{1}{n}$ or $d(x,y) > \frac{1}{n}$. If $d(x,y) > \frac{1}{n}$, then $\frac{1}{2}d(x,Tx) \leq \frac{1}{n} < d(x,y)$. Hence

$$d(Tx,Ty) \leq ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)).$$

Then

$$\begin{aligned}
d(x,y) &\leq d(x,Tx) + d(Tx,Ty) + d(y,Ty) \leq \frac{2}{n} + d(Tx,Ty) \\
&\leq \frac{2}{n} + ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)) \\
&\leq \frac{2}{n} + \frac{2b}{n} + ad(x,y) + c(d(x,y) + d(y,Ty) + d(x,y) + d(x,Tx)) \\
&\leq \frac{2 + 2b + 2c}{n} + (a + 2c)d(x,y) \\
&\leq \frac{2 + 2b + 2c}{n} + (1 - 2b)d(x,y).
\end{aligned}$$

So

$$d(x,y) \leq \frac{1 + b + c}{nb}.$$

Then for each $x, y \in K_n$, we have

$$d(x,y) \leq \max\left\{\frac{1}{n}, \frac{1 + b + c}{nb}\right\}.$$

Thus diam $(K_n) \to 0$ when $n \to \infty$. Since for each $n \in \mathbb{N}$, diam $(\overline{K_n}) = diam(K_n)$ and $(K_n)$ is a decreasing sequence of nonempty sets, so $(\overline{K_n})$ is a decreasing sequence of nonempty closed sets such that diam $(\overline{K_n}) \to 0$ as $n \to \infty$. Therefore $\bigcap_{n\in\mathbb{N}} \overline{K_n}$ is singleton, say, $\bigcap_{n\in\mathbb{N}} \overline{K_n} = \{x_0\}$. Then there exists a sequence $(x_n) \subseteq X$ with $x_n \in K_n$ for each $n \in \mathbb{N}$, such that $x_n \to x_0$. Now by Proposition 1 there exists $\mu \geq 1$, such that for each $n \in \mathbb{N}$,

$$d(x_n, Tx_0) \leq \mu d(x_n, Tx_n) + d(x_n, x_0).$$

Hence as $n \to \infty$ we obtain $d(x_0, Tx_0) = 0$, that is, $x_0$ is a fixed point for $T$.
To prove the uniqueness, suppose that $x, y \in X$ are two fixed point of $T$. Since $\frac{1}{2}d(x, Tx) = 0 \leq d(x, y)$ then ( note that $b > 0$ )

$$\begin{aligned} d(x,y) = d(Tx,Ty) &\leq ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)) \\ &\leq ad(x,y) + c(d(x,y) + d(y,Ty) + d(x,y) + d(x,Tx)) \\ &= (a+2c)d(x,y) \leq (1-2b)d(x,y), \end{aligned}$$

and so $x = y$. □

In the following example, we define a metric space $(X, d)$ and a mapping $T : X \to X$ which satisfies the assumptions of Theorem 3, but it is easy to see that $T$ is not in any classes of generalized nonexpansive mappings which are listed in Example 1. This example also shows that Theorem 3 is a real generalization of Theorem 2.1 of [6].

*Example* 2. Let $(X, d)$ be a metric space such that $X := \{1, 2, 3, 4, 5\}$ and $d(1, 2) = d(4, 5) = d(3, 4) = 1$, $d(1, 5) = d(2, 3) = d(1, 3) = d(2, 5) = 1.7$, $d(1, 4) = d(2, 4) = 2.2$ and $d(3, 5) = 1.8$. Let $T : X \to X$ define by $T1 = 5$, $T2 = 3$ and $T3 = T4 = T5 = 4$. It is straightforward to show that $T$ is $(\frac{1}{2}, -0.2, 0.3, 0.3)$-generalized nonexpansive mapping, and so by Theorem 3 has a unique fixed point. But we show that $T$ does not satisfy the assumptions of Theorem 2.1 of [6]. On the contrary, assume that $T$ is $(\frac{1}{2}, a, b, c)$-generalized nonexpansive mapping with $a \geq 0$, $b > 0$ and $c > 0$. Then, for $x = 1$ and $y = 2$ we have $\frac{1}{2}d(1, T1) \leq d(1, 2)$. Then (note that $a \geq 0$)

$$\begin{aligned} 1.8 = d(5,3) &= d(T1,T2) \\ &\leq ad(1,2) + b(d(1,T1) + d(2,T2)) + c(d(1,T2) + d(2,T1)) \\ &= a + (2b+2c)1.7 \\ &\leq (a+2b+2c)1.7 = 1.7, \end{aligned}$$

a contradiction.

## ACKNOWLEDGEMENTS

The authors would like to thank the associate editor and reviewers for their constructive comments, which helped us to improve the paper.

## REFERENCES

[1] J. Bogin, “A generalization of a fixed point theorem of Goebel, Kirk and Shimi,” *Canad. Math. Bull.*, vol. 19, no. 1, pp. 7–12, 1976, doi: 10.4153/CMB-1976-002-7. [Online]. Available: https://doi.org/10.4153/CMB-1976-002-7

[2] J. García-Falset, E. Llorens-Fuster, and T. Suzuki, “Fixed point theory for a class of generalized nonexpansive mappings,” *J. Math. Anal. Appl.*, vol. 375, no. 1, pp. 185–195, 2011, doi: 10.1016/j.jmaa.2010.08.069. [Online]. Available: https://doi.org/10.1016/j.jmaa.2010.08.069

[3] M. Greguš, Jr., "A fixed point theorem in Banach space," *Boll. Un. Mat. Ital. A (5)*, vol. 17, no. 1, pp. 193–198, 1980.

[4] G. A. Okeke and C. I. Ugwuogor, "Iterative construction of the fixed point of Suzuki's generalized nonexpansive mappings in Banach spaces," *Fixed Point Theory*, vol. 23, no. 2, pp. 633–651, 2022, doi: 10.24193/fpt-ro. [Online]. Available: https://doi.org/10.24193/fpt-ro

[5] R. Pandey, R. Pant, V. Rakočević, and R. Shukla, "Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications," *Results Math.*, vol. 74, no. 1, pp. Paper No. 7, 24, 2019, doi: 10.1007/s00025-018-0930-6. [Online]. Available: https://doi.org/10.1007/s00025-018-0930-6

[6] O. Popescu, "Two generalizations of some fixed point theorems," *Comput. Math. Appl.*, vol. 62, no. 10, pp. 3912–3919, 2011, doi: 10.1016/j.camwa.2011.09.044. [Online]. Available: https://doi.org/10.1016/j.camwa.2011.09.044

*Authors' addresses*

**Alireza Amini-Harandi**

Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, 81746-73441, Iran

*E-mail address:* `a.amini@sci.ui.ac.ir`

**Mahdi Goli**

Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, 81746-73441, Iran

*E-mail address:* `golimahdi@gmail.com`

**Hamid Reza Hajisharifi**

(**Corresponding author**) Department of Mathematics, Khansar Campus, University of Isfahan, Iran, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box19395-5746, Tehran, Iran

*E-mail address:* `h.r.hajisharifi@sci.ui.ac.ir`