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CONVERGENCE RELATION BETWEEN SOLUTIONS OF A P(X)-LANDAU-LIFSCHITZ TYPE AND P(X)-HARMONIC MAPS

BEI WANG

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Abstract. The author studies the asymptotic behavior of solutions u_ε of a $p(x)$-Landau-Lifschitz equation as ε tends to zero. Several kinds of convergence to the $p(x)$-harmonic map are presented in different senses.

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1. INTRODUCTION

Let $G \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary ∂G, and $B_1 = \{x \in \mathbb{R}^2; x_1^2 + x_2^2 < 1\}$. Denote $\mathbb{S}^1 = \{x \in \mathbb{R}^3; x_1^2 + x_2^2 = 1, x_3 = 0\}$ and $\mathbb{S}^2 = \{x \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$. The vector value function can be denoted as $u = (u_1, u_2, u_3) = (u', u_3)$. Let $g = (g', 0)$ be a smooth map from ∂G into $\mathbb{S}^1$. The p-Landau-Lifschitz-type energy functional is

$$E^{LL}(u) = \frac{1}{p}\int_G |\nabla u|^p dx + \frac{1}{2\varepsilon^p}\int_G u_3^2 dx$$

with a small parameter $\varepsilon > 0$. When $p = 2$, it was introduced in the study of some simplified model of high-energy physics, which controls the statics of planner ferromagnets and antiferromagnets (see [11] and [17]). The asymptotic behavior of minimizers of $E^{LL}(u)$ had been studied by Hang and Lin in [9]. When $p > 1$, the corresponding asymptotic properties were studied in [12] and [21]. These works show that the minimizers of $E_\varepsilon(u)$ converge to the p-harmonic maps with $\mathbb{S}^1$-value.

When $p = 2$, if the term $\frac{u_3^2}{2\varepsilon^2}$ replaced by $\frac{(1-|u|^2)^2}{4\varepsilon^2}$ and $\mathbb{S}^2$ replaced by $\mathbb{R}^2$, the problem becomes the simplified model of the Ginzburg-Landau theory for superconductors and was well studied in [3] and [4]. The energy functional is

$$E^{GL}(u) = \frac{1}{2}\int_G |\nabla u|^2 dx + \frac{1}{4\varepsilon^2}\int_G (1 - |u|^2)^2 dx.$$

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It is also showed that the properties of the harmonic maps can be studied via researching the minimizers of the functional with Ginzburg-Landau-type penalization term. Indeed, Chen and Struwe used the penalty method to establish the global existence of partial regular weak solutions of the harmonic map flow (see [5] and [7]). They also generalized the results to the case of $p > 1$ (cf. [6]). Afterwards, many papers proved that the limit of minimizers of p-Ginzburg-Landau functional is p-harmonic maps (cf. [1, 2, 13, 14, 18, 19, 22] and references therein). Misawa studied the p-harmonic maps by using the same idea of the penalty method in [16]. In 2009, Lei generalized the results to the case that p is a bounded function (cf. [13]). Now, the functional with penalization term is

$$E^{GL}(u,G) = \int_G \left[\frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{1}{4\varepsilon^{p(x)}}(1-|u|^2)^2\right]dx.$$

The main result is the convergence relation between the minimizers and $p(x)$-harmonic maps. Other results of $p(x)$-harmonic maps can be found in [8, 15] and [20].

In this paper, we are concerned with the $p(x)$-Landau-Lifschitz functional

$$E_\varepsilon(u,G) = \int_G \left[\frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{1}{2\varepsilon^{p(x)}}u_3^2\right]dx.$$

Here

$$2 < p_* = \min_{\overline{G}} p(x) \le \max_{\overline{G}} p(x) = p^* < \infty.$$

From the direct method in the calculus of variations, it is easy to see that the functional achieves its minimum in the function class

$$W_g^{1,p(x)}(G,\mathbb{S}^2) := \{u \in W^{1,p(x)}(G,\mathbb{S}^2); u - g \in W_0^{1,p(x)}(G,\mathbb{R}^3)\}.$$

Without loss of generality, we assume $u_3 \ge 0$, otherwise we may consider $|u_3|$ in view of the expression of the functional. We call u_ε a minimizer of $E_\varepsilon(u,G)$ in $W_g^{1,p(x)}(G,\mathbb{S}^2)$, if

$$E_\varepsilon(u_\varepsilon,G) = \min\{E_\varepsilon(u,G); u \in W_g^{1,p(x)}(G,\mathbb{S}^2)\}.$$

We will research the asymptotic properties of minimizers in $W_g^{1,p(x)}(G,\mathbb{S}^2)$ when $\varepsilon \to 0$, and shall prove the limit is the $p(x)$-harmonic map.

Theorem 1. *Let u_ε be a minimizer of $E_\varepsilon(u,G)$ in $W_g^{1,p(x)}(G,\mathbb{S}^2)$. Assume*

$$\deg(g', \partial G) = 0.$$

Then

$$\lim_{\varepsilon\to 0} u_\varepsilon = (u_p, 0), \quad in \quad W^{1,p(x)}(G,\mathbb{S}^2),$$

where u_p is the minimizer of $\int_G |\nabla u|^{p(x)}dx$ in $W_g^{1,p(x)}(G,\partial B_1)$.

Comparing with the assumption of Theorem 1, we will consider the problem under some weaker conditions.

Theorem 2. *Assume u_ε is a critical point of $E_\varepsilon(u,G)$ in $W_g^{1,p(x)}(G,\mathbb{S}^2)$. If*

$$E_\varepsilon(u_\varepsilon,K) \leq C \tag{1.1}$$

for some subdomain $K \subseteq G$. Then there exists a subsequence u_{ε_k} of u_ε such that as $k \to \infty$,

$$u_{\varepsilon_k} \to (u_p,0), \quad weakly \quad in \quad W^{1,p(x)}(K,\mathbb{R}^3), \tag{1.2}$$

where u_p is a critical point of $\int_K |\nabla u|^{p(x)}dx$ in $W^{1,p(x)}(K,\partial B_1)$ (which is named $p(x)$-harmonic map on K). Moreover, for any $\zeta \in C_0^\infty(K)$, when $\varepsilon \to 0$,

$$\int_K |\nabla u_{\varepsilon_k}|^{p(x)}\zeta dx \to \int_K |\nabla u_p|^{p(x)}\zeta dx, \tag{1.3}$$

$$\int_K \frac{1}{\varepsilon_k^{p(x)}} u_{\varepsilon_k 3}\zeta dx \to 0. \tag{1.4}$$

2. Proof of Theorem 1

A vector-valued function $u \in W_g^{1,p(x)}(G,\partial B_1)$ is named $p(x)$-harmonic map, if it is the critical point of $\int_G |\nabla u|^{p(x)}dx$. Namely, it is the weak solution of

$$-div(|\nabla u|^{p(x)-2}\nabla u) = u|\nabla u|^{p(x)} \tag{2.1}$$

on G, or for any $\phi \in C_0^\infty(G,\mathbb{R}^2)$, it satisfies

$$\int_G |\nabla u|^{p(x)-2}\nabla u \nabla\phi dx = \int_G u|\nabla u|^{p(x)}\phi dx. \tag{2.2}$$

By the argument of the weak low semi-continuity of the functional, we can deduce the strong convergence in $W^{1,p(x)}$ sense for some subsequence of the minimizer u_ε. To improve the conclusion of the convergence for all u_ε, we need the uniqueness of $p(x)$-harmonic maps. Therefore, we always assume $deg(g',\partial G) = 0$ in this section.

From $deg(g',\partial G) = 0$ and the smoothness of ∂G and g, we see that there is a smooth function $\phi_0 : \partial G \to \mathbb{R}$ such that

$$g = (\cos\phi_0, \sin\phi_0), \text{ on } \partial G. \tag{2.3}$$

Consider the Dirichlet problem

$$-div(|\nabla\Phi|^{p(x)-2}\nabla\Phi) = 0, \text{ in } G, \tag{2.4}$$

$$\Phi|_{\partial G} = \phi_0. \tag{2.5}$$

According to Proposition 2.4 in [13], there exists the unique weak solution Φ of (2.4) and (2.5) in $W^{1,p(x)}(G,\mathbb{R})$. Set

$$u_p = (\cos\Phi, \sin\Phi), \text{ on } \overline{G}. \tag{2.6}$$

Clearly, u_p is a $p(x)$-harmonic map on G.

Since $W_g^{1,p(x)}(G,\partial B_1)\neq\varnothing$ when $\deg(g',\partial G)=0$, we may consider the minimization problem

$$Min\left\{\int_G|\nabla u|^{p(x)}dx; u\in W_g^{1,p(x)}(G,\partial B_1)\right\}. \tag{2.7}$$

The solution is called the $p(x)$-energy minimizer. By the direct method, the solution of (2.7) exists. Obviously, the $p(x)$-energy minimizer is a $p(x)$-harmonic map. According to Proposition 2.5 in [13], the $p(x)$-harmonic map is unique in $W_g^{1,p(x)}(G,\partial B_1)$. So the $p(x)$-energy minimizer is also unique in $W_g^{1,p(x)}(G,\partial B_1)$.

In general, u_p is the unique $p(x)$-harmonic map as well as the unique $p(x)$-energy minimizer.

Proof of Theorem 1. Noticing that u_ε is the minimizer, we have

$$E_\varepsilon(u_\varepsilon,G)\le E_\varepsilon((u_p,0),G)\le C \tag{2.8}$$

with $C>0$ independent of ε. This means

$$\int_G|\nabla u_\varepsilon|^{p(x)}dx\le C, \tag{2.9}$$

$$\int_G u_{\varepsilon 3}^2 dx\le C\varepsilon^{p_*}. \tag{2.10}$$

Using (2.9), $|u_\varepsilon|=1$ and the embedding theorem, we see that there exists a subsequence u_{ε_k} of u_ε and $u_*\in W^{1,p(x)}(G,\mathbb{R}^3)$, such that as $\varepsilon_k\to 0$,

$$u_{\varepsilon_k}\to u_*,\quad \text{weakly}\quad \text{in}\quad W^{1,p(x)}(G,\mathbb{S}^2), \tag{2.11}$$

$$u_{\varepsilon_k}\to u_*,\quad \text{in}\quad C^\alpha(\overline{G},\mathbb{S}^2),\quad \alpha\in(0,1-2/p_*). \tag{2.12}$$

Obviously, (2.10) and (2.12) lead to $u_*\in W_g^{1,p(x)}(G,\mathbb{S}^1)$.

Applying (2.11) and the weak low semi-continuity of $\int_G|\nabla u|^{p(x)}dx$, we have

$$\int_G|\nabla u_*|^{p(x)}dx\le \underline{\lim}_{\varepsilon_k\to 0}\int_G|\nabla u_{\varepsilon_k}|^{p(x)}dx.$$

On the other hand, (2.8) implies

$$\int_G|\nabla u_{\varepsilon_k}|^{p(x)}dx\le\int_G|\nabla(u_p,0)|^{p(x)}dx.$$

Thus,

$$\int_G|\nabla u'_*|^{p(x)}dx\le\int_G|\nabla u_p|^{p(x)}dx.$$

This means that u'_* is also a $p(x)$-energy minimizer. Noting the uniqueness, we see $u_*=u_p$. Thus

$$\int_G|\nabla u_p|^{p(x)}dx\le \underline{\lim}_{\varepsilon_k\to 0}\int_G|\nabla u_{\varepsilon_k}|^{p(x)}dx\le\overline{\lim}_{\varepsilon_k\to 0}\int_G|\nabla u_{\varepsilon_k}|^{p(x)}dx\le\int_G|\nabla u_p|^{p(x)}dx.$$

When $\varepsilon_k\to 0$,

$$\int_G|\nabla u_{\varepsilon_k}|^{p(x)}\to\int_G|\nabla u_p|^{p(x)}.$$

Combining this with (2.11) yields

$$\lim_{k\to\infty} \nabla u_{\varepsilon_k} = \nabla(u_p, 0), \text{ in } L^{p(x)}(G, \mathbb{S}^2).$$

In addition, (2.12) implies that as $\varepsilon \to 0$,

$$u_{\varepsilon_k} \to (u_p, 0), \text{ in } L^{p(x)}(G, \mathbb{S}^2).$$

Then

$$\lim_{k\to\infty} u_{\varepsilon_k} = (u_p, 0), \text{ in } W^{1,p(x)}(G, \mathbb{S}^2).$$

Noticing the uniqueness of $(u_p, 0)$, we see the convergence above also holds for all u_ε.

3. Proof of Theorem 2

In this section, we always assume that u_ε is the critical point of the functional, and $E_\varepsilon(u_\varepsilon, K) \le C$ for some subdomain $K \subseteq G$, where C is independent of ε. The assumption is weaker than that of Theorem 1. So, all the results in this section will be derived in the weak sense.

The method in the calculus of variations shows that the minimizer u_ε of $E_\varepsilon(u, G)$ in $W_g^{1,p(x)}(G, S^2)$ is a weak solution of

$$-div(|\nabla u|^{p(x)-2}\nabla u) = u|\nabla u|^{p(x)} + \frac{1}{\varepsilon^{p(x)}}(uu_3^2 - u_3 e_3), \text{ on } G, \tag{3.1}$$

where $e_3 = (0,0,1)$. Namely, for any $\psi \in W_0^{1,p(x)}(G, \mathbb{R}^3)$, u_ε satisfies

$$\int_G |\nabla u|^{p(x)-2}\nabla u \nabla \psi dx = \int_G u\psi|\nabla u|^{p(x)}dx + \frac{1}{\varepsilon^{p(x)}}\int_G \psi(uu_3^2 - u_3 e_3)dx. \tag{3.2}$$

Proof of (1.2). $E_\varepsilon(u_\varepsilon, K) \le C$ means

$$\int_K |\nabla u_\varepsilon|^{p(x)} dx \le C, \tag{3.3}$$

$$\int_K u_{\varepsilon 3}^2 dx \le C\varepsilon^{p_*}, \tag{3.4}$$

where C is independent of ε. Combining the fact $|u_\varepsilon| = 1$ a.e. on $\overline{G}$ with (3.3) we know that there exist $u_p \in W^{1,p(x)}(K, \partial B_1)$ and a subsequence u_{ε_k} of u_ε, such that as $\varepsilon_k \to 0$,

$$u_{\varepsilon_k} \to (u_p, 0), \text{ weakly in } W^{1,p(x)}(K), \tag{3.5}$$

$$u_{\varepsilon_k} \to (u_p, 0), \text{ in } C^\alpha(\overline{K}), \tag{3.6}$$

for some $\alpha \in (0, 1 - \frac{2}{p_*})$. In the following we will prove that u_p is a weak solution of (2.1).

Let $B = B(x,3R) \subset K$. $\phi \in C_0^\infty(B(x,3R);[0,1])$, $\phi = 1$ on $B(x,R)$, $\phi = 0$ on $B \setminus B(x,2R)$ and $|\nabla\phi| \le C$, where C is independent of ε. Denote $u = u_{\varepsilon_k}$ in (3.2) and take $\psi = (0,0,\phi)$. Thus

$$\int_B |\nabla u|^{p(x)-2}\nabla u_3 \nabla\phi dx + \int_B \frac{1}{\varepsilon_k^{p(x)}}|u'|^2\phi u_3 dx = \int_B u_3\phi|\nabla u|^{p(x)}dx.$$

Applying (3.3) we can derive that

$$\int_B \frac{1}{\varepsilon_k^{p(x)}}|u'|^2\phi|u_3|dx \le \int_B |\nabla u|^{p(x)}\phi dx + \int_B |\nabla u|^{p(x)-1}|\nabla\phi|dx \le C. \tag{3.7}$$

From (3.6) it follows $|u'| \ge 1/2$ when ε_k is sufficiently small. Noting $\phi = 1$ on $B(x,R)$, we have

$$\int_{B(x,R)} \frac{1}{\varepsilon_k^{p(x)}}|u_3|dx \le C. \tag{3.8}$$

Taking $\frac{1}{k} = \varepsilon_k$, $F_k = \frac{1}{\varepsilon_k^{p(x)}}(u_{\varepsilon_k}u_{\varepsilon_k 3}^2 - u_{\varepsilon_k 3}e_3)$ in Lemma 3.11 of [10] (the proof is similar to Theorem 2.1 in [6]), noting $|F_k| = \frac{1}{\varepsilon_k^{p(x)}}|u_3|^2|u'|$ and applying (3.5) and (3.8), we obtain that

$$\lim_{\varepsilon_k\to 0}\nabla u_{\varepsilon_k} = \nabla u_p, \text{ in } L^q(B(x,R)),\ \forall q \in (1,p(x)).$$

Since $B(x,R)$ is an arbitrary disc in K, we can see that, for any $\xi \in C_0^\infty(B,\mathbb{R}^3)$ there holds

$$\lim_{\varepsilon_k\to 0}\int_B |\nabla u_{\varepsilon_k}|^{p(x)-2}\nabla u_{\varepsilon_k}\nabla\xi dx = \int_B |\nabla u_p|^{p(x)-2}\nabla u_p\nabla\xi dx. \tag{3.9}$$

Now, denote $u' = u'_{\varepsilon_k} = (u_1,u_2)$. Taking $\psi = (u_2,0,0)\zeta$ and $\psi = (0,u_1,0)\zeta$ in (3.2) respectively, where $\zeta \in C_0^\infty(B,\mathbb{R})$, we have

$$\begin{aligned}\int_B \frac{1}{\varepsilon_k^{p(x)}}u_3^2u_1u_2\zeta dx + \int_B u_1u_2\zeta|\nabla u|^{p(x)}dx &= \int_B |\nabla u|^{p(x)-2}\nabla u_1\nabla u_2\zeta dx \\ &\quad + \int_B u_2|\nabla u|^{p(x)-2}\nabla u_1\nabla\zeta dx.\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\int_B \frac{1}{\varepsilon_k^{p(x)}}u_3^2u_2u_1\zeta dx + \int_B u_2u_1\zeta|\nabla u|^{p(x)}dx &= \int_B |\nabla u|^{p(x)-2}\nabla u_2\nabla u_1\zeta dx \\ &\quad + \int_B u_1|\nabla u|^{p(x)-2}\nabla u_2\nabla\zeta dx.\end{aligned} \tag{3.11}$$

Equation (3.10) subtracts (3.11), then

$$0 = \int_B |\nabla u|^{p(x)-2}(u\wedge\nabla u)\nabla\zeta dx, \tag{3.12}$$

where $u \wedge \nabla u = u_1 \nabla u_2 - u_2 \nabla u_1$. On the other hand, since

$$\int_B u_2 |\nabla u|^{p(x)-2} \nabla u_1 \nabla \zeta dx - \int_B u_{p2} |\nabla u_p|^{p(x)-2} \nabla u_{p1} \nabla \zeta dx$$
$$= \int_B (|\nabla u|^{p(x)-2} \nabla u_1 - |\nabla u_p|^{p(x)-2} \nabla u_{p1}) u_{p2} \nabla \zeta dx + \int_B |\nabla u|^{p(x)-2} \nabla u_1 \nabla \zeta (u_2 - u_{p2}) dx,$$

using (3.3), (3.6) and (3.9), we obtain that

$$\lim_{\varepsilon_k \to 0} \int_B u_2 |\nabla u|^{p(x)-2} \nabla u_1 \nabla \zeta dx \to \int_B u_{p2} |\nabla u_p|^{p(x)-2} \nabla u_{p1} \nabla \zeta dx. \tag{3.13}$$

Similarly, we may also get that

$$\lim_{\varepsilon_k \to 0} \int_B u_1 |\nabla u|^{p(x)-2} \nabla u_2 \nabla \zeta dx = \int_B u_{p1} |\nabla u_p|^{p(x)-2} \nabla u_{p2} \nabla \zeta dx. \tag{3.14}$$

Clearly, (3.14) subtracting (3.13) yields

$$\lim_{\varepsilon_k \to 0} \int_B |\nabla u|^{p(x)-2} (u \wedge \nabla u) \nabla \zeta dx = \int_B |\nabla u_p|^{p(x)-2} (u_p \wedge \nabla u_p) \nabla \zeta dx.$$

Combining this with (3.12), we have

$$\int_B |\nabla u_p|^{p(x)-2} (u_p \wedge \nabla u_p) \nabla \zeta dx = 0. \tag{3.15}$$

Let $u_* = u_{p1} + i u_{p2} : B \to \mathbb{C}$, where $\mathbb{C}$ is the complex plane. Thus

$$|\nabla u_*|^2 = |\nabla u_p|^2. \tag{3.16}$$

It is easy to see that

$$\overline{u_*} \nabla u_* = \nabla(|u_*|^2) + (u_* \wedge \nabla u_*) i = 0 + (u_* \wedge \nabla u_*) i$$

since $|u_*|^2 = |u_{p1}|^2 + |u_{p2}|^2 = 1$. Substituting this into (3.15) yields

$$-i \int_B |\nabla u_*|^{p(x)-2} \overline{u_*} \nabla u_* \nabla \zeta dx = 0$$

for any $\zeta \in C_0^\infty(B, \mathbb{R})$. Taking $\zeta = Re(u_* \phi_j)$ and $\zeta = Im(u_* \phi_j)$ $(j = 1, 2)$ respectively, where $\phi = (\phi_1, \phi_2) \in C_0^\infty(B, \mathbb{R}^2)$, we can see that

$$\int_B |\nabla u_*|^{p(x)-2} \overline{u_*} \nabla u_* \nabla Re(u_* \phi) dx + i \int_B |\nabla u_*|^{p(x)-2} \overline{u_*} \nabla u_* \nabla Im(u_* \phi) dx = 0.$$

Namely

$$0 = \int_G |\nabla u_*|^{p(x)-2} \overline{u_*} \nabla u_* \nabla (u_* \phi) dx.$$

Noting $\overline{u_*} \nabla u_* = -u_* \nabla \overline{u_*}$, we obtain

$$0 = \int_B |\nabla u_*|^{p(x)-2} \nabla u_* \nabla \phi dx - \int_B |\nabla u_*|^{p(x)-2} u_* \nabla \overline{u_*} \nabla u_* \phi dx$$
$$= \int_B |\nabla u_*|^{p(x)-2} \nabla u_* \nabla \phi dx - \int_B |\nabla u_*|^p(x) u_* \phi dx := J.$$

By using (3.16) and $Re(J) = 0$, $Im(J) = 0$, we have

$$\int_B |\nabla u_p|^{p(x)-2} \nabla u_{p1} \nabla \phi dx = \int_B |\nabla u_p|^{p(x)} u_{p1} \phi dx \tag{3.17}$$

and

$$\int_B |\nabla u_p|^{p(x)-2} \nabla u_{p2} \nabla \phi dx = \int_B |\nabla u_p|^{p(x)} u_{p2} \phi dx.$$

Combining this with (3.17) yields that for any $\phi \in C_0^\infty(B, \mathbb{R}^3)$,

$$\int_B |\nabla u_p|^{p(x)-2} \nabla u_p \nabla \phi dx = \int_B |\nabla u_p|^{p(x)} u_p \phi dx.$$

This shows that u_p is a weak solution of (2.5). From (3.5), we know that (1.2) is proved.

Proof of (1.3). For simplification, we drop ε and ε_k from u_ε and u_{ε_k}. From (3.3) and (3.6) it is deduced that as $\varepsilon \to 0$,

$$\left| \int_K u_3^2 \zeta |\nabla u|^{p(x)} dx \right| \le \sup_K (1 - |u'|^2) \cdot \int_K |\nabla u|^{p(x)} dx \to 0, \tag{3.18}$$

$$\begin{aligned} \left| \int_K u' u_p \zeta |\nabla u|^{p(x)} dx - \int_K \zeta |\nabla u|^{p(x)} dx \right| &= \left| \int_K (u' u_p - u_p u_p) \zeta |\nabla u|^{p(x)} dx \right| \\ &\le \sup_K |u' - u_p| \cdot \left| \int_K u_p |\nabla u|^{p(x)} dx \right| \to 0, \end{aligned} \tag{3.19}$$

and

$$\left| \int_K (u - (u_p, 0)) \zeta |\nabla u|^{p(x)} dx \right| \le \sup_K |u - (u_p, 0)| \cdot \left| \int_K u_p |\nabla u|^{p(x)} dx \right| \to 0. \tag{3.20}$$

Similarly, (3.4) and (3.6) imply that as $\varepsilon \to 0$,

$$\left| \int_K \frac{1}{\varepsilon^{p(x)}} u_3^2 \zeta dx - \int_K \frac{1}{\varepsilon^{p(x)}} u_3^2 \zeta (1 - u_3^2) dx \right| \le \sup_K |1 - |u'|^2 \cdot \left| \int_K \frac{1}{\varepsilon^{p(x)}} u_3^2 dx \right| \to 0, \tag{3.21}$$

and

$$\left| \int_K \frac{1}{\varepsilon^{p(x)}} u_p \zeta u' u_3^2 dx - \int_K \frac{1}{\varepsilon^{p(x)}} \zeta u_3^2 dx \right| \le \sup_K |u' - u_p| \cdot \left| \int_K \frac{1}{\varepsilon^{p(x)}} u_p u_3^2 dx \right| \to 0. \tag{3.22}$$

Letting $\varepsilon \to 0$ in (3.2) we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_K \left[u \psi |\nabla u|^{p(x)} + \frac{1}{\varepsilon^{p(x)}} \psi (u u_3^2 - u_3 e_3) \right] dx &= \int_K |\nabla u_p|^{p(x)-2} \nabla (u_p, 0) \nabla \psi dx \\ &= \int_G (u_p, 0) \psi |\nabla u_p|^{p(x)} dx. \end{aligned} \tag{3.23}$$

Take $\psi = (0, 0, u_3 \zeta)$ where $\zeta \in C_0^\infty(K)$ we have

$$\lim_{\varepsilon \to 0} \int_K \left[u_3^2 \zeta |\nabla u|^{p(x)} + \frac{1}{\varepsilon^{p(x)}} u_3^2 \zeta (u_3^2 - 1) \right] dx = 0.$$

Combining this with (3.18) we derive

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}} u_3^2\zeta(u_3^2-1)dx = 0.$$

Substituting this into (3.21) yields

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}} u_3^2\zeta dx = 0. \tag{3.24}$$

Hence, as $\varepsilon \to 0$,

$$\left|\int_K \frac{1}{\varepsilon^{p(x)}} u u_3^2\zeta dx\right| \le \int_K \frac{1}{\varepsilon^{p(x)}} u_3^2\zeta dx \to 0.$$

Thus, for any $\psi \in W_0^{1,p}(K,\mathbb{R}^3)$, there holds

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}} u u_3^2\psi dx = 0. \tag{3.25}$$

In addition, substituting (3.24) into (3.22) leads to

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}} u_p\zeta u' u_3^2 dx = 0. \tag{3.26}$$

Take $\psi = (u_p\zeta, 0)$ in (3.23) we have

$$\lim_{\varepsilon\to 0}\int_K \left[u' u_p\zeta|\nabla u|^{p(x)}dx + \frac{1}{\varepsilon^{p(x)}} u_p\zeta u' u_3^2\right] dx = \int_K |\nabla u_p|^{p(x)}\zeta dx,$$

which, together with (3.26), implies

$$\lim_{\varepsilon\to 0}\int_K u' u_p\zeta|\nabla u|^{p(x)}dx = \int_K |\nabla u_p|^{p(x)}\zeta dx.$$

Combining this with (3.19) we can see (1.3) at last.
Proof of (1.4). Obviously, (3.20) and (1.3) show that as $\varepsilon \to 0$,

$$\begin{aligned}&\left|\int_K u|\nabla u|^{p(x)}\psi dx - \int_K (u_p,0)|\nabla u_p|^{p(x)}\psi dx\right| \\ &\le \left|\int_K (u-(u_p,0))|\nabla u|^{p(x)}\psi dx\right| + \left|\int_K (u_p,0)(|\nabla u|^{p(x)} - |\nabla u_p|^{p(x)})\psi dx\right| \to 0.\end{aligned}$$

Substituting this and (3.25) into (3.23), we see that the left hand side of (3.23) becomes

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_K \left[u\psi|\nabla u|^{p(x)} + \frac{1}{\varepsilon^{p(x)}}\psi(u u_3^2 - u_3 e_3)\right] dx \\ &= \int_K (u_p,0)|\nabla u_p|^{p(x)}\psi dx - \lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}}\psi u_3 e_3 dx.\end{aligned}$$

Comparing this with the right hand side of (3.23), we have

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}}\psi u_3 e_3 dx = 0.$$

This is (1.4). Theorem 2 is proved.

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Author's address

Bei Wang
Jiangsu Second Normal University, School of Mathematics and Information Technology, 210013, Nanjing, P. R. China
E-mail address: jsjywang@126.com