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CONFORMAL VECTOR FIELDS ON $N(k)$-PARACONTACT AND PARA-KENMOTSU MANIFOLDS

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Abstract. In this article, we initiate the study of conformal vector fields in paracontact geometry. First, we establish that if the Reeb vector field ζ is a conformal vector field on $N(k)$-paracontact metric manifold, then the manifold becomes a para-Sasakian manifold. Next, we show that if the conformal vector field $\mathbf{V}$ is pointwise collinear with the Reeb vector field ζ, then the manifold recovers a para-Sasakian manifold as well as $\mathbf{V}$ is a constant multiple of ζ. Furthermore, we prove that if a 3-dimensional $N(k)$-paracontact metric manifold admits a Killing vector field $\mathbf{V}$, then either it is a space of constant sectional curvature k or, $\mathbf{V}$ is an infinitesimal strict paracontact transformation. Besides these, we also investigate conformal vector fields on para-Kenmotsu manifolds.

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1. INTRODUCTION

Some intrinsic properties of contact metric manifolds can be explained beautifully by the existence of conformal vector fields (briefly, CVF) on a contact metric manifold. For example, it is well circulated that [18] if an n-dimensional Riemannian manifold admits a maximal, i.e., $\frac{(n+1)(n+2)}{2}$-parameter group of conformal motions, then it is conformally flat. It is well known that a Sasakian manifold which is conformally flat has a constant curvature 1 [12]. Again, it is to be noted that a complete and connected Sasakian manifold of dimension greater than three is isometric to sphere if it admits a conformal motion. This result was determined by Okumura in 1962 [11]. Later study of contact manifolds admitting conformal motions was extended to $N(k)$-contact metric manifolds by Sharma [15]. Sharma also introduced the notion of holomorphic planar CVF in Hermitian manifolds [14]. Planar CVF have also been studied in [7, 14].

A vector field $\mathbf{V}$ on $\mathbf{M}^{2n+1}$ satisfying the equation

$$\pounds_{\mathbf{V}} g = 2\rho g, \tag{1.1}$$

ρ being a smooth function and $\pounds$ is the Lie-derivative, is called a CVF. It is said to be non-trivial, if $\mathbf{V}$ is not Killing. If ρ vanishes, then the CVF $\mathbf{V}$ is named Killing. $\mathbf{V}$ is called homothetic, if ρ is constant. CVF have been studied by many authors such as [4, 5, 9, 10, 15, 16] and many others. The CVF $\mathbf{V}$ satisfies the followings [18]

$$\begin{aligned}
(\pounds_{\mathbf{V}}\nabla)(X_1,Y_1) &= (X_1\rho)Y_1-(Y_1\rho)X_1-g(X_1,Y_1)D\rho,\\
(\pounds_{\mathbf{V}}R)(X_1,Y_1)Z_1 &= g(\nabla_{X_1}D\rho,Z_1)Y_1-g(\nabla_{Y_1}D\rho,Z_1)X_1\\
&\quad +g(X_1,Z_1)\nabla_{Y_1}D\rho-g(Y_1,Z_1)\nabla_{X_1}D\rho,\\
(\pounds_{\mathbf{V}}S)(X_1,Y_1) &= -(2n-1)g(\nabla_{X_1}D\rho,Y_1)-(\triangle\rho)g(X_1,Y_1),\\
\pounds_{\mathbf{V}}r &= -4n\triangle\rho-2r\rho
\end{aligned}$$

for all vector fields X_1,Y_1,Z_1 on $\mathbf{M}^{2n+1}$, where $D\rho$ and $\triangle\rho = divD\rho$ respectively denote the gradient and Laplacian of ρ. Here, R being the curvature tensor, S indicates the Ricci tensor and r denotes the scalar curvature.

Definition 1. A vector field $\mathbf{V}$ satisfying the relation

$$\pounds_{\mathbf{V}}\eta = \sigma\eta,$$

σ being a scalar function, is named an infinitesimal paracontact transformation. It is called an infinitesimal strict paracontact transformation, if σ vanishes identically.

Sharma and Vrancken [16] generalized the theorem of Tanno [17] and proved:

Theorem 1. *If the CVF on $\mathbf{M}$ is an infinitesimal contact transformation, then it is an infinitesimal automorphism of* $\mathbf{M}$.

In this paper we observe that the above result of Theorem 1 is exactly the same in paracontact geometry and proof is the same as in [16].

Now, we recollect the subsequent outcome of Okumura [11], which reveals that the existence of a non-Killing vector field places a significant condition on a Sasakian manifold of dimension > 3.

Theorem 2. *If $\mathbf{M}^{2n+1}$, $n>1$ be a Sasakian manifold with a non-Killing CVF $\mathbf{V}$, then $\mathbf{V}$ is special concircular. $\mathbf{M}^{2n+1}$ is isometric to a unit sphere if it is connected and complete as well.*

Again, to generalize the foregoing result Sharma and Blair [15] proved the following:

Theorem 3. *On a $(k,0)$-contact metric manifold $\mathbf{M}^{2n+1}$, let $\mathbf{V}$ be a non-Killing CVF. $\mathbf{M}^{2n+1}$ is Sasakian and $\mathbf{V}$ is concircular for $n>1$, hence if $\mathbf{M}^{2n+1}$ is complete, it is isometric to a unit sphere. $\mathbf{M}^{2n+1}$ is either Sasakian or flat for $n=1$.*

To generalized the above result, Sharma and Vrancken [16] established the following theorem:

Theorem 4. *Let $\mathbf{M}^{2n+1}$ be a (k,μ)-contact metric manifold which admits a non-Killing CVF $\mathbf{V}$, for $n>1$,*

(i) *$\mathbf{M}^{2n+1}$ is Sasakian and $\mathbf{V}$ is concircular, in which case if $\mathbf{M}^{2n+1}$ is complete then it is isometric to a unit sphere, or*

(ii) *$\mu=1, k=-n-1$. In addition, $\mathbf{M}^{2n+1}$ is isometric to the unit sphere S^{2n+1} if it is compact.*

Very recently, De, Suh and Chaubey [4] studied CVF on almost co-Kähler manifolds.

Inspired by the foregoing studies we are interested to investigate CVF on paracontact geometry.

The present article is laid out as: In Section 2, we recollect some facts about paracontact manifolds, in particular $N(k)$-paracontact metric manifolds. In Section 3, we classify CVF on $N(k)$-paracontact metric manifolds. Finally, we investigate CVF on para-Kenmotsu manifolds.

2. Paracontact manifolds

An almost paracontact structure on an $(2n+1)$-dimensional manifold $\mathbf{M}$ consists of a (1,1)-tensor field ϕ, a vector field ζ and a one-form η obeying the subsequent conditions:

$$\phi^2 X_1 = X_1 - \eta(X_1)\zeta, \qquad \eta(\zeta)=1. \tag{2.1}$$

The manifold $\mathbf{M}^{2n+1}$ with an almost paracontact structure is called an almost paracontact manifold. From the foregoing definition we recovers that $\phi\zeta=0$, $\eta\circ\phi=0$ and rank of ϕ is $2n$. If the Nijenhuis tensor vanishes identically, then the almost paracontact manifold is named normal. $\mathbf{M}^{2n+1}$ is called an almost paracontact metric manifold if there exists a semi-Riemannian metric g such that

$$g(\phi X_1, \phi Y_1) = -g(X_1,Y_1) + \eta(X_1)\eta(Y_1)$$

for all $X_1, Y_1 \in \chi(\mathbf{M})$.

In $\mathbf{M}^{2n+1}$ the fundamental 2-form is defined by $\Phi(X_1,Y_1) = g(X_1,\phi Y_1)$. If $d\eta(X_1,Y_1) = g(X_1,\phi Y_1)$, then $\mathbf{M}^{2n+1},\phi,\zeta,\eta,g)$ is named paracontact metric manifold.

In $\mathbf{M}^{2n+1}$, one can introduce a symmetric, trace-free (1,1)-tensor $h=\frac{1}{2}\pounds_{\zeta}\phi$ fulfilling [2, 19]

$$\phi h + h\phi = 0, \qquad h\zeta = 0,$$
$$\nabla_{X_1}\zeta = -\phi X_1 + \phi h X_1, \tag{2.2}$$

for all $X_1 \in \chi(\mathbf{M})$. It is noted that the condition $h=0$ means that ζ is a Killing vector field and then (ϕ,ζ,η,g) is called K-paracontact structure. $\mathbf{M}^{2n+1}$ is named a para-Sasakian manifold if and only if [19]

$$(\nabla_{X_1}\phi)Y_1 = -g(X_1,Y_1)\zeta + \eta(Y_1)X_1$$

holds, for any X_1, Y_1. A para-Sasakian manifold satisfies the relation

$$R(X_1,Y_1)\zeta = -[\eta(Y_1)X_1 - \eta(X_1)Y_1] \tag{2.3}$$

for any X_1, Y_1, but in contrast to the contact metric geometry, the equation (2.3) does not mean that the paracontact manifold is para-Sasakian manifold. It is to be noted that every para-Sasakian manifold is K-paracontact, but the converse is not always true, it is true only for three dimensional case [1].

A $\mathbf{M}^{2n+1}$ is named a (k,μ)-paracontact manifold if the curvature tensor R obeys [2]

$$R(X_1,Y_1)\zeta = k[\eta(Y_1)X_1 - \eta(X_1)Y_1] + \mu[\eta(Y_1)hX_1 - \eta(X_1)hY_1]$$

for all vector fields $X_1, Y_1 \in \chi(\mathbf{M})$ and k, μ are real constants.

In particular, a (k,μ)-paracontact manifold turns into a $N(k)$-paracontact manifold for $\mu = 0$ and hence, the curvature tensor obeys

$$R(X_1,Y_1)\zeta = k[\eta(Y_1)X_1 - \eta(X_1)Y_1].$$

In a $N(k)$-paracontact metric manifold $(\mathbf{M}^3, \phi, \zeta, \eta, g)$, the subsequent results hold [3, 13]:

$$\begin{aligned} QX_1 &= (\frac{r}{2} - k)X_1 + (3k - \frac{r}{2})\eta(X_1)\zeta, \\ S(X_1,Y_1) &= (\frac{r}{2} - k)g(X_1,Y_1) + (3k - \frac{r}{2})\eta(X_1)\eta(Y_1), \\ S(X_1,\zeta) &= 2k\eta(X_1). \end{aligned} \tag{2.4}$$

Lemma 1 ([8], Theorem 3.1)**.** *Let $\mathbf{M}^{2n+1}$ be a paracontact metric manifold $R(X_1,Y_1)\zeta = -[\eta(Y_1)X_1 - \eta(X_1)Y_1]$, for all X_1, Y_1 on $\mathbf{M}^{2n+1}$. Then $\mathbf{M}^{2n+1}$ is para Sasakian if and only if ζ is a Killing vector field.*

Lemma 2. *Let $\mathbf{M}^{2n+1}$ be a $N(k)$-paracontact metric manifold and the Reeb vector field ζ is Killing. Then $\mathbf{M}^{2n+1}$ is a para-Sasakian manifold.*

Proof. In a $N(k)$-paracontact metric manifold,

$$R(X_1,Y_1)\zeta = k[\eta(Y_1)X_1 - \eta(X_1)Y_1]. \tag{2.5}$$

Contracting the above equation, we get

$$S(X_1,\zeta) = 2nk\eta(X_1),$$

which implies

$$Q\zeta = 2nk\zeta. \tag{2.6}$$

Now, if ζ is Killing in paracontact metric manifold, then it becomes a K-paracontact metric manifold. Hence $Q\zeta = -2n\zeta$. Therefore from (2.6), we get $k = -1$. Hence from (2.5), we obtain

$$R(X_1,Y_1)\zeta = -[\eta(Y_1)X_1 - \eta(X_1)Y_1].$$

Therefore, Lemma 1 implies that the manifold becomes a para-Sasakin manifold. This completes the proof. □

3. Conformal vector fields on $N(k)$-paracontact metric manifolds

Let us choose that the Reeb vector field ζ be a CVF on $\mathbf{M}^{2n+1}$. Then from (1.1), we get

$$(\pounds_{\zeta} g)(X_1, Y_1) = 2\rho g(X_1, Y_1), \tag{3.1}$$

which implies

$$g(\nabla_{X_1}\zeta, Y_1) + g(X_1, \nabla_{Y_1}\zeta) = 2\rho g(X_1, Y_1). \tag{3.2}$$

Contracting X_1 and Y_1 in (3.2) entails that

$$div\zeta = (2n+1)\rho. \tag{3.3}$$

Again, from (2.2), we infer

$$div\zeta = 0. \tag{3.4}$$

Equations (3.3) and (3.4) together imply

$$\rho = 0. \tag{3.5}$$

Using (3.5) in (3.1), we obtain

$$(\pounds_{\zeta} g)(X_1, Y_1) = 0,$$

which implies ζ is a Killing vector field. Hence from Lemma 2, we have:

Theorem 5. *If the Reeb vector field ζ of $\mathbf{M}^{2n+1}$ is a CVF, then $\mathbf{M}^{2n+1}$ becomes a para-Sasakian manifold.*

Suppose $\mathbf{V} = a\zeta$, where a is smooth function on $\mathbf{M}^{2n+1}$. Then from (2.1), we get

$$(\pounds_{a\zeta} g)(X_1, Y_1) = 2\rho g(X_1, Y_1), \tag{3.6}$$

which implies

$$g(\nabla_{X_1} a\zeta, Y_1) + g(X_1, \nabla_{Y_1} a\zeta) = 2\rho g(X_1, Y_1).$$

Using (2.2) in the above equation entails that

$$(X_1 a)\eta(Y_1) + (Y_1 a)\eta(X_1) - 2ag(h' X_1, 2) = 2\rho g(X_1, Y_1). \tag{3.7}$$

Contracting (3.7), we provide

$$\zeta a = (2n+1)\rho. \tag{3.8}$$

Putting $Y_1 = \zeta$ in (3.7) and using (3.8) gives

$$X_1 a = (1-2n)\rho\eta(X_1). \tag{3.9}$$

Above two equations together imply

$$\rho = 0.$$

Therefore from (3.9), we get $X_1 a = 0$, which means that a is a constant. Again from (3.6), we get

$$(\pounds_{\mathbf{V}} g)(X_1, Y_1) = 0,$$

which implies $\mathbf{V}$ is Killing, that is, ζ is Killing. Hence we have:

Theorem 6. *If a CVF* $\mathbf{V}$ *in* $\mathbf{M}^{2n+1}$ *is pointwise collinear with the Reeb vector field* ζ*, then* $\mathbf{M}^{2n+1}$ *becomes a para-Sasakian manifold and* $\mathbf{V}$ *is a constant multiple of the Reeb vector field* ζ*.*

We assume that the vector field $\mathbf{V}$ in $\mathbf{M}^3$ is Killing. Then

$$(\pounds_{\mathbf{V}} g)(X_1, Y_1) = 0 \tag{3.10}$$

and

$$(\pounds_{\mathbf{V}} S)(X_1, Y_1) = 0. \tag{3.11}$$

Definition of Lie-derivative infers that

$$(\pounds_{\mathbf{V}}\eta)X_1 = \pounds_{\mathbf{V}}\eta(X_1) - \eta(\pounds_{\mathbf{V}} X_1).$$

Equation (3.10) implies

$$(\pounds_{\mathbf{V}}\eta)X_1 = g(\pounds_{\mathbf{V}}\zeta, X_1). \tag{3.12}$$

Also, we have

$$\eta(\pounds_{\mathbf{V}}\zeta) = 0 \qquad \text{and} \qquad (\pounds_{\mathbf{V}}\eta)\zeta = 0. \tag{3.13}$$

Now, we take Lie-derivative of the equation (2.4) along the Killing vector field $\mathbf{V}$ and get

$$\begin{aligned}(\pounds_{\mathbf{V}} S)(X_1, Y_1) = &\frac{Vr}{2}[g(X_1, Y_1) - \eta(X_1)\eta(Y_1)] \\ &+ (3k - \frac{r}{2})[(\pounds_{\mathbf{V}}\eta)X_1\eta(Y_1) + (\pounds_{\mathbf{V}}\eta)Y_1\eta(X_1)].\end{aligned} \tag{3.14}$$

Using (3.11) and (3.12) in (3.14), we provide

$$\begin{aligned}&\frac{Vr}{2}[g(X_1, Y_1) - \eta(X_1)\eta(Y_1)] \\ &+ (3k - \frac{r}{2})[g(X_1, \pounds_{\mathbf{V}}\zeta)\eta(Y_1) + g(\pounds_{\mathbf{V}}\zeta, Y_1)\eta(X_1)] = 0.\end{aligned} \tag{3.15}$$

Setting $Y_1 = \zeta$ in (3.15) and using (3.13) gives

$$(6k - r)g(\pounds_{\mathbf{V}}\zeta, X_1) = 0,$$

which implies either $r = 6k$ or, $g(\pounds_{\mathbf{V}}\zeta, X_1) = 0$.

Case I: If $r = 6k$, then (2.4) implies

$$S(X_1, Y_1) = 2kg(X_1, Y_1).$$

Hence, we get

$$R(X_1, Y_1)Z_1 = k[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1],$$

which means that it is a space of constant sectional curvature k.

Case II: If $g(\pounds_{\mathbf{V}}\zeta, X_1) = 0$, then (3.12) implies

$$(\pounds_{\mathbf{V}}\eta)X_1 = 0,$$

which means that the vector field $\mathbf{V}$ is an infinitesimal strict paracontact transformation. Hence we have:

Theorem 7. *If M^3 admits a Killing vector field* $\mathbf{V}$*, then it is either space of constant sectional curvature k or the vector field* $\mathbf{V}$ *is an infinitesimal strict paracontact transformation.*

4. PARA-KENMOTSU MANIFOLDS

An almost paracontact manifold with the structure (ϕ,ζ,η,g) is named an almost para-Kenmotsu manifold, if

$$d\eta = 0, \qquad d\Phi = 2\eta\wedge\Phi.$$

In a para-Kenmotsu manifold the following relations hold [6]:

$$\begin{aligned} R(X_1,Y_1)\zeta &= \eta(X_1)Y_1-\eta(Y_1)X_1,\\ R(X_1,\zeta)Y_1 &= g(X_1,Y_1)\zeta-\eta(Y_1)X_1,\\ R(\zeta,X_1)Y_1 &= \eta(Y_1)X_1-g(X_1,Y_1)\zeta,\\ \eta(R(X_1,Y_1)Z_1) &= g(X_1,Z_1)\eta(Y_1)-g(Y_1,Z_1)\eta(X_1),\\ (\nabla_{X_1}\phi)Y_1 &= g(\phi X_1,Y_1)\zeta-\eta(Y_1)\phi X_1,\\ \nabla_{X_1}\zeta &= X_1-\eta(X_1)\zeta,\\ S(X_1,\zeta) &= -2n\eta(X_1). \end{aligned} \tag{4.1}$$

Also in a 3-dimensional para-Kenmotsu manifold, we have

$$QX_1 = (\frac{r}{2}+1)X_1-(\frac{r}{2}+3)\eta(X_1)\zeta,$$

which implies

$$S(X_1,Y_1) = (\frac{r}{2}+1)g(X_1,Y_1)-(\frac{r}{2}+3)\eta(X_1)\eta(Y_1). \tag{4.2}$$

5. CONFORMAL VECTOR FIELDS ON PARA-KENMOTSU MANIFOLDS

We suppose that the Reeb vector field ζ is a CVF. Then from (1.1), we get

$$(\pounds_\zeta g)(X_1,Y_1) = 2\rho g(X_1,Y_1), \tag{5.1}$$

which implies

$$g(\nabla_{X_1}\zeta,Y_1)+g(X_1,\nabla_{Y_1}\zeta) = 2\rho g(X_1,Y_1). \tag{5.2}$$

Using (4.1) in (5.2), we provide

$$2[g(X_1,Y_1)-\eta(X_1)\eta(Y_1)] = 2\rho g(X_1,Y_1). \tag{5.3}$$

Setting $X_1 = Y_1 = \zeta$ in (5.3) entails that

$$\rho = 0. \tag{5.4}$$

Using (5.4) in (5.1), we obtain

$$(\pounds_\zeta g)(X_1,Y_1) = 0,$$

which implies ζ is a Killing vector field. However the Reeb vector field ζ is not a Killing vector field on para-Kenmotsu manifolds. Hence we have:

Theorem 8. *In a para-Kenmotsu manifold the Reeb vector field ζ can not be a CVF.*

We assume that the vector field $\mathbf{V}$ in $\mathbf{M}^3$ is Killing. Then

$$(\pounds_{\mathbf{V}} g)(X_1, Y_1) = 0. \tag{5.5}$$

From the definition of Lie-derivative, we infer

$$(\pounds_{\mathbf{V}} \eta)X_1 = \pounds_{\mathbf{V}} \eta(X_1) - \eta(\pounds_{\mathbf{V}} X_1).$$

Equation (5.5) implies

$$(\pounds_{\mathbf{V}} \eta)X_1 = g(\pounds_{\mathbf{V}} \zeta, X_1). \tag{5.6}$$

Also,

$$\eta(\pounds_{\mathbf{V}} \zeta) = 0 \qquad \text{and} \qquad (\pounds_{\mathbf{V}} \eta)\zeta = 0. \tag{5.7}$$

Taking the Lie-derivative of (4.2), we get

$$\begin{aligned}(\pounds_{\mathbf{V}} S)(X_1, Y_1) = &\frac{Vr}{2}[g(X_1, Y_1) - \eta(X_1)\eta(Y_1)] \\ &- (3 + \frac{r}{2})[g(X_1, \pounds_{\mathbf{V}} \zeta)\eta(Y_1) + g(Y_1, \pounds_{\mathbf{V}} \zeta)\eta(X_1)].\end{aligned}$$

Since, $\mathbf{V}$ is Killing vector field, then

$$(\pounds_{\mathbf{V}} S)(X_1, Y_1) = 0.$$

Above two equations together imply

$$\begin{aligned}&\frac{Vr}{2}[g(X_1, Y_1) - \eta(X_1)\eta(Y_1)] \\ &- (3 + \frac{r}{2})[g(X_1, \pounds_{\mathbf{V}} \zeta)\eta(Y_1) + g(Y_1, \pounds_{\mathbf{V}} \zeta)\eta(X_1)] = 0.\end{aligned} \tag{5.8}$$

Putting $X_1 = \zeta$ in (5.8) gives

$$(3 + \frac{r}{2})[\eta(\pounds_{\mathbf{V}} \zeta)\eta(Y_1) + g(\pounds_{\mathbf{V}} \zeta, Y_1)] = 0. \tag{5.9}$$

Using (5.7) in (5.9), we obtain

$$(r + 6)g(\pounds_{\mathbf{V}} \zeta, Y_1) = 0,$$

which implies either $r + 6 = 0$ or, $g(\pounds_{\mathbf{V}} \zeta, Y_1) = 0$.

Case I: If we take $r + 6 = 0$, then from (4.2) entails that

$$S(X_1, Y_1) = -2g(X_1, Y_1),$$

which means the manifold is Einstein. Hence we have

$$R(X_1, Y_1)Z_1 = -[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1],$$

which is a space of constant sectional curvature -1.

Case II: If $g(\pounds_{\mathbf{V}}\zeta, Y_1) = 0$, which implies $\pounds_{\mathbf{V}}\zeta = 0$. Hence from (5.6), we get

$$(\pounds_{\mathbf{V}}\eta)Y_1 = 0.$$

Hence from Definition 1 we have:

Theorem 9. *If M^3 admits a Killing vector field $\mathbf{V}$, then it is either locally isometric to $\mathbb{H}^3(1)$ or the vector field $\mathbf{V}$ is an infinitesimal strict paracontact transformation.*

Now, we like to state a lemma before proving the next theorem which is similar to the proof of the result of Sharma's paper [16].

Lemma 3. *If $\mathbf{V}$ is a CVF on a para-Kenmotsu manifold, then*

(i) $(\pounds_{\mathbf{V}}\eta)\zeta = \rho$ *and*
(ii) $\eta(\pounds_{\mathbf{V}}\zeta) = -\rho$.

Theorem 10. *Let $\mathbf{V}$ be a CVF on a para-Kenmotsu manifold $\mathbf{M}^{2n+1}$. If $\mathbf{V}$ is an infinitesimal paracontact transformation, then $\mathbf{V}$ is a homothetic vector field.*

Proof. Let $\mathbf{V}$ be a CVF, then by definition

$$(\pounds_{\mathbf{V}}g)(X_1, Y_1) = 2\rho g(X_1, Y_1),$$

which implies

$$\pounds_{\mathbf{V}}g(X_1, Y_1) - g(\pounds_{\mathbf{V}}X_1, Y_1) - g(X_1, \pounds_{\mathbf{V}}Y_1) = 2\rho g(X_1, Y_1). \tag{5.10}$$

Setting $Y = \zeta$ in (5.10), we provide

$$\pounds_{\mathbf{V}}\zeta = (\sigma - 2\rho)\zeta.$$

Using Lemma 3 in the foregoing equation gives

$$\sigma = \rho. \tag{5.11}$$

Equation (5.11) and Definition 1 together imply

$$\pounds_{\mathbf{V}}\eta = \rho\eta \qquad \text{and} \qquad \pounds_{\mathbf{V}}\zeta = -\rho\zeta.$$

The above equation implies

$$(\pounds_{\mathbf{V}}d\eta)(X_1, Y_1) = \frac{1}{2}[(X_1\rho)\eta(Y_1) - (Y_1\rho)\eta(X_1)] + \rho d\eta(X_1, Y_1). \tag{5.12}$$

Since in para-Kenmotsu manifold $d\eta = 0$, then (5.12) implies

$$(X_1\rho)\eta(Y_1) = (Y_1\rho)\eta(X_1),$$

which implies

$$X_1\rho = (\zeta\rho)\eta(X_1).$$

If we take $\zeta\rho = 0$, then the above equation implies $X_1\rho = 0$, which means ρ is a constant. Hence from (5.11), we get $\sigma = \rho$ = constant. This completes the proof. □

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