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# INFINITELY MANY SOLUTIONS FOR A $p(x)$-KIRCHHOFF TYPE EQUATION WITH STEKLOV BOUNDARY VALUE

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*Abstract.* In the present article deal with the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff type problem with Steklov boundary-value. By variational approach and theory of the variable exponent Sobolev spaces, under appropriate assumptions on $f$, we obtain existence of infinitely solutions and at least one nontrivial weak solution.

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## 1. INTRODUCTION

The purpose of this article is to study the following nonlinear Steklov boundary value problem

$$\begin{cases} M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) \Delta_{p(x)} u = |u|^{p(x)-2} u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial v} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases} \tag{E}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $\lambda$ is a positive parameter $p \in C\left(\overline{\Omega}\right)$, $\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$ is a $p(x)$-Laplacian operator, $M\colon (0,\infty) \to (0,\infty)$ is a continuous Kirchohoff function and $f\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\frac{\partial u}{\partial v}$ is the outer unit normal derivative on $\partial\Omega$.

Problem (**E**) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [17]. In 1883, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where $L$, $E$, $\rho$, $P_0$, $h$ are constants. The above equation is an extension of the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. Moreover, equation $(\mathbf{E})$ can be used for modelling several physical and biological systems, where $u$ describes a process which depends on the average of itself, such as the population density [10].

Lions [19] has proposed an abstract framework for the Kirchhoff-type equations. After this work of Lions, various equations of Kirchhoff-type have attracted much attention [6, 9]. The study of Kirchhoff-type equations has initially been extended to the case involving the $p$-Laplacian operator, and then the equations containing the $p(x)$-Laplacian operator [11, 16].

The study of variation problems with variable exponent has extremely been attract in recently. Because such problems are used to model dynamical phenomena arising from the study of electrorheological fluids [20], elastic mechanics [25], like image processing and stationary thermo-rheological viscous flows of non-Newtonian fluids [2, 8, 21] and in the mathematical description of the processes filtration of an idea barotropic gas through a porous medium [5].

The Steklov problems involving $p(x)$-Laplacian have been worked by some of the authors [15, 22, 24]. Especially, the authors have studied the problems of type $(\mathbf{E})$ when $M(t) = 1$. For example, in [7], the author investigated the existence and multiplicity of solutions for Steklov problem with non-standard growth condition without using the Ambrosetti-Rabinowitz type condition. In [3], the author proved the existencel of solutions by using Ekeland variational principle together with min-max method. In [4], the authors obtained the existence and multiplicity of solutions for the nonlinear Steklov boundary value problem, using Mountain Pass, Fountain and Ricceri three critical points theorems. Moreover, in [1], they showed the existence and multiplicity of solutions using variational methods under suitable assumptions on the nonlinearity,.

Inspired by the papers above mentioned , we studied the Steklov problem involving the $p(x)$-Kirchhoff type operator. The present article is composed of two sections. In Section 2, we present some necessary preliminary knowledge of variable exponent Lebesgue-Sobolev spaces. In Section 3, using the variational method, we give the main results and their proofs.

## 2. PRELIMINARIES

To discuss problem $(\mathbf{E})$, we define some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ which will be used later. For more details, see [18].

Set

$$C_+\left(\overline{\Omega}\right) = \left\{p : \ p \in C\left(\overline{\Omega}\right), \ p(x) > 1 \text{ for all } x \in \overline{\Omega}\right\}.$$

For any $p(x) \in C_+\left(\overline{\Omega}\right)$, we write

$$1 < p^- := \max_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ := \min_{x\in\overline{\Omega}} p(x) < \infty.$$

Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega)=\left\{ u \mid u\colon \Omega\to\mathbb{R} \text{ is a measureable and } \int_\Omega |u(x)|^{p(x)}\, dx<\infty\right\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)}:=|u|_{p(x)}=\inf\left\{\eta>0:\ \int_\Omega\left|\frac{u(x)}{\eta}\right|^{p(x)}dx\le 1\right\}.$$

Also, we can define $C_+(\partial\Omega)$ and $p-,p^+$ for any $p(x)\in C(\partial\Omega)$, and denote

$$L^{p(x)}(\partial\Omega)=\left\{u \mid u\colon \partial\Omega\to\mathbb{R} \text{ is a measureable and } \int_{\partial\Omega}|u(x)|^{p(x)}d\sigma<\infty\right\},$$

with the norm

$$|u|_{L^{p(x)}(\partial\Omega)}=|u|_{p(x)}:=\inf\left\{\delta>0:\ \int_{\partial\Omega}\left|\frac{u(x)}{\delta}\right|^{p(x)}d\sigma\le 1\right\},$$

where $d\sigma$ is the measure on the boundary.

**Proposition 1** ([18, Theorem 2.1]). *For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$ we have the following Hölder-type inequality*

$$\left|\int_\Omega uv dx\right|\le\left(\frac{1}{p^-}+\frac{1}{(p^-)'}\right)|u|_{p(x)}|v|_{p'(x)},$$

*where $L^{p'(x)}(\Omega)$ denotes the conjugate space of $L^{p(x)}(\Omega)$ and* $\frac{1}{p'(x)}+\frac{1}{p(x)}=1.$

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\varphi_{p(x)}\colon L^{p(x)}(\Omega)\to\mathbb{R}$ defined by

$$\varphi_{p(x)}(u)=\int_\Omega|u(x)|^{p(x)}dx \quad \text{for all } u\in L^{p(x)}(\Omega),$$

and it satisfies the following proposition.

**Proposition 2** ([14, Theorem 1.3]). *For all $u,u_n\in L^{p(x)}(\Omega)$, $n=1,2,\dots$ we have*

(i) $|u|_{p(x)}>1\,(=1,>1)\Leftrightarrow\varphi_{p(x)}(u)>1\,(=1,>1)$,
(ii) $\min\left(|u|^{p^-}_{p(x)},|u|^{p^+}_{p(x)}\right)\le\varphi_{p(x)}(u)\le\max\left(|u|^{p^-}_{p(x)},|u|^{p^+}_{p(x)}\right)$,
(iii) $|u_n-u|_{p(x)}\to 0\,(\to\infty)\Leftrightarrow\varphi_{p(x)}(u_n-u)\to 0\,(\to\infty)$.

**Proposition 3** ([12, Proposition 2.4]). *Let $\phi(u)=\int_{\partial\Omega}|u(x)|^{p(x)}d\sigma$ for all $u$, $u_n\in L^{p(x)}(\partial\Omega)$, $n=1,2,\dots$ we have*

(i) $|u|_{L^{p(x)}(\partial\Omega)}>1\Rightarrow|u|^{p^-}_{L^{p(x)}(\partial\Omega)}\le\phi(u)\le|u|^{p^+}_{L^{p(x)}(\partial\Omega)}$,
(ii) $|u|_{L^{p(x)}(\partial\Omega)}<1\Rightarrow|u|^{p^+}_{L^{p(x)}(\partial\Omega)}\le\phi(u)\le|u|^{p^-}_{L^{p(x)}(\partial\Omega)}$,
(iii) $|u_n-u|_{p(x)}\to 0\ (\to\infty)\Leftrightarrow\phi(u_n-u)\to 0\ (\to\infty)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is denined by

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega):\ |\nabla u|\in L^{p(x)}(\Omega)\right\},$$

and equipped with the norm

$$\|u\|_{1,p(x)}=\inf\left\{\kappa>0:\int_{\Omega}\left(\left|\frac{\nabla u(x)}{\kappa}\right|^{p(x)}+\left|\frac{u(x)}{\kappa}\right|^{p(x)}\right)dx\le 1\right\},$$

or

$$\|u\|_{1,p(x)}=|u|_{p(x)}+|\nabla u|_{p(x)},\quad \forall u\in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm

$$\|u\|=|\nabla u|_{p(x)}\quad \text{for all } u\in W_0^{1,p(x)}(\Omega).$$

**Proposition 4** ([18, Theorem 3.1]; [14, Theorem 2.3]; [12, Theorem 2.1]; [13, Lemma 3.1])**.**

(i) *If $1<p^-\le p^+<\infty$, then the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *If $q(x)\in C_+\left(\overline{\Omega}\right)$ and $q(x)<p^*(x)$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where*

$$p^*(x):=\begin{cases}\frac{Np(x)}{N-p(x)} & \text{if } N>p(x),\\ \infty & \text{if } N\le p(x).\end{cases}$$

(iii) *If $q(x)\in C_+(\partial\Omega)$ and $q(x)<p^{\partial}(x)$ for all $x\in\partial\Omega$ then the trace embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\partial\Omega)$ is compact and continuous, where*

$$p^{\partial}(x):=\begin{cases}\frac{(N-1)p(x)}{N-p(x)} & \text{if } N>p(x)\\ \infty & \text{if } N\le p(x).\end{cases}$$

(iv) ***(Poincaré inequality).*** *There is a positive constant $C>0$ such that*

$$|u|_{p(x)}\le C\|u\|,\quad \forall u\in W_0^{1,p(x)}(\Omega).$$

**Proposition 5** ([13, Lemma 2.1])**.** *Let $p(x)$ and $q(x)$ be measurable functions such that $1\le p(x)q(x)\le\infty$ and $p(x)\in L^{\infty}(\Omega)$ for a.e. $x\in\Omega$. Let $u\in L^{q(x)}(\Omega)$, $u\ne 0$. Then*

$$\min\left(|u|^{p^-}_{p(x)q(x)},|u|^{p^+}_{p(x)q(x)}\right)\le\left||u|^{p(x)}\right|_{q(x)}\le\max\left(|u|^{p^-}_{p(x)q(x)},|u|^{p^+}_{p(x)q(x)}\right).$$

*In particular, if $p(x)=p$ is constant, then $||u|^p|_{q(x)}=|u|^p_{pq(x)}$. Throughout this paper, we assume that $f$ and $M$ satisfy the following assumptions:*

$(\mathbf{M}_0)$ *$M\colon (0,\infty)\to(0,\infty)$ is a continuous function such that*

$$m_1 s^{\alpha-1} \leq M(s) \leq m_2 s^{\alpha-1}, \quad \forall\, s>0$$

*where $m_1$, $m_2$ and $\alpha$ are real numbers such that $0<m_1\leq m_2$ and $\alpha>1$.*

$(\mathbf{f}_1)$ *$f\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies Carathéodory condition and*

$$|f(x,t)| \leq c_1\left(1+|t|^{q(x)-1}\right), \quad \forall (x,t)\in\partial\Omega\times\mathbb{R},$$

*where $c_1>0$ is positive constant and $q(x)\in C_+(\partial\Omega)$ such that $p^+<q^- < q(x) < p^{\partial}(x)$ for all $x\in\partial\Omega$.*

$(\mathbf{f}_2)$ *$f(x,t)=o\left(|t|^{\alpha p^+-1}\right)$ as $t\to 0$, uniformly for $x\in\partial\Omega$ and $\alpha p^+<q^-$.*

$(\mathbf{f}_3)$ *$f(x,-t)=-f(x,t)$ for all $(x,t)\in\partial\Omega\times\mathbb{R}$.*

$(\mathbf{AR})$ *Ambrosetti-Rabinowitz's Condition holds, i.e. there exists $M>0$ and $\theta>\frac{m_2\alpha(p^+)^\alpha}{m_1(p^-)^{\alpha-1}}$ such that*

$$0<\theta F(x,t)\leq f(x,t)t, \quad |t|\geq M \quad \text{for all } x\in\partial\Omega.$$

*Moreover, we will use $X$ instead of the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$.*

## 3. Main results and proofs

We present the main results of the paper:

**Theorem 1.** *Assume that $(\mathbf{M}_0)$, $(\mathbf{AR})$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$ and $p^+<\alpha p^-$ hold. Then there exists $\lambda^*>0$ such that for any $\lambda\in(0,\lambda^*)$, problem $(\mathbf{E})$ has at least one nontrivial weak solution.*

**Theorem 2.** *Assume that $(\mathbf{M}_0)$, $(\mathbf{AR})$, $(\mathbf{f}_1)$, $(\mathbf{f}_2)$, $(\mathbf{f}_3)$ and $p^+<\alpha p^-$ hold. Then there exists $\lambda^*>0$ such that for any $\lambda\in(0,\lambda^*)$, $I_\lambda$ has a sequence of critical points $\{u_n\}$ such that $I_\lambda(u_n)\to\infty$ as $n\to\infty$ and problem $(\mathbf{E})$ has infinite many pairs of weak solutions.*

**Proposition 6** ([4, Theorem 2.9])**.** *Let $f\colon \partial\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying $(\mathbf{f}_1)$. For each $u\in X$ set $\Psi(u)=\int_{\partial\Omega}F(x,u)\,d\sigma$. Then $\Psi(u)\in C^1(X,\mathbb{R})$ and*

$$\left\langle \Psi'(u),\upsilon\right\rangle = \int_{\partial\Omega} f(x,u)\,\upsilon\, d\sigma \quad \text{for all } \upsilon\in X.$$

*Moreover, the operator $\Psi\colon X\to X^*$ is compact.*

**Proposition 7** ([14, Proposition 3.1])**.** *If one denotes*

$$\psi(u)=\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx$$

*then* $\psi \in C^1(X,\mathbb{R})$ *and the derivative operator of* $\psi$*, denoted by* $\psi'$*, is*

$$\left\langle \psi'(u), v \right\rangle = \int_\Omega |\nabla u|^{p(x)-2} uvdx \quad \forall u, v \in X$$

*and one has*

(i) $\psi' : X \to X^*$ *is a continuous, bounded, strictly monotone operator and homeomorphism,*

(ii) $\psi'$ *is a mapping of* $(S_+)$ *type, that is if* $u_n \rightharpoonup u$ *in* $X$ *and* $\limsup_{n\to\infty} \langle \psi'(u_n), u_n - u\rangle \leq 0$ *implies, then* $u_n \to u$ *in* $X$*, where* $X = W_0^{1,p(x)}(\Omega)$.

We say that $u \in X$ is a weak solution of (**E**) if

$$M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega |u|^{p(x)-2} uvdx = \lambda \int_{\partial\Omega} f(x,u)\, v\, d\sigma$$

where $v \in X$. We associate to the problem (**E**) the energy functional, defined as $I_\lambda : X \to \mathbb{R}$,

$$I_\lambda(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) + \int_\Omega \frac{|u|^{p(x)}}{p(x)}dx - \lambda \int_{\partial\Omega} F(x,u)\, d\sigma,$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$ and $F(x,t) = \int_0^t f(x,s)\, ds$. Moreover, from $(\mathbf{f}_1)$ and $(\mathbf{M}_0)$, Proposition 6 and Proposition 7, it is easy to see that the functional $I_\lambda \in C^1(X,\mathbb{R})$ and we can infer that critical points of functional $I_\lambda$ are the weak solutions for problem (**E**). Then, we have

$$\begin{aligned}\langle I'_\lambda(u), \upsilon\rangle = {} & M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \upsilon dx \\ & + \int_\Omega |u|^{p(x)-2} u\upsilon dx - \lambda \int_{\partial\Omega} f(x,u)\, \upsilon d\sigma\end{aligned}$$

for any $u, \upsilon \in X$.

**Definition 1.** Let $X$ be a Banach spaces and $I_\lambda \in C^1(X,\mathbb{R})$. We say that $I_\lambda$ satisfies Palais-Smale condition (**PS**) if any sequence $\{u_n\}$ in $X$ such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

**Lemma 1.** *If* $(\mathbf{M}_0)$*,* $(\mathbf{f}_1)$*,* (**AR**) *and* $p^+ < \alpha p^-$ *hold, then for any* $\lambda \in (0,\infty)$ *the functional* $I_\lambda$ *satisfies* (**PS**) *condition.*

*Proof.* Let us assume that there exists a sequence $\{u_n\}$ in $X$ such that

$$|I_\lambda(u_n)| \leq c \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Initially we prove that $\{u_n\}$ is bounded in $X$. Considering $\|u_n\| > 1$ for $n$ large enough and using ($\mathbf{M}_0$), (**AR**), (3.1) we obtain

$$\begin{aligned} c+1 &\geq I_\lambda(u_n) - \frac{1}{\theta}\left\langle I'_\lambda(u_n), u_n\right\rangle \\ &\geq \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}}\right)\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^\alpha + \frac{1}{p^+}\int_\Omega |u_n|^{p(x)}dx \\ &\quad - \frac{1}{\theta}\int_\Omega |u_n|^{p(x)}dx - \lambda\left(\int_{\partial\Omega}\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\,d\sigma\right) \\ &\geq \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}}\right)\|u_n\|^{\alpha p^-} - \frac{c_1}{\theta}\|u_n\|^{p^+}. \end{aligned}$$

When we divide the last inequality by $\|u_n\|^{\alpha p^-}$,

$$\frac{c+1}{\|u_n\|^{\alpha p^-}} \geq \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}}\right) - \frac{c_1}{\theta}\frac{\|u_n\|^{p^+}}{\|u_n\|^{\alpha p^-}}$$

and pass to the limit as $n \to \infty$, we have

$$0 \geq \frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}}$$

or

$$\theta \leq \frac{m_2\alpha(p^+)^\alpha}{m_1(p^-)^{\alpha-1}}.$$

Since $\theta > \frac{m_2\alpha(p^+)^\alpha}{m_1(p^-)^{\alpha-1}}$ in the condition (**AR**), we obtain a contradiction. Thus $\{u_n\}$ is bounded in $X$; from Proposition 4, there exists $u$ in $X$ such that, up to a subsequence, $\{u_n\}$ converges weakly to $u$ in $X$. Later, we will show that $u_n \to u$ (converges strongly) in $X$. By relation (3.1), we have that $\left\langle I'_\lambda(u_n), u_n - u\right\rangle \to 0$. Therefore, we write

$$\begin{aligned} \left\langle I'_\lambda(u_n), u_n - u\right\rangle &= M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \\ &\quad + \int_\Omega |u_n|^{p(x)-2}(u_n-u)\,dx \\ &\quad - \lambda\int_{\partial\Omega} f(x,u_n)(u_n-u)\,d\sigma \to 0 \end{aligned} \tag{3.2}$$

By Proposition 1, Proposition 4, Proposition 4 and ($\mathbf{f}_1$) we obtain

$$\begin{aligned} \left|\int_{\partial\Omega} f(x,u_n)(u_n-u)\,d\sigma\right| &\leq \left|\int_{\partial\Omega}\left(c_1 + c_1|u_n|^{q(x)-1}\right)(u_n-u)\,d\sigma\right| \\ &\leq c_1\int_{\partial\Omega}|(u_n-u)|\,d\sigma + c_2\left||u_n|^{q(x)-1}\right|_{L^{q'(x)}}|u_n-u|_{L^{q(x)}(\partial\Omega)}, \end{aligned}$$

where $c_2 > 0$ is a constant. If we consider the compact embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$, that is, $|u_n - u|_{L^{q(x)}(\partial\Omega)} \to 0$ as $n \to \infty$, we get

$$\int_{\partial\Omega} f(x, u_n)(u_n - u)\, d\sigma \to 0. \tag{3.3}$$

Similarly, by Proposition 1, Proposition 4, Proposition 5, we can write

$$\int_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u)\, dx \to 0. \tag{3.4}$$

On the other hand, we use (3.1), (3.3) and (3.4) in the above inequality (3.2) we obtain

$$M\left(\int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx\right) \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \to 0.$$

Moreover, from $(\mathbf{M}_0)$, we conclude that

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \to 0.$$

Eventually from Proposition 7 we obtain that $u_n \to u$ (strongly) in $X$. Therefore $I_\lambda$ satisfies the $(\mathbf{PS})$ condition. □

**Lemma 2.** *Assume that* $(\mathbf{M}_0)$, $(\mathbf{AR})$, $(\mathbf{f}_1)$ *and* $(\mathbf{f}_2)$ *hold. Then, the following statements hold:*

(i) *There exist positive real numbers* $\mu$, $\tau$ *and* $\lambda^*$ *for any* $\lambda \in (0, \lambda^*)$ *such that*

$$I_\lambda(u) \geq \tau > 0, \quad \forall u \in X \quad \text{with } \|u\| = \mu,$$

(ii) *There exists* $u_1 \in X$ *such that* $\|u_1\| > \mu$ *and* $I_\lambda(u_1) < 0$.

*Proof.*

(i) Let us assume that $\|u\| < 1$. Since we have the continuous embeddings $X \hookrightarrow L^{p^-}(\Omega)$, $X \hookrightarrow L^{\alpha p^+}(\partial\Omega)$ and $X \hookrightarrow L^{q^-}(\partial\Omega)$ from Proposition 4, there exist positive constants $c_3$, $c_4$ and $c_5$ for all $u \in X$ such that

$$|u|_{L^{\alpha p^+}(\partial\Omega)} \leq c_3 \|u\|, |u|_{L^{q^-}(\partial\Omega)} \leq c_4 \|u\|$$

and

$$|u|_{L^{p^-}(\Omega)} \leq c_5 \|u\|. \tag{3.5}$$

In addition, using $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, we write

$$|F(x,t)| \leq \varepsilon |t|^{\alpha p^+} + c_\varepsilon |t|^{q(x)}, \ \forall (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3.6}$$

By taking into account (3.5) and (3.6), we get

$$I_\lambda(u) \geq \frac{m_1}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^+} + \frac{c_5^{p^-}}{p^+} \|u\|^{p^+} - \lambda\varepsilon \int_{\partial\Omega} |u|^{\alpha p^+} d\sigma - \lambda \int_{\partial\Omega} |u|^{q(x)} d\sigma$$

$$\geq \left(\frac{m_1}{\alpha\left(p^{+}\right)^{\alpha}}+\frac{c_5^{p^-}}{p^{+}}\right)\|u\|^{\alpha p^{+}}-\lambda \varepsilon c_3^{\alpha p^{+}}\|u\|^{\alpha p^{+}}-\lambda c_{\varepsilon} c_4^{q^-}\|u\|^{q^-}.$$

By the above inequality, if we choose $\varepsilon > 0$ small enough such that $0 < 2\lambda \varepsilon c_3^{\alpha p^+}$ $< \left(\frac{m_1}{\alpha(p^+)^{\alpha}}+\frac{c_5^{p^-}}{p^+}\right)$, we obtain

$$I_{\lambda}(u) \geq \left(\left(\frac{m_1}{2\alpha\left(p^{+}\right)^{\alpha}}+\frac{c_5^{p^-}}{2p^{+}}\right)\|u\|^{\alpha p^{+}-q^{-}}-\lambda c_6\right)\|u\|^{q^-}.$$

On the other hand, we remark that $\alpha p^+ < q$

$$\lambda^{*}=\frac{1}{4c_6}\left(\frac{m_1}{\alpha\left(p^{+}\right)^{\alpha}}+\frac{c_5^{p^-}}{p^{+}}\right)\mu^{\alpha p^{+}-q^{-}}.$$

Then, there exist $\tau > 0$ and $\mu > 0$ for any $\lambda \in (0, \lambda^*)$ such that

$$I_{\lambda}(u) \geq \tau > 0, \quad \forall u \in X \quad \text{with } \|u\| = \mu \in (0,1).$$

This completes the proof.

(ii) By (**AR**), one easily deduces $c_7 > 0$ such that

$$F(x,t) \geq c_7 |t|^{\theta}, \ \forall\ (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3.7}$$

Using Proposition 2, Proposition 4 and (3.7) for any $\omega \in X \setminus \{0\}$ and $t > 1$ large enough, we have

$$\begin{aligned} I_{\lambda}(t\omega) &= \widehat{M}\left(\int_{\Omega}\frac{|\nabla t\omega|^{p(x)}}{p(x)}dx\right)+\int_{\Omega}\frac{|t\omega|^{p(x)}}{p(x)}dx-\lambda\int_{\partial\Omega}F(x,t\omega)\,d\sigma \\ &\leq \frac{m_2}{\alpha\left(p^{-}\right)^{\alpha}}t^{\alpha p^{+}}\left(\int_{\Omega}|\nabla\omega|^{p(x)}dx\right)^{\alpha}+\frac{t^{p^{+}}}{p^{-}}\int_{\Omega}|\omega|^{p(x)}dx \\ &\quad -t^{\theta}\lambda c_7\int_{\partial\Omega}|\omega|^{\theta}d\sigma. \end{aligned}$$

Since $\theta > \alpha p^+$, we obtain $\lim_{t\to\infty} I_{\lambda}(t\omega) = -\infty$. Then, we can take $u_1 = t\omega$ such that $\|u_1\| > \mu$ and $I_{\lambda}(u_1) < 0$. The proof of Lemma 2 is complete.

□

*Proof of Theorem 1.* From Lemma 1, Lemma 2 and $I_{\lambda}(0) = 0$, $I_{\lambda}$ satisfies the Mountain Pass theorem [23]. Therefore, $I_{\lambda}$ has at least one nontrivial weak.

We will use the following “Fountain theorem”to prove Theorem 2.

Since $X$ is a separable and reflexive Banach space, then there exist $e_j \subset X$ and $e_j^* \subset X^*$ such that

$$X = \overline{span\left\{e_j :\ j = 1,2,\dots\right\}}, \qquad X^* = \overline{span\left\{e_j^* : j = 1,2,\dots\right\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, \text{ if } i \neq j \\ 0, \text{ if } i = j \end{cases}$$

where $\langle .,. \rangle$ denotes the duality product between $X$ and $X^*$. For convenience, we have

$$X_j = span\{e_j\}, \qquad Y_k = \oplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}.$$

□

**Lemma 3** ([7, Theorem 2.6]). *Assume that $I_\lambda \in C^1(X,\mathbb{R})$ is an even functional satisfying the condition* (**PS**). *Moreover, If there exist $\rho_k > \gamma_k > 0$, for each $k = 1,2,\ldots$ such that*

($\mathbf{A}_1$) $\inf_{u \in Z_k, \|u\| = \gamma_k} I_\lambda(u) \to \infty$ *as* $k \to \infty$,

($\mathbf{A}_2$) $\max_{u \in Y_k, \|u\| = \rho_k} I_\lambda(u) \leq 0$,

*then $I_\lambda$ has an unbounded sequence of critical values.*

**Lemma 4** ([7, Lemma 2.7]). *If $q(x) \in C_+(\partial\Omega)$ and $q(x) < p^\partial(x)$ for any $x \in \partial\Omega$, denote*

$$\beta_k = \sup\left\{ |u|_{L^{q(x)}(\partial\Omega)} : \|u\| = 1, \ u \in Z_k \right\},$$

*then* $\lim_{k\to\infty} \beta_k = 0$.

*Proof of Theorem 2*. It is enough to prove that $I_\lambda$has an unbounded sequence of critical points. Since $I_\lambda$ satisfies (**PS**) condition from Lemma 1 and $I_\lambda$ is an even functional from the assumptions ($\mathbf{f}_3$), we only need to show whether it satisfies the conditions ($\mathbf{A}_1$) and ($\mathbf{A}_2$) in Lemma 3.

($\mathbf{A}_1$) For any $u \in Z_k$ with $\|u\| > 1$, by ($\mathbf{M}_0$), ($\mathbf{f}_1$) and $\alpha p^- > p^+$, we write

$$\begin{aligned} I_\lambda(u) &\geq \frac{m_1}{\alpha (p^+)^\alpha} \|u\|^{\alpha p^-} + \frac{1}{p^+} \|u\|^{p^-} - \lambda c_1 \int_{\partial\Omega} \left(1 + |t|^{q(x)-1}\right) d\sigma \\ &\geq \left( \frac{m_1}{\alpha (p^+)^\alpha} + \frac{1}{p^+} \right) \|u\|^{p^-} - \lambda c_1 \max\left\{ |u|^{q^+}_{L^{q(x)}(\partial\Omega)}, |u|^{q^-}_{L^{q(x)}(\partial\Omega)} \right\} - c_8. \end{aligned}$$

It follows that

$$\begin{aligned} I_\lambda(u) &\geq \begin{cases} \left( \frac{m_1}{\alpha (p^+)^\alpha} + \frac{1}{q^-} \right) \|u\|^{p^-} - \lambda c_9 - c_8 & \text{if } |u|_{L^{q(x)}(\partial\Omega)} \leq 1 \\ \left( \frac{m_1}{\alpha (p^+)^\alpha} + \frac{1}{q^-} \right) \|u\|^{p^-} - \lambda c_9 \beta_k^{q^+} \|u\|^{q^+} - c_8 & \text{if } |u|_{L^{q(x)}(\partial\Omega)} > 1 \end{cases} \\ &\geq \left( \frac{m_1}{\alpha (p^+)^\alpha} + \frac{1}{q^-} \right) \|u\|^{p^-} - \lambda c_9 \beta_k^{q^+} \|u\|^{q^+} - c_{10}. \end{aligned}$$

For $\|u\| = \gamma_k = \left(\lambda c_9 q^+ \beta_k^{q^+}\right)^{\frac{1}{p^- - q^+}}$, we obtain

$$I_\lambda(u) \geq \left( \frac{m_1}{\alpha (p^+)^\alpha} + \frac{1}{q^-} - \frac{1}{q^+} \right) \gamma_k^{p^-} - c_{10}$$

$$\geq \frac{m_1}{\alpha (p^+)^\alpha} \gamma_k^{p^-} - c_{10},$$

where $c_8$, $c_9$ and $c_{10}$ are positive constants. Since $\beta_k \to 0$ and $p^- < q^+$ we obtain $\gamma_k \to \infty$ as $k \to \infty$. Consequently,

$$I_\lambda (u) \to \infty \quad \text{as } \|u\| \to \infty \quad \text{for } u \in Z_k.$$

The statement of $(\mathbf{A}_1)$ is satisfied.

$(\mathbf{A}_2)$ From $(\mathbf{AR})$, we have

$$F(x,t) \geq c_{11} |t|^\theta - c_{12},$$

where $c_{11}$, $c_{12}$ are positive constants. Let $u \in Y_k$ with $\|u\| = \rho_k > 1$. We write

$$I_\lambda (u) \leq \frac{m_2}{\alpha (p^-)^\alpha} \|u\|^{\alpha p^+} + \frac{1}{p^-} \|u\|^{p^+} - \lambda \int_{\partial\Omega} F(x,u) d\sigma$$
$$\leq \frac{m_2}{\alpha (p^-)^\alpha} \|u\|^{\alpha p^+} + \frac{1}{p^-} \|u\|^{p^+} - \lambda c_{11} \int_{\partial\Omega} |u|^\theta d\sigma + c_{12}.$$

Since the space $Y_k$ has finite dimension, all norms are equivalents. Hence, we obtain

$$I_\lambda (u) \leq \frac{m_2}{\alpha (p^-)^\alpha} \|u\|^{\alpha p^+} - \lambda c_{11}^\theta \|u\|^\theta + c_{12}.$$

Finally, we have

$$I_\lambda (u) \to -\infty \text{ as } n \to +\infty \quad \text{for any } u \in Y_k$$

because $\theta > \alpha p^+$. We show that there exists $\rho_k > \gamma_k > 0$ such that

$$\max_{u \in Y_k, \|u\| = \rho_k} I_\lambda (u) \leq 0 \quad \text{for each } k = 1, 2, \ldots$$

Therefore, the statement of $(\mathbf{A}_2)$ is satisfied. We complete the proof of Theorem 2.

□

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