Miskolc Mathematical Notes
Vol. 23 (2022), No. 1, pp. 85–92
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.4101

EXISTENCE OF SOLUTION FOR A DIRICHLET BOUNDARY VALUE PROBLEM INVOLVING THE $p(x)$ LAPLACIAN VIA A FIXED POINT APPROACH

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Received 23 January, 2022

Abstract. In this paper, we study the existence of a non-trivial solution in $W_0^{1,p(x)}(\Omega)$ for the problem

$$\begin{cases} \Delta_{p(x)}u = f(x,u,\nabla u) \text{ in } \Omega, \\ u = 0 \text{ in } \Omega. \end{cases}$$

The proof is based on Schaefer's fixed point theorem.

2010 *Mathematics Subject Classification:* 34B15; 34A12; 54H25

Keywords: $p(x)$-Laplacian, generalized Sobolev space, variable exponent, fixed point.

1. INTRODUCTION

Let us consider the problem

$$\begin{cases} -\Delta_{p(x)}u = f(x,u,\nabla u), & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 2,\ p(x) \in C(\overline{\Omega})$ is log-Hölder continuous with values in $(1,+\infty)$,

$$-\Delta_{p(x)}(u) = div\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right)$$

and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying some growth conditions.

According to the behavior of f, the existence results of weak solutions for Dirichlet boundary value problems involving $p(x)$-Laplacian have been established in [3, 4, 6, 10–12] where variational and topological methods were used in [10] while the proof relies on simple variational arguments based on Mountain-Pass Theorem in [12].

Our contribution in this area is to prove the existence of nontrivial solutions for the Problem (1.1) using variable exponent and a fixed point theorem, in particular, Schaefer's fixed point theorem. This paper is divided into three sections. In the following section, we will introduce some tools which we need to prove our main result in the last section.

2. Preliminaries

Let us summary the most important results on variable exponent Sobolev space and the basic properties of the $p(x)$-Laplacian and the Nemytskii operators. For more details, we orient the reader to [3, 5–8, 10–14] and references therein. We also recall Schaefer's and Schauder's fixed point theorems [1, 2, 9, 15].

$\Omega \subset \mathbb{R}^N, N \geq 2$ denotes an open bounded subset with smooth boundary $\partial\Omega$ and set

$$C_+(\overline{\Omega}) := \{g : g \in C(\overline{\Omega}),\ g(x) > 1 \text{ for all } x \in \overline{\Omega}\},$$
$$D = \{u : \Omega \to \mathbb{R}; \text{ is a measurable real-valued function}\}.$$

For any $p \in C_+(\overline{\Omega})$, we denote $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty$ and

$$L^{p(x)}(\Omega) = \left\{ u \in D : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

On this space, the so-called Luxemburg norm is defined by the following formula

$$|u|_{p(x)} = \inf \left\{ \tau > 0 : \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}.$$

$\left(L^{p(x)}(\Omega), |u|_{p(x)}\right)$ are Banach spaces. There exists a fact that they are reflexive if and only if $1 < p^- \leq p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturality: if $0 < |\Omega| < \infty$ and $p_1(\cdot)$, $p_2(\cdot)$ are variable exponents such that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$. Then, there exists a continuous embedding from $L^{p_2(x)}(\Omega)$ to $L^{p_1(x)}(\Omega)$.

Moreover, if $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, then we have Hölder-type inequality [3]

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)},\ \ u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega). \tag{2.1}$$

Later we need to use the modular and its properties, which is a mapping

$$\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$$

defined by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx.$$

Proposition 1. *(See [3]) For all $u, v \in L^{p(x)}(\Omega)$, we have*

$$|u|_{p(x)} < 1 \ (resp. = 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 \ (resp. = 1, > 1). \tag{2.2}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}. \tag{2.3}$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}. \tag{2.4}$$

$$\rho_{p(x)}(u-v) \to 0 \Leftrightarrow |u-v|_{p(x)} \to 0. \tag{2.5}$$

From (2.3) and (2.4), it follows that

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1 \tag{2.6}$$

and

$$\rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} + |u|_{p(x)}^{p^-}. \tag{2.7}$$

The variable Sobolev space $W^{1,p(x)}(\Omega)$ is defined as follows:

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

and it is equipped with the norm

$$\|u\|_{1,p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

$W_0^{1,p(x)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)}, & \text{if } \ p(x) < N \\ +\infty, & \text{if } \ p(x) \geq N. \end{cases}$$

We have the following proposition.

Proposition 2. *(See [5, 7, 12])*

(1) *$W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.*
(2) *If $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{\alpha(x)}(\Omega)$ is continuous and compact.*
(3) *There exists a constant $\theta_0 > 0$ such that*

$$|u|_{p(x)} \leq \theta_0 |\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{2.8}$$

According to this last assertion, $|\nabla u|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

We close this section with the following fixed point theorem which will be used in this paper.

Theorem 1. *(Classical Schaefer's fixed point theorem) [15]*
Let X be a normed space, T a continuous mapping of X into X which is compact on each bounded subset of X. Then, either

- *the equation $u = \lambda T u$ has a solution for $\lambda = 1$, or*
- *the set of all such solutions u, for $\ 0 < \lambda < 1\ $ is unbounded.*

3. Main Result

In this section, we first consider the following problem

$$\begin{cases} -\Delta_{p(x)}u = f(x,u,\nabla u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $p(x) \in C(\overline{\Omega})$ is log-Hölder continuous with values in $(1,+\infty)$, $1 < p^- \leq p(x) \leq p^+ < \infty$, $p'(x)$ the conjugate of $p(x)$, $-\Delta_{p(x)}(u) = \mathrm{div}\big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\big)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Caratheodory function satisfying the following growth condition.

$$(A_1) \quad |f(x,s,t)| \leq \phi(x) + c|s|^{\eta(x)-1} + c|t|^{\eta(x)-1}, \ , \ \forall (x,s,t) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

where

$$\phi(x) \in L^{p'(x)}(\Omega),\ c > 0, \eta(x) \in C_+(\overline{\Omega}),\ \eta(x) < p^*(x),\ 1 < \eta^- \leq \eta(x) \leq \eta^+ < p^-.$$

Let θ_0 be such that $|u|_{p(x)} \leq \theta_0 |\nabla u|_{p(x)}$, for all $u \in W_0^{1,p(x)}(\Omega)$ and θ_1 the constant of the embedding of $L^{p(x)}(\Omega)$ into $L^{\eta(x)}(\Omega)$ such that

$$(A_2)\ \ 0 < c < \frac{1}{4\lambda^{p^+-1}\left((\theta_0\theta_1)^{p^-} + \theta_1^{p^-}\right)},\ |\phi|_{p'(x)} < \frac{1 - 4c\lambda^{p^+-1}\left((\theta_0\theta_1)^{p^-} + \theta_1^{p^-}\right)}{2\lambda^{p^+-1}\theta_0},$$

for some $\lambda \in [0\ 1]$. If u is a weak solution of the Problem (1.1), then for all $v \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx = \int_\Omega f(x,u,\nabla u) v \, dx. \tag{3.1}$$

Let $K = W_0^{1,p(x)}(\Omega)$ and K^* its dual, that is $(W_0^{1,p(x)}(\Omega))^* = K^*$. We define the operators $(-\Delta_{p(x)})$ and Φ as follows:

$$-\Delta_{p(x)} : \ K \to K^*$$
$$\langle -\Delta_{p(x)}u, v\rangle = \int_\Omega |\nabla|(u)|^{p(x)-2}\nabla u \nabla v \, dx, \ \ \forall u,v \in K$$

and

$$\Phi : \ K^* \to K$$
$$\langle \Phi u, v\rangle = \int_\Omega f(x,u,\nabla u) v \, dx, \ \ \forall u,v \in K.$$

If (A_1) is satisfied, then the following propositions give the most properties of $-\Delta_{p(x)}$ and Φ.

Proposition 3. *[3, 6]*

(1) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)'$ *is a homeomorphism from* $W_0^{1,p(x)}(\Omega)$ *to* $\left(W_0^{1,p(x)}(\Omega)\right)'$.

(2) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)'$ *is a continuous, bounded and strictly monotone operator.*

(3) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)'$ *is a mapping of type* S_+.

(4) *The operator* $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)'$ *has a continuous inverse mapping* $\left(-\Delta_{p(x)}\right)^{-1} : \left(W_0^{1,p(x)}(\Omega)\right)' \to W_0^{1,p(x)}(\Omega)$.

(5) $\left(-\Delta_{p(x)}\right)^{-1}$ *is bounded and satisfies condition* S_+.

Proposition 4. *[3] The operator* Φ *is compact.*

One can see that if u is a weak solution for the Problem (1.1) means that

$$\langle -\Delta_{p(x)}u, v\rangle = \langle \Phi u, v\rangle, \ \ \forall v \in K.$$

On the other hand, we have the following equivalence:

$$-\Delta_{p(x)}u = \Phi(u) \Longleftrightarrow u = (-\Delta_{p(x)})^{-1}\Phi u.$$

Let T be the operator defined by

$$T : K \to K, \ \ Tu = (-\Delta_{p(x)})^{-1}\Phi u.$$

Now, we state our main result which will be proved by using Schaefer's fixed point theorem.

Theorem 2. *Under assumptions* (A_1) *and* (A_2)*, the Problem* (1.1) *admits at least a nontrivial solution.*

It is easily seen that a solution of the Problem (1.1) is a fixed point of the operator T. So, we will prove that T is well defined and all the conditions of Theorem 1 are satisfied.

Proof. (1) T is well defined. Indeed, the operator T can be considered as follows:

$$T : \ K \xrightarrow{\Phi} K^* \xrightarrow{(-\Delta_{p(x)})^{-1}} K.$$

(2) T is compact. In fact, let (u_n) be a bounded sequence in the reflexive space K. Then, there exist u_0 and a subsequence which we also denote (u_n) such that u_n converges weakly to u_0 in K. By the compactness of Φ (Lemma 2 in [3]), we have $\Phi(u_n) \to \Phi(u_0)$ in K^*. Using the continuity of the operator $(-\Delta_{p(x)})^{-1}$, then we get

$$(-\Delta_{p(x)})^{-1}\overline{F}(u_n) \to (-\Delta_{p(x)})^{-1}\overline{F}(u_0).$$

That is, $T(u_n)$ converges strongly to $T(u_0)$. We mention that T is also continuous.

(3) We claim that the set

$$B=\left\{u\in W_0^{1,p(x)}(\Omega):\ u=\lambda Tu,\ \lambda\in[0\ 1]\right\}$$

is bounded. In fact, we will prove that for all $u\in B$, there exists $R>0$ such that $|\nabla u|_{p(x)}\le R$. Let $u\in B$. Then, we have to consider two cases:

• If $|\nabla u|\le 1$, then B is bounded.

• If $|\nabla u|>1$, by using modular's properties, we proceed as follows: If $u=\lambda Tu$ and assuming that $\lambda\neq 0$, we have $\dfrac{u}{\lambda}=Tu=(-\Delta_{p(x)})^{-1}\Phi u$. Then we obtain

$$-\Delta_{p(x)}\left(\frac{u}{\lambda}\right)=\Phi u$$

and consequently,

$$\int_\Omega |\nabla\left(\frac{u}{\lambda}\right)|^{p(x)-2}\nabla\left(\frac{u}{\lambda}\right)\nabla v\,dx=\int_\Omega f(x,u,\nabla u)v\,dx,$$

for all $v\in K$. Using Hölder-type inequality, Proposition 1 and Proposition 2, we have

$$\begin{aligned}
\frac{1}{\lambda^{p^+}}\rho_{p(x)}(\nabla u)&\le\rho_{p(x)}\nabla\left(\frac{u}{\lambda}\right)\le\frac{1}{\lambda}\int_\Omega f(x,u,\nabla u)u\,dx,\\
\rho_{p(x)}(\nabla u)&\le\lambda^{p^+-1}\int_\Omega f(x,u(x))u(x)\,dx,\\
&\le\lambda^{p^+-1}\left[\int_\Omega|\phi(x)u(x)|+c\int_\Omega|u(x)|^{\eta(x)}+c\int_\Omega|\nabla u(x)|^{\eta(x)}\,dx\right],\\
&\le\lambda^{p^+-1}\left[2|\phi|_{p'(x)}|u|_{p(x)}+c\rho_{\eta(x)}(u)+c\rho_{\eta(x)}(\nabla u)\right],\\
&\le\lambda^{p^+-1}\Big[2|\phi|_{p'(x)}\theta_0\left(\rho_{p(x)}(\nabla u)+1\right)+c\left(|u|_{\eta(x)}^{\eta^+}+|u|_{\eta(x)}^{\eta^-}\right)\\
&\qquad+c\left(|\nabla u|_{\eta(x)}^{\eta^+}+|\nabla u|_{\eta(x)}^{\eta^-}\right)\Big],
\end{aligned}$$

and so,

$$\begin{aligned}
\left(1-2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}\right)\rho_{p(x)}(\nabla u)&\le 2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}\\
&\quad+4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-}+(\theta_1)^{p^-}\right]\left|\nabla u\right|_{p(x)}^{p^-},\\
\left(1-2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}\right)\left|\nabla u\right|_{p(x)}^{p^-}&\le 2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}\\
&\quad+4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-}+(\theta_1)^{p^-}\right]\left|\nabla u\right|_{p(x)}^{p^-},
\end{aligned}$$

$$\left(1-2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}-4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-}+(\theta_1)^{p^-}\right]\right)\left|\nabla u\right|_{p(x)}^{p^-}$$
$$\leq 2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}.$$

It follows from assumption (A_2) that

$$1-4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-}+(\theta_1)^{p^-}\right]>0,$$

and

$$2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}<1-4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-}+(\theta_1)^{p^-}\right],$$

hence we obtain

$$1-2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}-4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-}+(\theta_1)^{p^-}\right]>0$$

and

$$|\nabla u|_{p(x)}^{p^-}\leq\frac{2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}}{1-2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}-4\lambda^{p^+-1}c(\theta_0\theta_1)^{p^-}+(\theta_1)^{p^-}}=R>0.$$

That is, B is bounded. By Schaefer's fixed point theorem, the operator T has a fixed point u which is the solution of the given Problem (1.1).

□

Remark 1. If we assume that there exists $x_0\in\Omega$ such that $f(x_0,0,t)\neq 0$ in Theorem 2, then the solution of the Problem (1.1) is nontrivial.

ACKNOWLEDGEMENTS

The authors express their gratitude to the anonymous referees for their helpful suggestions and corrections.

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