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ADDITIVE MAPPINGS SATISFYING CERTAIN ALGEBRAIC EQUATIONS IN PRIME RINGS

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Abstract. In this paper we give a classification of endomorphisms and additive mappings of a prime ring satisfying certain algebraic identities. Moreover, we provide an example proving that the primeness hypothesis imposed in our theorems is not superfluous.

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1. INTRODUCTION

Throughout this article, R will represent an associative ring with center $Z(R)$. For any $x,y \in R$, the symbol $[x,y]$ will denote the commutator $xy-yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy+yx$. Recall that a ring R is said to be prime if for any $a,b \in R$, $aRb=\{0\}$ implies $a=0$ or $b=0$. R is 2-torsion free if whenever $2x=0$, with $x \in R$, implies $x=0$.

Recently many authors have obtained commutativity of prime and semiprime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the rings (for example, see [4], [5], [6], [8], [7], [3], [9][10]). In [2] it is proved that a prime ring R must be commutative if it admits derivation d satisfying any one of the properties $d(xy)-xy \in Z(R)$, $d(xy)+xy \in Z(R)$, $d(xy)-yx \in Z(R)$, $d(xy)+yx \in Z(R)$, $d(x)d(y)-xy \in Z(R)$, $d(x)d(y)+xy \in Z(R)$, for all $x,y \in R$. Further some authors have studied the situations replacing d with a generalized derivation F. More precisely, they proved that the prime ring R must be commutative if R is equipped with a generalized derivation F associated with a nonzero derivation d satisfying any one of the following conditions:

(i) $F(xy)-xy \in Z(R)$ for all $x,y \in I$,
(ii) $F(xy)+xy \in Z(R)$ for all $x,y \in I$,
(iii) $F(xy)-yx \in Z(R)$ for all $x,y \in I$,
(iv) $F(xy)+yx \in Z(R)$ for all $x,y \in I$,
(v) $F(x)F(y)-xy \in Z(R)$ for all $x,y \in I$,

(vi) $F(x)F(y)+xy \in Z(R)$ for all $x,y \in I$, where I is a nonzero two sided ideal of R.

Inspired by these results, Ali et al. [1] have studied the situations replacing generalized derivation F with a multiplicative generalized derivation. Moreover, The authors in [3] considered the situation when the derivation d is replaced by an endomorphism g satisfying any one of the following conditions:

(i) $g(xy)-xy \in Z(R)$,
(ii) $g(xy)+xy \in Z(R)$,
(iii) $g(xy)-yx \in Z(R) \in Z(R)$, (iv) $g(xy)+yx \in Z(R)$, for all $x,y \in R$.

The present paper is motivated by the previous results and we here continue this line of investigation by examining a more general situations of algebraic identities involving an endomorphism g and an additive mappings ϕ satisfying any one of the following properties :

$$g(xy)-\phi(x)y \in Z(R) \quad \text{for all } x,y \in R$$

$$g(xy)-\phi(y)x \in Z(R) \quad \text{for all } x,y \in R$$

More precisely, we will discuss existence of such mappings and gives classifications of these mappings in case of prime ring.

1.1. *Preliminary considerations*

Lemma 1. *Let R be a prime ring, g an endomorphism of R and ϕ an additive mapping of R. If $g(x)g(y)-\phi(x)y=0$ for all $x,y \in R$, then one of the following assertions holds :*

(1) $g=\phi=I_R$;
(2) $\phi=0$ *and* $g(R^2)=0$.

Proof. Assume that

$$g(x)g(y)-\phi(x)y=0 \quad \text{for all } x,y \in R. \tag{1.1}$$

Substituting yu for y in equation (1.1) we find that

$$g(x)g(y)g(u)-\phi(x)yu=0 \quad \text{for all } u,x,y \in R.$$

Multiplying equation (1.1) by $g(u)$ and comparing with the last relation, one can see that

$$\phi(x)y(g(u)-u)=0 \quad \text{for all } u,x,y \in R$$

and therefore

$$\phi(x)R(g(u)-u)=0 \quad \text{for all } u,x \in R.$$

Since R is prime, then either $\phi=0$ or $g=I_d$.

If $\phi=0$, then equation (1.1) reduces to

$$g(x)g(y)=0 \quad \text{for all } x,y \in R$$

and thus $g(R^2)=0$.

If $g = I_d$, then the hypothesis becomes

$$(\phi(x) - x)y = 0 \quad \text{for all } x, y \in R$$

which, because of the primeness of R, implies that $\phi = I_d$. □

Lemma 2. *Let R be a prime ring, g an endomorphism of R and ϕ an additive mapping of R. If $g(x)g(y) - \phi(y)x = 0$ for all $x, y \in R$, then one of the following assertions holds :*

(1) $\phi = 0$ *and* $g(R^2) = 0$;

(2) $\phi = g = 0$.

(3) $\phi = g = I_R$ *and R is a commutative integral domain.*

Proof. Assume that

$$g(x)g(y) - \phi(y)x = 0 \quad \text{for all } x, y \in R. \tag{1.2}$$

Replacing x by xr in equation (1.2) we obtain

$$g(x)g(r)g(y) - \phi(y)xr = 0 \quad \text{for all } r, x, y \in R.$$

Using equation (1.2) one can see that

$$g(x)\Big(g(r)g(y) - g(y)r\Big) = 0 \quad \text{for all } r, x, y \in R.$$

Once again, using (1.2) we find that

$$g(x)\Big(\phi(y) - g(y)\Big)r = 0 \quad \text{for all } r, x, y \in R.$$

Accordingly,

$$g(x)\Big(\phi(y) - g(y)\Big) = 0 \quad \text{for all } x, y \in R.$$

Substituting xr for x and using the hypothesis, we obviously get

$$\phi(r)x\Big(\phi(y) - g(y)\Big) = 0 \quad \text{for all } r, x, y \in R$$

and thus

$$\phi(r)R\Big(\phi(y) - g(y)\Big) = 0 \quad \text{for all } r, y \in R.$$

Since R is prime, then either $\phi = 0$ in which case the relation (1.2) implies that $g(x)g(y) = 0$ and thus $g(R^2) = 0$ or $g = \phi$. In the last case the hypothesis forces

$$g(x)g(y) - g(y)x = 0 \quad \text{for all } x, y \in R. \tag{1.3}$$

Writing xr instead of x we find that

$$g(x)g(r)g(y) - g(y)xr = 0 \quad \text{for all } r, x, y \in R. \tag{1.4}$$

Invoking (1.3), equation (1.4) yields

$$g(r)xg(y) - g(y)xr = 0 \quad \text{for all } r, x, y \in R. \tag{1.5}$$

Substituting rt for r in equation (1.5), it is obvious to see that

$$g(r)g(t)xg(y)-g(y)xrt=0 \quad \text{for all } r,t,x,y\in R. \tag{1.6}$$

By view of (1.3), the last equation becomes

$$g(t)rxg(y)-g(y)xrt=0 \quad \text{for all } r,t,x,y\in R. \tag{1.7}$$

Taking $r=rt$ in the last relation and using it we are forced to conclude that

$$g(t)r[t,xg(y)]=0 \quad \text{for all } r,t,x,y\in R \tag{1.8}$$

which implies that $g(t)=0$ or $[t,xg(y)]=0$ for all $t,x,y\in R$.

Now if $[t,xg(y)]=0$ for all $t,x,y\in R$ then replacing x by xr we obtain

$$[t,x]Rg(y)=0 \quad \text{for all } t,x,y\in R$$

and thus $[t,x]=0$ or $g=0$. Hence $g(t)=0$ or $[t,x]=0$ for all $t,x\in R$; using Brauer's trick we conclude that $g=0$ or R is a commutative integral domain. □

Theorem 1. *Let R be a noncommutative prime ring, g an endomorphism of R and ϕ an additive mapping of R. The following assertions are equivalent:*

(1) $g(x)g(y)-\phi(x)y\in Z(R)$ *for all* $x,y\in R$;
(2) $g=\phi=I_R$ *or* $\big(\phi=0$ *and* $g(R^2)\subseteq Z(R)\big)$.

Proof. For the nontrivial implication, assume that

$$g(x)g(y)-\phi(x)y\in Z(R) \quad \text{for all } x,y\in R. \tag{1.9}$$

Since $[g(x)g(y),g(z)]=0$ for all $z\in Z(R)$, then equation (1.9) implies that

$$[\phi(x)y,g(z)]=0 \quad \text{for all } x,y\in R. \tag{1.10}$$

Substituting yt for y, one can see that

$$\phi(x)y[t,g(z)]=0 \quad \text{for all } t,x,y\in R \tag{1.11}$$

in such a way that

$$\phi(x)R[t,g(z)]=0 \quad \text{for all } t,x\in R.$$

By view of the primeness of R we conclude that $\phi=0$ or $g(z)\in Z(R)$ for all $z\in Z(R)$.

If $\phi=0$ then the hypothesis becomes

$$g(x)g(y)\in Z(R) \quad \text{for all } x,y\in R$$

which means that $g(R^2)\subseteq Z(R)$.

If $g(Z(R))\subseteq Z(R)$; replacing y by yz in equation (1.9), where $z\in Z(R)$, we get

$$[g(x)g(y),r]g(z)-[\phi(x)y,r]z=0 \quad \text{for all } r,x,y\in R. \tag{1.12}$$

Using (1.9), equation (1.12) yields

$$[\phi(x)y,r](g(z)-z)=0 \quad \text{for all } r,x,y\in R. \tag{1.13}$$

Writing tr instead of r in the last equation, we obtain

$$[\phi(x)y,r]R(g(z)-z)=0 \quad \text{for all } r,x,y\in R \tag{1.14}$$

which implies that $[\phi(x)y,r]=0$ or $g(z)=z$.

If $[\phi(x)y,r]=0$ for $r,x,y\in R$, then substituting yt for y we find that

$$\phi(x)y[t,r]=0 \quad \text{for all } r,t,x,y\in R.$$

Hence $\phi=0$, because of the noncommutativity of R, which leads to $g(R^2)\subseteq Z(R)$.

If $g(z)=z$ for all $z\in Z(R)$, then considering (1.9) with $x=y=z$, we have

$$(\phi(z)-z)z\in Z(R) \quad \text{for all } z\in Z(R)$$

which implies that $\phi(z)\in Z(R)$. Replacing x by z in equation (1.9), where $z\in Z(R)$, we obtain

$$z[g(y),r]-\phi(z)[y,r]=0 \quad \text{for all } r,y\in R. \tag{1.15}$$

Substitution yx for y and using (1.15) we arrive at

$$\Big((g(y)-y)[x,r]+[y,r](g(x)-x)\Big)\phi(z)=0 \quad \text{for all } r,x,y\in R \tag{1.16}$$

in such a way that $\phi(z)=0$ in which case the hypothesis becomes $zg(y)\in Z(R)$ for all $y\in R$ and $z\in Z(R)$ and thus $g(R)\subseteq Z(R)$ or

$$(g(y)-y)[x,r]+[y,r](g(x)-x)=0 \quad \text{for all } r,x,y\in R.$$

Replacing y by ry we obviously get

$$(g(r)-r)g(y)[x,r]=0 \quad \text{for all } r,x,y\in R. \tag{1.17}$$

Substituting rt for r one can see that

$$(g(r)-r)g(y)R[x,r]=0 \quad \text{for all } r,x,y\in R \tag{1.18}$$

and thus

$$g(r)g(y)-rg(y)=0 \quad \text{for all } r,y\in R. \tag{1.19}$$

Writing yt instead of y and using (1.19), we obtain

$$(g(r)-r)yg(t)=0 \quad \text{for all } r,t,y\in R. \tag{1.20}$$

Hence $g=I_R$ because $g\neq 0$ and thus equation (1.9) reduces to

$$(\phi(x)-x)y\in Z(R) \quad \text{for all } x,y\in R. \tag{1.21}$$

Taking yt instead of y we have

$$(\phi(x)-x)yt\in Z(R) \quad \text{for all } t,x,y\in R \tag{1.22}$$

thereby obtaining

$$(\phi(x)-x)y=0 \quad \text{for all } x,y\in R \tag{1.23}$$

and thus $\phi=I_R$. □

Corollary 1. *Let R be a noncommutative prime ring, g an epimorphism of R and ϕ an additive mapping of R. The following assertions are equivalent:*

(1) $g(x)g(y)-\phi(x)y\in Z(R)$ *for all* $x,y\in R$;

(2) $g=\phi=I_R$.

Corollary 2. *[[3], Lemma 2] Let R be a prime ring and g a nonidentity endomorphism of R. If $g(x)g(y)-xy\in Z(R)$ for all $x,y\in R$, then R is a commutative integral domain.*

Theorem 2. *Let R be a noncommutative prime ring, g an endomorphism of R and ϕ an additive mapping of R. The following assertions are equivalent:*

(1) $g(x)g(y)-\phi(y)x\in Z(R)$ *for all* $x,y\in R$;
(2) $g=\phi=0$ *or* $\big(\phi=0$ *and* $g(R^2)\subseteq Z(R)\big)$.

Proof. For the nontrivial implication, suppose that

$$g(x)g(y)-\phi(y)x\in Z(R)\quad \text{for all } x,y\in R. \tag{1.24}$$

Since $[g(x)g(y),g(z)]=0$ for all $z\in Z(R)$, then equation (1.24) yields

$$[\phi(y)x,g(z)]=0\quad \text{for all } x,y\in R. \tag{1.25}$$

Substituting xt for x, one can see that

$$\phi(y)x[t,g(z)]=0\quad \text{for all } t,x,y\in R \tag{1.26}$$

in such a way that

$$\phi(y)R[t,g(z)]=0\quad \text{for all } t,y\in R.$$

In light of primeness, we conclude that $\phi=0$ in which case the hypothesis becomes $g(x)g(y)\in Z(R)$ for all $x,y\in R$ and thus $g(R^2)\subseteq Z(R)$ or $g(z)\in Z(R)$ for all $z\in Z(R)$. In the last case; replacing x and y by z in equation (1.24), where $z\in Z(R)$, we conclude that

$$\phi(z)\in Z(R)\ \text{ for all } z\in Z(R).$$

Writing z instead of y and xy instead of x in equation (1.24), we get

$$g(x)g(y)g(z)-xy\phi(z)\in Z(R)\quad \text{for all } x,y\in R. \tag{1.27}$$

Multiplying equation (1.24) by $g(z)$ and using equation (1.27) one can see that

$$\phi(y)xg(z)-xy\phi(z)\in Z(R)\quad \text{for all } x,y\in R. \tag{1.28}$$

Replacing x by xr we obtain

$$\phi(y)xrg(z)-xry\phi(z)\in Z(R)\quad \text{for all } r,x,y\in R$$

and thus

$$\phi(y)xg(z)r-xy\phi(z)r+xyr\phi(z)-xry\phi(z)\in Z(R)\quad \text{for all } r,x,y\in R.$$

Taking $A(x,y,z)=\phi(y)xg(z)-xy\phi(z)$ we get

$$A(x,y,z)r+x[y,r]\phi(z)\in Z(R)\quad \text{for all } r,x,y\in R.$$

Commuting the last equation with u and using equation (1.28), we find that

$$A(x,y,z)[r,u]+[x[y,r],u]\phi(z)=0\quad \text{for all } r,u,x,y\in R. \tag{1.29}$$

Writing ur instead of u and using equation (1.29), we may write

$$[x[y,r],r]\phi(z)=0 \quad \text{for all } r,x,y\in R. \tag{1.30}$$

So that $[x[y,r],r]=0$ which leads to a contradiction because R is noncommutative. Hence $\phi(z)=0$; taking $y=z$ in equation (1.24), where $z\in Z(R)$, we have

$$g(x)g(z)\in Z(R) \text{ for all } x\in R.$$

Substituting xy for x, it follows that

$$g(x)g(z)g(y)\in Z(R) \text{ for all } x,y\in R$$

which leads to $g(R)\subseteq Z(R)$ or $g=0$ in which case the hypothesis forces $\phi=0$ or $g(z)=0$ for all $z\in Z(R)$. In the last case replacing x by xz in (1.24), we obtain

$$\phi(y)xz\in Z(R) \text{ for all } x,y\in R$$

and thus for $x=xr$ one can see that

$$\phi(y)xrz\in Z(R) \text{ for all } r,x,y\in R.$$

Hence $\phi(y)x=0$ and therefore $\phi=0$ which leads to $g(R^2)\subseteq Z(R)$. □

Remark 1. Proceeding on the same lines with necessary variations, one can prove that the same conclusion remains valid for the identity $g(x)g(y)+\phi(x)y\in Z(R)$ for all $x,y\in R$.

Corollary 3. *Let R be a prime ring, g an epimorphism of R and ϕ an additive mapping of R. If $g(x)g(y)-\phi(y)x\in Z(R)$ for all $x,y\in R$, then R is a commutative integral domain.*

Corollary 4. *[[3], Lemma 3] Let R be a prime ring and g an endomorphism of R. If $g(x)g(y)-yx\in Z(R)$ for all $x,y\in R$, then R is a commutative integral domain.*

The following example proves that the `primeness` hypothesis in Theorem 1 and 2 is necessary.

Example 1. Let us consider $R=\mathbb{R}[X]\times M_n(\mathbb{R})$. It is straightforward to check that R is a noncommutative semi-prime ring. Moreover, let $g:R\longrightarrow R$ be the endomorphism defined on R by $g(P,M)=(P(0),0)$ and ϕ the additive mapping defined by $\phi(P,M)=(P',0)$. Hence g and ϕ satisfy the conditions of Theorem 1 and 2 but $\phi\neq 0$ and $\phi\neq I_R$.

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