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EXISTENCE THEOREMS FOR EQUATIONS AND SYSTEMS IN R^N WITH k_i-HESSIAN OPERATOR

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Abstract. We establish the existence of positive radial entire solutions for nonlinear equations and systems. Our main results obtained with the use of the Schauder-Tychonov fixed point theorem will complete the works of Kusano-Swanson and Holanda.

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1. INTRODUCTION

We consider the existence of entire radial solutions for the following nonlinear equation

$$S_{k_1}\left(\Lambda\left(D^2u_1\right)\right)-\alpha_1 S_{k_2}\left(\Lambda\left(D^2u_1\right)\right)=\lambda_1 f\left(|x|,u_1,|\nabla u_1|\right),\quad x\in\mathbb{R}^N, \tag{1.1}$$

and the nonlinear system

$$\begin{cases} S_{k_1}\left(\Lambda\left(D^2u_1\right)\right)-\alpha_1 S_{k_2}\left(\Lambda\left(D^2u_1\right)\right)=\lambda_1 f_1\left(|x|,u_1,u_2,|\nabla u_1|,|\nabla u_2|\right), & x\in\mathbb{R}^N,\\ S_{k_1}\left(\Lambda\left(D^2u_2\right)\right)-\alpha_2 S_{k_2}\left(\Lambda\left(D^2u_2\right)\right)=\lambda_2 f_2\left(|x|,u_1,u_2,|\nabla u_1|,|\nabla u_2|\right), & x\in\mathbb{R}^N, \end{cases} \tag{1.2}$$

where $N\geq 3$, $\alpha_i\in(0,\infty)$, $\lambda_i\in\mathbb{R}$, $k_i\in\{1,2,...,N\}$ with $k_1>2k_2$, D^2u_i is the Hessian matrix of a C^2 function u_i defined over $\mathbb{R}^N$, $\Lambda\left(D^2u_i\right)=\left(\Lambda_1^i,...,\Lambda_N^i\right)$ is the vector of eigenvalues of D^2u_i, $S_{k_i}\left(\Lambda\left(D^2u_i\right)\right)$ is the k_i- Hessian operator defined by

$$S_{k_i}\left(\Lambda\left(D^2u_i\right)\right)=\sum_{1\leq j_1<...<j_{k_i}\leq N}\Lambda_{j_1}^i\cdot...\cdot\Lambda_{j_{k_i}}^i,\quad i=1,2,$$

and the nonlinearities f, f_1 and f_2 satisfy some of the conditions that will be specified later.

Historically, when $\alpha_1\in(0,\infty)$, $k_1=N$ and $k_2=1$, (1.1) is referred in differential geometry as the equation for prescribed generalized Gaussian curvature, as pointed in the paper written with a long time ago by Kusano and Swanson [5] (for further details, see Pogorelov [6, Chap. 10-13]). In this situation, with $k_1=N$ and $k_2=1$, from mathematical point of view the authors [5] carried out a systematic study of (1.1) and gave sufficient conditions on f to admit solutions in $\mathbb{R}^N$.

Recently, in the case $\alpha_1 = \alpha_2 = 0$, the question of existence of solutions for the more general equations and systems of the form (1.1) and (1.2) was studied by Holanda [2]. So, the case $\alpha_1, \alpha_2 \in (0,\infty)$ is a whole new ball game. Observing the results obtained until now and practical character of the problems of the type (1.1)/(1.2), there are several reasons to restate the research of these more general equations and systems.

In this work, we shall consider directly the equation (1.1), the system (1.2) and we intend to obtain the same results as observed by Kusano and Swanson [5], in the particular case mentioned. We also point that, our results can be adapted to a more larger classes of problems that was studied by Holanda [2].

The outline of the paper is as follows. Section 2 present the hypotheses on the nonlinearities and our main results. In Section 3 we give some auxiliary results. Our main Theorems will be proved in Section 4.

2. The main results

2.1. *The scalar equation (1.1)*

We work with the following classes of functions f:

(C1) $|f(t,x,z)|$ is monotone nondecreasing function with respect to x and z for fixed values of other variables. Meaning, $x \to |f(t,x,z)|$ and $z \to |f(t,x,z)|$ are monotone nondecreasing, for fixed (t,z) and fixed (t,x) respectively;

(C2) $F(m) = \sup_{t\in[0,\infty)} \left|f\left(t, m\left(1+t^2\right), 2mt\right)\right| < \infty$ for all $m > 0$;

(C3) $\lim_{m\to\infty} \frac{F(m)}{m^{k_1}} = 0$;

(C4)
$$H(c) = \sup_{t\in[0,\infty)} \left|f\left(t, c + 2\alpha_{N;k_1,k_2} t^2, 4\alpha_{N;k_1,k_2} t\right)\right| < \infty$$
for all $c \in \mathbb{R}$, where $\alpha^1_{N;k_1,k_2}$ is defined by
$$\alpha^1_{N;k_1,k_2} = \left(\alpha_1 k_1 C^{k_2-1}_{N-1} / k_2 C^{k_1-1}_{N-1}\right)^{1/(k_1-k_2)}, \quad C^{k_2-1}_{N-1} = (N-1)!/(k_2-1)!(N-k_2)!;$$

(C5) $\lim_{c\to-\infty} H(c) = 0$.

The main results of the paper for the equation (1.1) are the following theorems.

Theorem 1. *Suppose that $f\colon [0,\infty)\times(0,\infty)\times[0,\infty) \to \mathbb{R}$ is a continuous function that satisfies the assumptions (C1) and (C2). Then, there exists $\lambda^1_0 > 0$ such that for all $\lambda_1 \in \left[-\lambda^1_0, \lambda^1_0\right]$ the equation (1.1) has an infinitude of positive radial entire solutions $u_1(x) = u_1(|x|)$ such that*
$$0 < \lim_{|x|\to\infty} \inf \frac{u_1(|x|)}{|x|^2} \quad \textit{and} \quad \lim_{|x|\to\infty} \sup \frac{u_1(|x|)}{|x|^2} < \infty. \tag{2.1}$$

Theorem 2. *Assume that $f\colon\ [0,\infty)\times(0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function that satisfies the assumptions (C1), (C2) and (C3). Then, for all $\lambda_1\in[0,\infty)$ the equation (1.1) has an infinitude of positive radial entire solutions $u_1(x)=u_1(|x|)$ satisfying (2.1).*

Theorem 3. *If $f\colon\ [0,\infty)\times\mathbb{R}\times[0,\infty)\to\mathbb{R}$ is a continuous function that satisfies the assumptions (C1), (C4) and (C5) then, for all $\lambda_1\in\mathbb{R}$ the equation (1.1) has an infinitude of radial entire solutions $u_1(x)=u_1(|x|)$ which are positive in a neighborhood of infinity and satisfy (2.1).*

2.2. *The system of equations (1.2)*

Let $i=1,2$ and consider the following classes of functions f_1 and f_2:

(CS1) $|f_i(t,s,l,q,w)|$ is monotone nondecreasing function with respect to t, s, l, q and w for fixed values of other variables;

(CS2) $F_i(m_1,m_2)=\sup_{t\in[0,\infty)}\left|f_i\left(t,m_1\left(1+t^2\right),m_2\left(1+t^2\right),2m_1t,2m_2t\right)\right|<\infty$ for all $m_1,m_2>0$;

(CS3)
$$\lim_{m_1\to\infty}\frac{F_1(m_1,m_2)}{(m_1)^{k_1}}=0\quad\text{for all } m_2>0$$
and
$$\lim_{m_2\to\infty}\frac{F_2(m_1,m_2)}{(m_2)^{k_1}}=0\quad\text{for all } m_1>0;$$

(CS4)
$$H_i(c_1,c_2)=\sup_{t\in[0,\infty)}\left|f_i\left(t,c_1+2\alpha^1_{N;k_1,k_2}t^2,c_2+2\alpha^2_{N;k_1,k_2}t^2,4\alpha^1_{N;k_1,k_2}t,4\alpha^2_{N;k_1,k_2}t\right)\right|<\infty$$
for all $c_1,c_2\in\mathbb{R}$, where $\alpha^i_{N;k_1,k_2}$ is defined by
$$\alpha^i_{N;k_1,k_2}=\left(\alpha_i k_1 C^{k_2-1}_{N-1}/k_2 C^{k_1-1}_{N-1}\right)^{1/(k_1-k_2)},\quad C^{k_2-1}_{N-1}=(N-1)!/(k_2-1)!(N-k_2)!;$$

(CS5) $\lim_{c_1\to-\infty}H_1(c_1,c_2)=0$ for all $c_2\in\mathbb{R}$ and $\lim_{c_2\to-\infty}H_2(c_1,c_2)=0$ for all $c_1\in\mathbb{R}$.

Regarding the system (1.2), Our main results are the following theorems.

Theorem 4. *Let*
$$D=[0,\infty)\times(0,\infty)\times(0,\infty)\times[0,\infty)\times[0,\infty).$$
Suppose that f_1, $f_2\colon D\to\mathbb{R}$ are continuous functions that satisfy the assumptions (CS1) and (CS2). Then, there exists $\lambda^i_0>0$ such that for all $\lambda_i\in\left[-\lambda^i_0,\lambda^i_0\right]$ the system (1.2) has an infinitude of positive radial entire solutions $(u_1(x),u_2(x))=(u_1(|x|),u_2(|x|))$ such that
$$0<\lim_{|x|\to\infty}\inf\frac{u_i(|x|)}{|x|^2}\quad\textit{and}\quad\lim_{|x|\to\infty}\sup\frac{u_i(|x|)}{|x|^2}<\infty,\quad i=1,2.\tag{2.2}$$

Theorem 5. *Let*

$$D=[0,\infty)\times(0,\infty)\times(0,\infty)\times[0,\infty)\times[0,\infty).$$

Assume that $f_1,\ f_2\colon D\to[0,\infty)$ are continuous functions that satisfy the assumptions (CS1), (CS2) and (CS3). Then, for all $\lambda_i\in[0,\infty)$ the system (1.2) has an infinitude of positive radial entire solutions $(u_1(x),u_2(x))=(u_1(|x|),u_2(|x|))$ satisfying (2.2).

Theorem 6. *Let*

$$D=[0,\infty)\times\mathbb{R}\times\mathbb{R}\times[0,\infty)\times[0,\infty).$$

Suppose that $f_1,\ f_2\colon D\to\mathbb{R}$ are continuous functions that satisfy the assumptions (CS1), (CS4) and (CS5). Then, for all $\lambda_i\in\mathbb{R}$ the system (1.2) has an infinitude of radial entire solutions $(u_1(x),u_2(x))=(u_1(|x|),u_2(|x|))$ which are positive in a neighborhood of infinity and satisfy (2.2).

Comparing the results in the paper of [5] with our Theorems 1-3, we consider that our work is a consistent generalization of [5] from the mathematical point view and on the other hand Theorems 4-6 are excellent sources of inspiration for the futures works in treating a more general classes of equations and systems that was considered by Holanda [2].

3. Preliminary considerations

To prove our theorems, we make to introduce the following useful result.

Lemma 1. *Setting*

$$\varphi(t)=t^{k_1}-t^{k_2}\quad \text{for } t\in\mathbb{R},\ t_0=(k_2/k_1)^{1/(k_1-k_2)},$$

the following hold:

1. $\varphi(t_0)=\frac{k_2-k_1}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)}<0$, $\varphi(1)=0$ *and* $\varphi(\infty):=\lim_{t\to\infty}\varphi(t)=\infty$*;*
2. $\varphi\colon[t_0,\infty)\to[\varphi(t_0),\infty)$ *is strictly increasing for all* $t>t_0$ *and in fact has a uniquely defined inverse function* $\phi\colon[\varphi(t_0),\infty)\to[t_0,\infty)$ *with* $\phi(0)=1$*;*
3. $\phi\colon[\varphi(t_0),\infty)\to[t_0,\infty)$ *is analytic and concave. In particular,* $\phi(t)\geq 1$ *for all* $t\geq 0$, $\phi(\infty):=\lim_{t\to\infty}\phi(t)=\infty$ *and for* $t>\varphi(t_0)$ *we have that* ϕ *is strictly increasing and it hold*
$$\phi'(t)=\frac{1}{k_1(\phi(t))^{k_1-1}-k_2(\phi(t))^{k_2-1}}>0$$
and
$$\phi''(t)=-\frac{k_1(k_1-1)(\phi(t))^{k_1-2}-k_2(k_2-1)(\phi(t))^{k_2-2}}{\left[k_1(\phi(t))^{k_1-1}-k_2(\phi(t))^{k_2-1}\right]^3}<0;$$
4. $\phi(s\xi)\leq\xi^{1/k_1}\phi(s)$ *for all* $s\geq 0$ *and* $\xi\geq 1$.

Proof of Lemma 1.

1. By a direct calculation

$$\varphi(t_0) = \left(\frac{k_2}{k_1}\right)^{\frac{k_1}{k_1-k_2}} - \left(\frac{k_2}{k_1}\right)^{\frac{k_2}{k_1-k_2}} = \left(\frac{k_2}{k_1}\right)^{\frac{k_1-k_2+k_2}{k_1-k_2}} - \left(\frac{k_2}{k_1}\right)^{\frac{k_2}{k_1-k_2}}$$
$$= \left(\frac{k_2}{k_1}\right)^{\frac{k_2}{k_1-k_2}} \left[\left(\frac{k_2}{k_1} - 1\right)\right] = \frac{k_2-k_1}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)},$$

and from $k_1 > k_2$ we obtain $\varphi(t_0) < 0$. Clearly $\varphi(1) = 0$ and $\varphi(\infty) := \lim_{t\to\infty}\varphi(t) = \infty$.

2. Since φ is differentiable, we have

$$\varphi'(t) = k_1 t^{k_1-1} - k_2 t^{k_2-1} = t^{k_2-1}\left(k_1 t^{k_1-k_2} - k_2\right) \geq 0 \quad \text{for} \quad t \geq t_0$$

and so φ is strictly increasing on (t_0,∞) with the range $[\varphi(t_0),\infty)$. Whereas φ is bijective with the inverse function satisfying $\phi(0) = 1$.

3. It is important to note that the last part of the result follow from standard inversion theorem, from where

$$\phi'(t) = \frac{1}{(\varphi'\circ\phi)(t)}.$$

On the other hand

$$\frac{k_2}{k_1} \geq \frac{k_2(k_2-1)}{k_1(k_1-1)} \text{ and } \phi(t) > 0, \quad \text{for } t > \varphi(t_0),$$

implies $\phi''(t) < 0$ for all $t > \varphi(t_0)$.

4. By the basic fact $\sigma^{k_1} \geq \sigma$ for all $\sigma \geq 1$, we have that for each $t \geq 1$ and $\sigma \geq 1$

$$\varphi(t\sigma) = (t\sigma)^{k_1} - (t\sigma)^{k_2} = t^{k_1}\sigma^{k_1} - t^{k_2}\sigma^{k_2}$$
$$= \sigma^{k_1}\left(t^{k_1} - t^{k_2}\sigma^{k_2-k_1}\right) \geq \sigma^{k_1}\left(t^{k_1} - t^{k_2}\right) = \sigma^{k_1}\varphi(t)$$

where, we have used $\sigma^{k_2-k_1} \leq 1$. It follows that

$$t\sigma = \phi(\varphi(t\sigma)) \geq \phi\left(\sigma^{k_1}\varphi(t)\right).$$

Let $s = \varphi(t)$ and $\xi = \sigma^{k_1}$, i.e., $t = \Phi(s)$ and $\xi^{1/k_1} = \sigma$, one can see that **4.** holds.

□

To end the section, for the readers' convenience, we recall the radial form of the k-Hessian operator, see for example [1,3,4].

Lemma 2. *Assume $u \in C^2[0,R)$ is radially symmetric with $u'(0) = 0$. Then, the function u defined by $u(x) = u(r)$ where $r = |x| < R$ is $C^2(B_R)$, and*

$$\Lambda\left(D^2u(r)\right) = \begin{cases} \left(u''(r), \frac{u'(r)}{r}, \dots, \frac{u'(r)}{r}\right) & \text{for } r \in (0,R), \\ \left(u''(0), u''(0), \dots, u''(0)\right) & \text{for } r = 0; \end{cases}$$

$$S_k\left(\Lambda\left(D^2u(r)\right)\right) = \begin{cases} C_{N-1}^{k-1}u''(r)\left(\frac{u'(r)}{r}\right)^{k-1} + C_{N-1}^{k-1}\frac{N-k}{k}\left(\frac{u'(r)}{r}\right)^k & \text{for } r \in (0,R), \\ C_N^k\left(u''(0)\right)^k & \text{for } r = 0; \end{cases}$$

where the prime denotes differentiation with respect to r.

4. Proofs of the main results

Proofs of Theorems 1-3. Setting $r = |x|$ we prove the existence of a radial solution $u_1(r) \in C^2$ to the problem (1.1). Observe that we can rewrite (1.1) as follows:

$$C_{N-1}^{k_1-1}\left[\frac{r^{N-k_1}}{k_1}\left(u_1'(r)\right)^{k_1}\right]' - \alpha_1 C_{N-1}^{k_2-1}\left[\frac{r^{N-k_2}}{k_2}\left(u_1'(r)\right)^{k_2}\right]' = \lambda_1 r^{N-1} f\left(r, u_1(r), \left|u_1'(r)\right|\right). \quad (4.1)$$

Then, the radial solution of (4.1) is a solution u, of the ordinary differential equation (4.1) with the initial conditions

$$u_1(0) = c_0 \quad \text{and} \quad u_1'(0) = 0. \quad (4.2)$$

For $r > 0$ it follows that

$$\frac{C_{N-1}^{k_1-1}r^{N-k_1}}{k_1}\left(u_1'(r)\right)^{k_1} - \frac{\alpha_1 C_{N-1}^{k_2-1}r^{N-k_2}}{k_2}\left(u_1'(r)\right)^{k_2} = \int_0^r \lambda_1 s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds,$$

or, equivalently

$$\left(\frac{u_1'(r)}{\alpha_{N;k_1,k_2}^1 r}\right)^{k_1} - \left(\frac{u_1'(r)}{\alpha_{N;k_1,k_2}^1 r}\right)^{k_2} = \int_0^r \frac{\lambda_1 k_1 r^{-N} s^{N-1} f\left(s, u_1(s), |u_1'(s)|\right)}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} ds, \quad r > 0. \quad (4.3)$$

From the definition of ϕ in Lemma 1, we see that (4.3) is equivalent to

$$\frac{u_1'(r)}{\alpha_{N;k_1,k_2}^1 r} = \phi\left(\frac{\lambda_1 r^{-N}}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}}\int_0^r s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds\right), \quad r > 0. \quad (4.4)$$

Since $\lim_{r\to 0+} u_1'(r) = 0 = u_1^{'}(0) = 0$ via L'Hôpital's rule and (4.2), the equation (4.4) can be extended by continuity

$$u^{'}(r) = \alpha^1_{N;k_1,k_2} r\phi\left(\frac{\lambda_1 k_1 r^{-N}}{C^{k_1-1}_{N-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \int_0^r s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds\right), \quad r \geq 0,$$

for any C^1-function u. Then, (4.1) with the initial conditions (4.2) can be equivalently written as an integral equation

$$\begin{cases} u_1(r) = c_0 + \alpha^1_{N;k_1,k_2} \\ \qquad \int_0^r t\phi\left(\frac{\lambda_1 k_1 t^{-N}}{C^{k_1-1}_{N-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \int_0^t s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds\right) dt, \quad r \geq 0 \\ u_1(0) = c_0 \text{ and } u_1'(0) = 0. \end{cases} \tag{4.5}$$

To establish the existence of a solution to this problem (4.5), we use the Schauder-Tychonov fixed point theorem and hence $u_1(x) := u_1(r)$ is a radial entire solution of (1.1). □

Next, we are ready to prove our main results.

Proof of the Theorem 1. Our conditions permit to choose $\lambda_0^1 > 0$ such that

$$\frac{\lambda_0^1 k_1}{NC^{k_1-1}_{N-1}} \frac{F\left(2\alpha^1_{N;k_1,k_2}\right)}{\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \leq \frac{1}{k_1^{k_2}}.$$

Denote by C^1 the Fréchet space of all C^1- functions in $[0,\infty)$, with the topology of uniform convergence of functions and their first derivatives on compact subintervals of $[0,\infty)$. Let $c_0 \in \left(0, 2\alpha^1_{N;k_1,k_2}\right)$ arbitrarily fixed and we consider the closed convex set

$$K = \left\{u_1 \in C^1 \left| c_0 \leq u_1(r) \leq c_0 + 2\alpha^1_{N;k_1,k_2} r^2,\ 0 \leq u_1'(r) \leq 4\alpha^1_{N;k_1,k_2} r,\ r \geq 0\right.\right\}. \tag{4.6}$$

We define the mapping $T: K \to C^1$ by

$$(Tu_1)(r) = c_0 + \alpha^1_{N;k_1,k_2} \int_0^r t\phi(w(t))\,dt, \quad r \geq 0,\ u \in K, \tag{4.7}$$

where

$$w(t) = \frac{\lambda_1 k_1 t^{-N}}{C^{k_1-1}_{N-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \int_0^t s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds, \quad \lambda_1 \in \left[-\lambda_0^1, \lambda_0^1\right].$$

From the above settings, the assumptions (C1) and (C2), if $u_1 \in K$, then for all $r \geq 0$ and $\lambda_1 \in \left[-\lambda_0^1, \lambda_0^1\right]$ we have

$$\begin{aligned}
|w(t)| &\leq \frac{|\lambda_1| k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} \left|f\left(t, 2\alpha_{N;k_1,k_2}^1\left(1+t^2\right), 4\alpha_{N;k_1,k_2}^1 t\right)\right| \int_0^t s^{N-1} ds \\
&= \frac{|\lambda_1| k_1}{N C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} \left|f\left(t, 2\alpha_{N;k_1,k_2}^1\left(1+t^2\right), 4\alpha_{N;k_1,k_2}^1 t\right)\right| \\
&\leq \frac{\lambda_0^1 k_1}{N C_{N-1}^{k_1-1}} \frac{F\left(2\alpha_{N;k_1,k_2}^1\right)}{\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} \leq \frac{1}{k_1^{k_2}} < \frac{k_1-k_2}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)},
\end{aligned}$$

showing that T is well-defined on K.

Next, we observe that $\varphi(2) = 2^{k_1} - 2^{k_2} = 2^{k_2}\left(2^{k_1-k_2}-1\right) \geq 1$, from where

$$2 = \phi(\varphi(2)) \geq \phi(1).$$

Also, with the use of Lemma 1, if $u \in K$ we have

$$\begin{aligned}
\alpha_{N;k_1,k_2}^1 \phi(w(t)) &\leq \alpha_{N;k_1,k_2}^1 \phi\left(\frac{1}{k_1^{k_2}}\right) \\
&\leq \alpha_{N;k_1,k_2}^1 \phi\left(\left(\frac{1}{k_1}+1\right)^{k_2}\right) \\
&\leq \alpha_{N;k_1,k_2}^1 \phi\left(\left(\frac{1}{k_1}+1\right)^{k_2}\right) \\
&\leq \alpha_{N;k_1,k_2}^1 \left(\frac{1}{k_1}+1\right) \phi(1) \\
&\leq 2\alpha_{N;k_1,k_2}^1 \left(\frac{1}{k_1}+1\right) \leq 4\alpha_{N;k_1,k_2}^1, \quad t \geq 0,
\end{aligned}$$

and hence

$$c_0 \leq (Tu_1)(r) \leq c_0 + 2\alpha_{N;k_1,k_2}^1 r^2, \quad r \geq 0.$$

Furthermore

$$0 \leq (Tu_1)'(r) \leq \alpha_{N;k_1,k_2}^1 r \phi(w(r)) \leq 4\alpha_{N;k_1,k_2}^1 r, \quad r \geq 0, \tag{4.8}$$

implying that T maps K into itself.

Next, we prove the continuity of T in C^1- topology. To do this, let $\{u_1^n(r)\}_{n\geq 0}$ be a sequence in K converging to $u_1 \in K$ in this topology, and define

$$w_n(t) = \frac{\lambda k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} \int_0^t s^{N-1} f\left(s, u_1^n(s), \left|(u_1^n)'(s)\right|\right) ds,\ t \geq 0,\ \lambda_1 \in \left[-\lambda_0^1, \lambda_0^1\right].$$

Using (4.8) we have

$$|w_n(t) - w(t)| \leq \lambda_0^1 \left(\alpha_{N;k_1,k_2}^1\right)^{k_1} \sup_{0\leq r\leq t} \left|f\left(r, u_1^n(r), \left|(u_1^n)'(r)\right|\right) - f\left(r, u_1(r), \left|u_1'(r)\right|\right)\right|$$

and

$$\left|(Tu_1^n)'(t) - (Tu_1^n)'(t)\right| = \alpha_{N;k_1,k_2}^1 t\left|\phi(w_n(t)) - \phi(w(t))\right|.$$

The continuity of ϕ implies that $(Tu_1^n)'(t) \to (Tu_1)'(t)$ as $n \to \infty$ and the convergence is uniformly on every compact subinterval of $[0,\infty)$. Likewise, from (4.7) $(Tu_1^n)(t) \to (Tu_1)(t)$ as $n \to \infty$ uniformly on such subintervals, implying the continuity of T in C^1.

To prove that $T(K)$ has compact closure in C^1 via Arzela-Ascoli's theorem, we note that $(Tu_1)(r) \in C^2([0,\infty))$ for all $u_1 \in K$ and

$$\begin{aligned}(Tu_1)''(r) &= \left((Tu_1)'(r)\right)' = \alpha_{N;k_1,k_2}(r\phi(w(r)))' \\ &= \alpha_{N;k_1,k_2}^1 \phi(w(r)) + \alpha_{N;k_1,k_2}^1 r\phi'(w(r))w'(r) \\ &= \alpha_{N;k_1,k_2}^1 \phi(w(r)) + \lambda_1 \frac{k_1\left(\alpha_{N;k_1,k_2}^1\right)^{1-k_1}}{C_{N-1}^{k_1-1}} \phi'(w(r))\left[f\left(r, u_1(r), \left|u_1'(r)\right|\right)\right. \\ &\quad \left. - Nr^{-N}\int_0^r s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds\right]\end{aligned}$$

for all $r \geq 0$. Then, Lemma 1 imply the uniform bound

$$\left|(Tu_1)''(r)\right| \leq \alpha_{N;k_1,k_2}^1 \phi\left(\frac{1}{k_1^{k_2}}\right) + 2\lambda_0^1 k_1 \left(\alpha_{N;k_1,k_2}^1\right)^{1-k_1} \phi'\left(-\frac{1}{k_1^{k_2}}\right), \quad r \geq 0.$$

Thus

$$\sup\left|(Tu_1)''(r)\right| < \infty \quad \text{for any } r \in [a,b] \subset [0,\infty).$$

By the Mean Value Theorem

$$\left|(Tu_1)'(r_2) - (Tu_1)'(r_1)\right| \leq \sup_{r\in[a,b]} \left|(Tu_1)''(r)\right| |r_2 - r_1|,$$

with

$$r_1, r_2 \in [a,b] \subset [0,\infty),$$

we obtain that $(TK)' = \{(Tu_1)' | u_1 \in K\}$ is locally equicontinuous on any compact of $[0,\infty)$. Similarly, TK is locally equicontinuous on compact sets from $[0,\infty)$, and the local uniform boundedness of TK and $(TK)'$ is easily verified. Therefore, from Ascoli's Theorem, it follows that TK is relatively compact in the C^1-topology.

Finally, we can then apply the Schauder-Tychonov fixed point theorem to conclude that there exists an element $u_1 \in K$ such that $Tu_1 = u_1$. We have proved that u_1 satisfies (4.5), yielding a positive entire solution $u_1(x) = u_1(|x|)$ of equation (1.1) in $\mathbb{R}^N$.

To finish the proof of Theorem 1, the fact that $u_1(r)$ satisfies (2.1) follows from the inequalities

$$c_0 + \alpha^1_{N;k_1,k_2} (k_2/k_1)^{1/(k_1-k_2)} \frac{r^2}{2} \leq u_1(r) \leq c_0 + 2\alpha^1_{N;k_1,k_2} r^2, \quad r \geq 0, \tag{4.9}$$

where the left inequality in (4.9) is a consequence of the fact

$$\phi(t) > (k_2/k_1)^{1/(k_1-k_2)} \quad \text{for } t > \frac{k_2-k_1}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)},$$

and the right inequality in (4.9) is obvious from (4.6). On the other hand since any $c_0 \in \left(0, 2\alpha^1_{N;k_1,k_2}\right)$ will serve as an initial value $u_1(0) = c_0$ in (4.5), there exists an infinitude of positive radial entire solutions of equation (1.1). $\square$

Proof of Theorems 2 and 3. The proofs of Theorems 2 and 3 are virtually the same as that for Theorem 1, and the details will be omitted. For any further comments regarding the details, see the paper of Kusano and Swanson [5]. $\square$

Proofs of Theorems 4-6. Setting $r = |x|$ we prove the existence of a radial solution $(u_1, u_2) \in C^2 \times C^2$ to the problem (1.2). Denote

$$G_1(|x|, u_1, u_2) = f_1(|x|, u_1, u_2, |\nabla u_1|, |\nabla u_2|)$$

and

$$G_2(|x|, u_1, u_2) = f_2(|x|, u_1, u_2, |\nabla u_1|, |\nabla u_2|).$$

We observe that we can rewrite (1.2) as follows:

$$\begin{cases} C^{k_1-1}_{N-1}\left[\frac{r^{N-k_1}}{k_1}\left(u_1'(r)\right)^{k_1}\right]' - \alpha_1 C^{k_2-1}_{N-1}\left[\frac{r^{N-k_2}}{k_2}\left(u_1'(r)\right)^{k_2}\right]' = \\ \lambda_1 r^{N-1} G_1(r, u_1(r), u_2(r)), \\ C^{k_1-1}_{N-1}\left[\frac{r^{N-k_1}}{k_1}\left(u_2'(r)\right)^{k_1}\right]' - \alpha_2 C^{k_2-1}_{N-1}\left[\frac{r^{N-k_2}}{k_2}\left(u_2'(r)\right)^{k_2}\right]' = \\ \lambda_2 r^{N-1} G_2(r, u_1(r), u_2(r)). \end{cases} \tag{4.10}$$

Then, the radial solution of (4.10) is a solution u of the ordinary differential system (4.10) with the initial conditions

$$(u_1(0), u_2(0)) = (c_1, c_2) \text{ and } \left(u_1'(0), u_2'(0)\right) = (0,0). \tag{4.11}$$

It follows that

$$\begin{cases} u_1'(r) = \alpha^1_{N;k_1,k_2} r\phi\left(\frac{\lambda_1 r^{-N}}{C^{k_1-1}_{N-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \int_0^r s^{N-1} G_1(s,u_1(s),u_2(s))\,ds\right), & r \geq 0, \\ u_2'(r) = \alpha^2_{N;k_1,k_2} r\phi\left(\frac{\lambda_2 r^{-N}}{C^{k_1-1}_{N-1}\left(\alpha^2_{N;k_1,k_2}\right)^{k_1}} \int_0^r s^{N-1} G_2(s,u_1(s),u_2(s))\,ds\right), & r \geq 0. \end{cases}$$

for any C^1-function (u_1,u_2). Then, for $r \geq 0$, (4.10) with the initial conditions (4.11) can be equivalently written as an integral system of equations

$$\begin{cases} u_1(r) = c_1 + \alpha^1_{N;k_1,k_2} \int_0^r t\phi\left(\frac{\lambda_1 k_1 t^{-N}}{C^{k_1-1}_{N-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \int_0^t s^{N-1} G_1(s,u_1(s),u_2(s))\,ds\right) dt, \\ u_2(r) = c_2 + \alpha^2_{N;k_1,k_2} \int_0^r t\phi\left(\frac{\lambda_2 k_1 t^{-N}}{C^{k_1-1}_{N-1}\left(\alpha^2_{N;k_1,k_2}\right)^{k_1}} \int_0^t s^{N-1} G_2(s,u_1(s),u_2(s))\,ds\right) dt, \\ (u_1(0),u_2(0)) = (c_1,c_2) \text{ and } (u_1'(0),u_2'(0)) = (0,0). \end{cases} \tag{4.12}$$

To establish the existence of a solution to this problem (4.12), we use the Schauder-Tychonov fixed point theorem and hence $(u_1(x),u_2(x)) := (u_1(r),u_2(r))$ is a radial entire solution of (1.2). □

Next, we are ready to prove our main results.

Proof of the Theorem 4. Let $i = 1,2$. We choose $\lambda_0^i > 0$, such that

$$\frac{\lambda_0^i k_1}{N C^{k_1-1}_{N-1}\left(\alpha^i_{N;k_1,k_2}\right)^{k_1}} F_i\left(2\alpha^1_{N;k_1,k_2}, 2\alpha^2_{N;k_1,k_2}\right) \leq \frac{1}{k_1^{k_2}}.$$

For simplicity, denote by C^1 the Fréchet space of all C^1- functions in $[0,\infty)$, with the topology of uniform convergence of functions and their first derivatives on compact subintervals of $[0,\infty)$.

For a fixed choice of c_i in $\left(0, 2\alpha^i_{N;k_1,k_2}\right)$ and λ_i a small positive parameter, the solutions of (1.2) are fixed point of the compact operator $(T_1,T_2) : K_1 \times K_2 \to C^1 \times C^1$ defined by

$$(T_i u_i)(r) = c_i + \alpha^i_{N;k_1,k_2} \int_0^r t\phi(w_i(t))\,dt, \quad r \geq 0,\ u \in K, \tag{4.13}$$

where

$$w_i(t) = \frac{\lambda_i k_1 t^{-N}}{C^{k_1-1}_{N-1}\left(\alpha^i_{N;k_1,k_2}\right)^{k_1}} \int_0^t s^{N-1} G_i(s,u_1(s),u_2(s))\,ds, \quad \lambda_i \in \left[-\lambda_0^i, \lambda_0^i\right],$$

on the closed convex set

$$K_i = \left\{u \in C^1 \,\middle|\, c_i \leq u_i(r) \leq c_i + 2\alpha^i_{N;k_1,k_2} r^2,\ 0 \leq u_i'(r) \leq 4\alpha^i_{N;k_1,k_2} r,\ r \geq 0\right\}. \tag{4.14}$$

Similar to the proof of Theorem 1, if $(u_1,u_2) \in K_1 \times K_2$, for all $r \geq 0$ and $\lambda_i \in \left[-\lambda_0^i, \lambda_0^i\right]$ we have

$$|w_i(t)| \leq \frac{\lambda_0^i k_1}{N C_{N-1}^{k_1-1}} \frac{F_i\left(2\alpha_{N;k_1,k_2}^1, 2\alpha_{N;k_1,k_2}^2\right)}{\left(\alpha_{N;k_1,k_2}^i\right)^{k_1}} \leq \frac{1}{k_1^{k_2}} < \frac{k_1-k_2}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)}.$$

From the above analysis, (T_1,T_2) is well-defined on $K_1 \times K_2$. Also, if $u_i \in K_i$ we have

$$\alpha_{N;k_1,k_2}^i \phi(w_i(t)) \leq 4\alpha_{N;k_1,k_2}^i, \quad t \geq 0,$$

and hence

$$c_i \leq (T_i u_i)(r) \leq c_i + 2\alpha_{N;k_1,k_2}^i r^2, \quad r \geq 0.$$

Therefore

$$0 \leq (T_i u_i)'(r) \leq \alpha_{N;k_1,k_2}^i r \phi(w_i(r)) \leq 4\alpha_{N;k_1,k_2}^i r, \quad r \geq 0, \tag{4.15}$$

implying that (T_1,T_2) maps $K_1 \times K_2$ into itself.

Next, we prove the continuity of (T_1,T_2) in $C^1 \times C^1$- topology. To do this, let $\{(u_1^n(r), u_2^n(r))\}_{n\geq 0}$ be a sequence in $K_1 \times K_2$ converging to $(u_1,u_2) \in K_1 \times K_2$ in this topology, and define

$$w_i^n(t) = \frac{\lambda_i k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^i\right)^{k_1}} \int_0^t s^{N-1} G_i(s, u_1^n(s), u_2^n(s))\, ds, \quad t \geq 0,\ \lambda_i \in [-\lambda_0, \lambda_0].$$

Using (4.15) we have

$$|w_i^n(t) - w_i(t)| \leq \lambda_0^i \alpha_{N;k_1,k_2}^{k_1} \sup_{0\leq r\leq t} |G_i(r, u_1^n(r), u_2^n(r)) - G_i(r, u_1(r), u_2(r))|$$

and

$$\left|(T_i u_i^n)'(t) - (T_i u_i)'(t)\right| = \alpha_{N;k_1,k_2}^i t\, |\phi(w_i^n(t)) - \phi(w_i(t))|.$$

Due to the continuity of ϕ we get the convergences $(T_i u_i^n)'(t) \to (T_i u_i)'(t)$ as $n \to \infty$ uniformly on every compact subinterval of $[0,\infty)$. Likewise, from (4.13) $(T_i u_i^n)(t) \to (T_i u_i)(t)$ as $n \to \infty$ uniformly on such subintervals. So (T_1,T_2) is continuous in $C^1 \times C^2$-topology.

We prove that $T_i K_i$ has compact closure in C^1. We note that $(T_i u_i)(r) \in C^2([0,\infty))$ for all $u_i \in K_i$ and

$$(T_i u_i)''(r) = \alpha_{N;k_1,k_2}^i \phi(w_i(r)) + \lambda_i \frac{k_1\left(\alpha_{N;k_1,k_2}^i\right)^{1-k_1}}{C_{N-1}^{k_1-1}} \phi'(w_i(r)) [G_i(r, u_1(r), u_2(r))$$
$$- N r^{-N} \int_0^r s^{N-1} G_i(s, u_1(s), u_2(s))\, ds],$$

for all $r \geq 0$. Lemma 1 imply the uniform bound

$$\left|(T_i u_i)''(r)\right| \leq \alpha^i_{N;k_1,k_2} \phi\left(\frac{1}{k_1^{k_2}}\right) + 2\lambda_0 k_1 \left(\alpha^i_{N;k_1,k_2}\right)^{1-k_1} \phi'\left(-\frac{1}{k_1^{k_2}}\right), \quad r \geq 0,$$

and so $(T_i K_i)' = \{(T_i u_i)' | u_i \in K_i\}$ is locally equicontinuous in $[0,\infty)$. Similarly $T_i K_i$ is locally equicontinuous, and the local uniform boundedness of $T_i K_i$ and $(T_i K_i)'$ is easily verified. Therefore, from Ascoli's Theorem, it follows that $T_i K_i$ and $(T_i K_i)'$ are relatively compact in any compact interval of $[0,\infty)$. Consequently, by a diagonal sequential process, we conclude that $T_i K_i$ is relatively compact in the C^1-topology. Finally, we apply the Schauder-Tychonov fixed point theorem to conclude that there exists an element $(u_1,u_2) \in K_1 \times K_2$ such that $(T_1 u_1, T_2 u_2) = (u_1,u_2)$.

Then (u_1,u_2) satisfies (4.12), yielding a positive entire solution

$$(u_1(x), u_2(x)) := (u_1(|x|), u_2(|x|)),$$

to the original system (1.2).

The fact that $u_i(r)$ satisfies (2.2) follows from the inequalities

$$c_i + \alpha^i_{N;k_1,k_2} (k_2/k_1)^{1/(k_1-k_2)} \frac{r^2}{2} \leq u_i(r) \leq c_i + 2\alpha^i_{N;k_1,k_2} r^2, \quad r \geq 0, \tag{4.16}$$

where the left inequality in (4.16) is a consequence of the fact

$$\phi(t) > (k_2/k_1)^{1/(k_1-k_2)} \quad \text{for} \quad t > \frac{k_2-k_1}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)},$$

and the right inequality in (4.16) is obvious from (4.14). On the other hand since any

$$(c_1,c_2) \in \left(0, 2\alpha^1_{N;k_1,k_2}\right) \times \left(0, 2\alpha^2_{N;k_1,k_2}\right),$$

will serve as an initial value $(u_1(0), u_2(0)) = (c_1,c_2)$ in (4.12), there exists an infinitude of positive radial entire solutions of system (1.2) and, thus, Theorem 4 is proved. □

Proof of Theorems 5 and 6. Details of the proofs of Theorems 5 and 6 are omitted, since are virtually the same as that for Theorem 4, see also the paper of Kusano and Swanson [5] for any further comments, resulting a fixed point (u_1,u_2) of the mapping (T_1,T_2) defined by (4.13) in the set (4.14). □

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