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ALMOST QUASI-YAMABE SOLITONS ON CONTACT METRIC MANIFOLDS

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Abstract. In this article, we study contact metric manifolds admitting almost quasi-Yamabe solitons (g,V,m,λ). First we prove that there does not exist a nontrivial almost quasi-Yamabe soliton whose potential vector field V is pointwise collinear with the Reeb vector field ξ on a contact metric manifold. For V being orthogonal to ξ, we consider the three dimensional cases. Next we consider a non-Sasakian contact metric (κ,μ)-manifold admitting a nontrivial closed almost quasi-Yamabe soliton and give a classification. Finally, for a closed almost quasi-Yamabe soliton on K-contact manifolds, we prove that either the soliton is trivial or $r-\lambda=m$ if $r-\lambda$ is nonnegative and attains a maximum on M, where r is the scalar curvature.

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1. INTRODUCTION

Yamabe soliton, introduced by R. Hamilton, is a Riemannian metric g of a complete Riemannian manifold (M,g) satisfying

$$\frac{1}{2}\mathcal{L}_V g = (r-\lambda)g \tag{1.1}$$

for $\lambda \in \mathbb{R}$ and a smooth vector field V, where $\mathcal{L}_V$ is the Lie derivative along V and r is the scalar curvature of M. For $\lambda = 0$ the Yamabe soliton is steady, for $\lambda < 0$ is expanding, and for $\lambda > 0$ is shrinking. In particular, if the potential vector field V is a gradient field, the Yamabe soliton is said to be a Yamabe gradient soliton. Yamabe solitons have been studied under some conditions (cf.[4,8,11,16,18]). In the Yamabe soliton equation (1.1), if λ is a smooth function, (g,V,λ) is called an almost Yamabe soliton, introduced by E. Barbosa and E. Ribeiro in [1], and T. Seko and S. Maeta in [23] completely classified almost Yamabe solitons on hypersurfaces in Euclidean spaces.

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Later many researchers generalized the notion of Yamabe soliton. For instance, Huang and Li [17] proposed the concept of quasi-Yamabe gradient soliton, namely the Riemannian metric g satisfies the equation

$$\nabla^2 f - \frac{1}{m} df \otimes df = (r-\lambda)g \tag{1.2}$$

for some $f \in C^\infty(M)$, $\lambda \in \mathbb{R}$ and a constant $m > 0$. Such a soliton is also considered by Neto [20] and Wang [27]. V. Pirhadi and A. Razavi in [22] modified λ to be a smooth function and obtained some formulas and a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient. Furthermore, Neto-Oliveira [21] defined the generalized quasi Yamabe gradient soliton by replacing $\frac{1}{m}$ by a smooth function in equation (1.2). Recently, Blaga [2] and Chen-Deshmukh [9] studied more generalized quasi Yamabe solitons. In this article, we consider almost quasi-Yamabe soliton, which is defined as follows:

Definition 1. A Riemannian metric is said to be *almost quasi-Yamabe soliton* if there exist a constant $m > 0$, a smooth vector field V and a C^∞ function λ such that

$$\frac{1}{2}\mathcal{L}_V g - \frac{1}{m} V^\flat \otimes V^\flat = (r-\lambda)g \tag{1.3}$$

holds, where $V^\flat$ is the 1-form associated to V and r stands for the scalar curvature. Denote the almost quasi Yamabe soliton by (g,V,m,λ).

If the 1-form $V^\flat$ is closed, the almost quasi-Yamabe soliton (g,V,m,λ) is said to be *closed*. Using the terminology of Yamabe solitons, we call an almost quasi-Yamabe soliton *shrinking, steady or expanding*, respectively, if $\lambda < 0, \lambda = 0$, or $\lambda > 0$. When $V \equiv 0$, an almost quasi-Yamabe soliton is said to be *trivial*. Otherwise, it will be called *nontrivial*. It is mentioned that an almost quasi-Yamabe soliton (g,V,m,λ) is reduced to an almost Yamabe soliton when $m = \infty$. If $V = Df$ is a gradient vector field, it is called an *almost quasi-Yamabe gradient soliton*, denoted by (g, f, m, λ). Notice that equation (1.2) recovers the Yamabe gradient soliton when $m = \infty$.

For the odd-dimensional manifold, we notice that Sharma [24] proved that a 3-dimensional Sasakian manifold with a Yamabe soliton has constant scalar curvature, and V is Killing. Venkatesha-Naik [26] further generalized Sharma's results to a 3-dimensional contact metric manifold with commuting Ricci operator. For other results the reader can see [12, 13, 25, 28].

In the present paper, we consider almost quasi-Yamabe solitons on contact metric manifolds and it is organized as follows: In Section 2, we recall some definitions and related conclusions on contact metric manifolds. In Section 3, we first prove an nonexistence for a general contact metric manifold with a nontrivial almost quasi-Yamabe soliton whose potential vector field is pointwise collinear with the Reeb vector field. For V being orthogonal to the Reeb vector field, we also obtain two results. In the following Section 4 and Section 5, we study respectively contact metric (κ,μ)-manifolds and K-contact manifolds admitting closed almost quasi-Yamabe solitons.

2. Preliminaries

Let M^{2n+1} be a $(2n+1)$-dimensional smooth manifold. If there exists a global 1-form η (called contact form) on M such that $\eta\wedge(d\eta)^n \neq 0$ everywhere, M^{2n+1} is said to be a *contact manifold*. The contact form induces a unique vector field ξ, called *Reeb vector field*, satisfying $\eta(\xi)=1$ and $d\eta(\xi,\cdot)=0$. Every contact manifold has an almost contact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field such that $\phi^2=-I+\eta\otimes\xi$, $\eta\circ\phi=0$, $\phi\circ\xi=0$.

A Riemannian metric g on M can be defined by

$$d\eta(X,Y)=g(\phi X,Y),\quad g(X,\xi)=\eta(X)$$

for any $X,Y\in\mathfrak{X}(M)$. We note that the Riemannian metric g, ϕ and contact form η can be related each other by

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y).$$

We refer to (ϕ,ξ,η,g) as a contact metric structure and to the manifold M^{2n+1} carrying such a structure as a *contact metric manifold*.

We define the tensor $h=\frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes the Lie derivative and satisfies

$$\operatorname{trace}(h)=0,\quad h\xi=0,\quad \phi h=-h\phi,\quad g(hX,Y)=g(X,hY),\tag{2.1}$$

$$\operatorname{trace}(\phi h)=0.\tag{2.2}$$

Furthermore, we also have

$$\nabla_X\xi=-\phi X-\phi hX\tag{2.3}$$

and $\nabla_\xi\phi=0$. A contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ for which Reeb vector field ξ is Killing, i.e. $\mathcal{L}_\xi g=0$, is called a *K-contact manifold*. If $h=0$ then we have $\mathcal{L}_\xi g=0$, that means that M^{2n+1} is a K-contact manifold. For a K-contact manifold $(M^{2n+1},\phi,\xi,\eta,g)$ the following equations were proved in [3]:

$$Q\xi=2n\xi,\tag{2.4}$$

$$R(X,\xi)\xi=-\phi^2X\tag{2.5}$$

for any vector field X on M. An almost contact structure (ϕ,ξ,η) is said to be *normal* if the corresponding complex structure J on $M\times\mathbb{R}$ is integrable. A normal contact metric manifold is said to be a *Sasakian manifold*. A contact metric manifold is Sasakian if and only if $R(X,Y)\xi=\eta(Y)X-\eta(X)Y$ for all vector fields X,Y on the manifold.

In addition, Blair-Koufogiorgos-Papantoniou [6] defined the notion of *contact metric (κ,μ)-manifold*, i.e. the curvature tensor of a contact metric manifold satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2.6}$$

for any vector fields X,Y, where κ,μ are constants.

The following two lemmas will be used in the sequel proofs.

Lemma 1. *For an almost quasi-Yamabe gradient soliton (M,g,f,m,λ), the curvature tensor R can be expressed as*

$$R(X,Y)Df = -\frac{r-\lambda}{m}\{X(f)Y - Y(f)X\} + X(r-\lambda)Y - Y(r-\lambda)X \tag{2.7}$$

for any vector fields X,Y on M.

Proof. Since equation (1.2) may be exhibited as

$$\nabla_Y Df = \frac{1}{m}Y(f)Df + (r-\lambda)Y, \tag{2.8}$$

we get

$$\nabla_X\nabla_Y Df = \frac{1}{m}\{X(Y(f))Df + Y(f)\nabla_X Df)\} + X(r-\lambda)Y + (r-\lambda)\nabla_X Y.$$

Using the previous two equations, a direct calculation gives

$$\begin{aligned} R(X,Y)Df &= \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df \\ &= \frac{r-\lambda}{m}\{Y(f)X - X(f)Y\} + X(r-\lambda)Y - Y(r-\lambda)X. \end{aligned}$$

□

Lemma 2 ([10])**.** *For an almost quasi-Yamabe gradient soliton (M^{2n+1},g,f,m,λ), the following equation holds:*

$$\Delta(r-\lambda) = \frac{2}{m}g(Df,D(r-\lambda)) + \frac{2n+1}{m}(r-\lambda)^2 - \frac{1}{4n}g(Df,Dr) - \frac{1}{2n}(r-\lambda)r. \tag{2.9}$$

3. Contact metric manifolds with almost quasi-Yamabe solitons

Theorem 1. *There does not exist a nontrivial almost quasi-Yamabe soliton (g,V,m,λ) with $V = \eta(V)\xi$ on a contact metric manifold.*

Proof. We set $V = F\xi$ for a non-zero function F. By (2.3), we have

$$\nabla_X V = X(F)\xi - F(\phi X + \phi hX). \tag{3.1}$$

Using (3.1), formula (1.3) becomes

$$(r-\lambda)g(X,Y) - \frac{1}{2}(X(F)\eta(Y) + Y(F)\eta(X)) + Fg(\phi hX,Y) + \frac{F^2}{m}\eta(X)\eta(Y) = 0. \tag{3.2}$$

Now replacing X and Y by ϕX and ϕY, respectively, implies

$$(r-\lambda)\phi X - FhX = 0.$$

Taking the inner product of the above relation with ϕX and contracting over X, we get $r-\lambda = 0$ by (2.2), which further implies $h = 0$ by the previous relation.

Now letting $Y = \xi$ in (3.2) gives

$$\Big(-\xi(F)+\frac{2F^2}{m}\Big)\eta(X) = X(F).$$

Further putting $X = \xi$ implies $\xi(F) = \frac{F^2}{m}$. Thus the above relation yields $DF = \frac{F^2}{m}\xi$. For any vector fields X,Y, it follows from (2.3) that

$$g(\nabla_X DF,Y) = g(2F\frac{X(F)}{m}\xi - \frac{F^2}{m}\phi X,Y).$$

Since $g(\nabla_X DF,Y) = g(\nabla_Y DF,X)$, we have

$$2F\frac{X(F)}{m}\eta(Y) - 2F\frac{Y(F)}{m}\eta(X) = \frac{2F^2}{m}g(\phi X,Y).$$

Replacing X and Y by ϕX and ϕY, respectively, we deduce $F = 0$, which is a contradiction. We thus complete the proof. □

For V being orthogonal to the Reeb vector field ξ, we intend to consider a three dimensional non-Sasakian contact metric manifold (i.e. $h \neq 0$). It is well-known that there exits a local orthonormal frame field $\mathcal{E} = \{e,\phi e,\xi\}$ such that $he = \nu e$ and $h\phi e = -\nu\phi e$, where ν is a positive non-vanishing smooth function of M.

First of all, we have the following lemma:

Lemma 3 ([15])**.** *In the open subset U, the Levi-Civita connection ∇ is given by*

$$\begin{aligned}
&\nabla_\xi e = a\phi e, && \nabla_\xi \phi e = -ae, && \nabla_\xi \xi = 0,\\
&\nabla_e \xi = -(1+\nu)\phi e, && \nabla_e e = b\phi e, && \nabla_e \phi e = -be+(1+\nu)\xi,\\
&\nabla_{\phi e}\xi = (1-\nu)e, && \nabla_{\phi e}\phi e = ce, && \nabla_{\phi e} e = -c\phi e+(\nu-1)\xi,
\end{aligned}$$

where a is a smooth function,

$$b = \frac{1}{2\nu}[\phi e(\nu)+A] \quad \textit{with} \quad A = Ric(e,\xi), \tag{3.3}$$

$$c = \frac{1}{2\nu}[e(\nu)+B] \quad \textit{with} \quad B = Ric(\phi e,\xi). \tag{3.4}$$

The components of Ricci operator Q are given by

$$\begin{cases}
Qe = \Big(\frac{1}{2}r-1+\nu^2-2a\nu\Big)e+\xi(\nu)\phi e+A\xi,\\
Q\phi e = \xi(\nu)e+\Big(\frac{1}{2}r-1+\nu^2+2a\nu\Big)\phi e+B\xi,\\
Q\xi = Ae+B\phi e+2(1-\nu^2)\xi.
\end{cases} \tag{3.5}$$

The scalar curvature

$$r = \mathrm{trace}(Q) = 2(1-\nu^2-b^2-c^2+2a+e(c)+\phi e(b)). \tag{3.6}$$

Moreover, it follows from Lemma 3 that

$$\begin{cases} [e,\phi e] = \nabla_e\phi e - \nabla_{\phi e}e = -be + c\phi e + 2\xi, \\ [e,\xi] = \nabla_e\xi - \nabla_\xi e = -(a+\nu+1)\phi e, \\ [\phi e,\xi] = \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a-\nu+1)e. \end{cases} \tag{3.7}$$

Theorem 2. *If a non-Sasakian contact metric manifold admits a non-trivial almost quasi-Yamabe gradient soliton (g, f, m, λ) whose potential vector field is orthogonal to the Reeb vector field, then (g, f, m, λ) is a steady quasi-Yamabe gradient soliton and M is locally isometric to $E(2)$.*

Proof. Since the potential vector field Df is orthogonal to ξ, we may write $Df = f_1 e + f_2\phi e$, where f_1, f_2 are two smooth functions on M. For any vector field X, equation (1.2) may be expressed as

$$\nabla_X Df - \frac{1}{m}X(f)Df = (r-\lambda)X. \tag{3.8}$$

Choosing $X=\xi$ in (3.8) and using Lemma 3, we have

$$(\xi(f_1) - f_2 a)e + (\xi(f_2) + f_1 a)\phi e = (r-\lambda)\xi.$$

This shows

$$r=\lambda, \quad \xi(f_1) - f_2 a = 0, \quad \xi(f_2) + f_1 a = 0. \tag{3.9}$$

Similarly, putting $X = e$ in (3.8) and using Lemma 3, we obtain

$$e(f_1) - bf_2 - \frac{1}{m}f_1^2 = 0, \tag{3.10}$$

$$e(f_2) + bf_1 - \frac{1}{m}f_1 f_2 = 0, \tag{3.11}$$

$$(1+\nu)f_2 = 0. \tag{3.12}$$

Putting $X = \phi e$ in (3.8) and using Lemma 3, we obtain

$$\phi e(f_1) + cf_2 - \frac{1}{m}f_1 f_2 = 0, \tag{3.13}$$

$$\phi e(f_2) - cf_1 - \frac{1}{m}f_2^2 = 0, \tag{3.14}$$

$$0 = (\nu-1)f_1. \tag{3.15}$$

Since Df is nonzero and $\nu > 0$, we know $f_2 = 0$ and $\nu = 1$ from (3.12) and (3.15). Moreover, we deduce from (3.10), (3.14) and the third term of (3.9) that $a = b = c = 0$. Because $\nu = 1$, it follows from (3.3) and (3.4) that $A = B = 0$. This implies $Q\xi = 0$. Making use of (3.6) we obtain $r = 0$. Moreover, (3.20) becomes

$$[e,\phi e] = 2\xi, \quad [\phi e,\xi] = 0, \quad [\xi, e] = 2\phi e.$$

We complete the proof by Milnor's classification theorem ([19]). □

Theorem 3. *Let M^3 be non-Sasakian contact metric manifold with $Q\phi = \phi Q$. If (g,V,m,λ) is a nontrivial almost quasi-Yamabe soliton whose potential vector field V is orthogonal to ξ, then (g,V,m,λ) is a steady quasi-Yamabe soliton and M is flat.*

Proof. We write $V = f_1 e + f_2\phi e$, where f_1, f_2 are two smooth functions on M. By the assumpitons, we have $A = B = 0$ and $\xi(\nu) = a = 0$ (see [14, Proposition 2.5]). Therefore (3.5) becomes

$$\begin{cases} Qe = \left(\frac{1}{2}r - 1 + \nu^2\right)e, \\ Q\phi e = \left(\frac{1}{2}r - 1 + \nu^2\right)\phi e, \\ Q\xi = 2(1-\nu^2)\xi. \end{cases}$$

Next making use of the above formulas and Lemma 3 we compute

$$(\nabla_\xi Q)\xi = -2\xi(\nu^2)\xi = 0,$$
$$(\nabla_e Q)e = \nabla_e(Qe) - Q\nabla_e e = e\left(\frac{1}{2}r - 1 + \nu^2\right)e,$$
$$(\nabla_{\phi e} Q)\phi e = \nabla_{\phi e}(Q\phi e) - Q\nabla_{\phi e}\phi e = \phi e\left(\frac{1}{2}r - 1 + \nu^2\right)\phi e.$$

Since $\frac{1}{2}Dr = \mathrm{div}Q$, we obtain

$$e(\nu) = 0, \quad \phi e(\nu) = 0.$$

That shows $b = c = 0$ from (3.3) and (3.4).

For any vector fields X, Y, equation (1.3) may be expressed as

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) - \frac{2}{m}V(X)V(Y) = 2(r-\lambda)g(X,Y). \tag{3.16}$$

Letting $X = Y = \xi$ in (3.16) gives $r = \lambda$. Putting $X = e$ and $Y = \xi$ in (3.16) and using Lemma 3, we obtain

$$\xi(f_1) + f_2(1+\nu) = 0. \tag{3.17}$$

Putting $X = \phi e$ and $Y = \xi$ and using Lemma 3, we obtain

$$\xi(f_2) + f_1(\nu - 1) = 0. \tag{3.18}$$

Choosing $X = e$ and $Y = \phi e$ we get

$$\phi e(f_1) + e(f_2) - \frac{2}{m}f_1 f_2 = 0. \tag{3.19}$$

On the other hand, since $a = b = c = 0$, the Lie bracket (3.7) may be expressed as

$$[e,\xi] = -(1+\nu)\phi e, \quad [\phi e, \xi] = (1-\nu)e. \tag{3.20}$$

Applying the first term of (3.20) on f_2 and using (3.18), we obtain

$$\xi(e(f_2)) - (1+\nu)\phi e(f_2) = -e(f_1)(\nu - 1).$$

Applying the second term of (3.20) on f_1 and using (3.17), we obtain

$$\xi(\phi e(f_1)) + (1-\nu)e(f_1) = \phi e(f_2)(-1-\nu).$$

Therefore, the previous two equations together with (3.19) give

$$\xi(f_1 f_2) = 0.$$

Using (3.17) and (3.18), we thus derive

$$f_2^2(\nu+1) + f_1^2(\nu-1) = 0.$$

Differentiating this along ξ and using (3.17) and (3.18) again, we have

$$\nu^2 - 1 = 0.$$

This shows that $\nu = 1$ and (3.6) yields $\lambda = r = 0$. Moreover, it is clear that $Q\xi = 0$. We complete the proof by [5, Remark 3.1]. □

4. Contact metric (κ,μ)-manifolds with closed almost quasi-Yamabe solitons

In this section we suppose that $(M^{2n+1}, \phi, \xi, \eta, g)$ is a contact metric (κ,μ)-manifold, namely the curvature tensor satisfies (2.6). Furthermore, the following relations are provided (see [6]) :

$$QX = (2(n-1) - n\mu)X + (2(n-1)+\mu)hX + (n(2\kappa+\mu) - 2(n-1))\eta(X)\xi, \quad (4.1)$$

$$h^2 = (\kappa-1)\phi^2. \quad (4.2)$$

Using (2.1), it follows from (4.1) that the scalar curvature $r = 2n(2(n-1)+\kappa-n\mu)$ and $Q\xi = 2n\kappa\xi$. By (4.2), we find easily that $\kappa \leq 1$ and $\kappa = 1$ if and only if M is a Sasakian manifold. In particular, for $\kappa = \mu = 0$, Blair proved the following result.

Theorem 4 ([3, Theorem 7.5]). *A contact metric manifold M^{2n+1} satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

For a non-Sasakian (κ,μ)-manifold M, Boeckx [7] introduced an invariant

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$$

and proved the following conclusion:

Theorem 5 ([7, Corollary 5]). *Let M be a non-Sasakian (κ,μ)-manifold. Then it is locally isometric, up to a D-homothetic transformation, to the unit tangent sphere bundle of some space of constant curvature (different from 1) if and only if $I_M > -1$.*

Making use of the above theorems we obtain

Theorem 6. *Let M^{2n+1} be a non-Sasakian (κ,μ)-manifold. If M admits a non-trivial closed almost quasi-Yamabe soliton (g,V,m,λ), then M is flat for $n=1$ and for $n>1$, M is either locally isometric, up to a D-homothetic transformation, to the unit tangent sphere bundle of some space of constant curvature, or locally isometric to $E^{n+1}\times S^n(4)$.*

Proof. In view of equation (1.3), we obtain

$$\nabla_Y V = (r-\lambda)Y + \frac{1}{m}g(V,Y)V \tag{4.3}$$

for any vector Y. Since the scalar curvature $r = 2n(2(n-1)+\kappa-n\mu)$ is constant, using (4.3) we compute

$$\begin{aligned} R(X,Y)V &= \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V \\ &= -X(\lambda)Y + \frac{1}{m}g(\nabla_X V,Y)V + \frac{1}{m}g(V,Y)\nabla_X V \\ &\quad + Y(\lambda)X - \frac{1}{m}g(\nabla_Y V,X)V - \frac{1}{m}g(V,X)\nabla_Y V \\ &= Y(\lambda)X - X(\lambda)Y + \frac{r-\lambda}{m}g(V,Y)X - \frac{r-\lambda}{m}g(V,X)Y. \end{aligned} \tag{4.4}$$

Taking an inn product of the above formula with ξ and using (2.6), we have

$$\begin{aligned} &-\kappa(\eta(Y)g(X,V)-\eta(X)g(Y,V)) - \mu(\eta(Y)g(hX,V)-\eta(X)g(hY,V)) \\ &= \frac{r-\lambda}{m}[g(V,Y)\eta(X)-g(V,X)\eta(Y)] + Y(\lambda)\eta(X) - X(\lambda)\eta(Y). \end{aligned}$$

Now replacing X and Y by ϕX and ξ, respectively, yields

$$\kappa g(\phi X,V) + \mu g(h\phi X,V) = \phi X(\lambda) + \frac{r-\lambda}{m}g(\phi X,V)$$

for any vector field X. This is equivalent to

$$\Big(\kappa - \frac{r-\lambda}{m}\Big)\phi V + \mu\phi hV = \phi D\lambda. \tag{4.5}$$

On the other hand, contracting (4.4) over Y and using (4.1) we obtain

$$\begin{aligned} &(2(n-1)-n\mu)V + (2(n-1)+\mu)hV + (n(2\kappa+\mu)-2(n-1))\eta(V)\xi \\ &= 2n\Big(D\lambda + \frac{r-\lambda}{m}V\Big). \end{aligned} \tag{4.6}$$

Now applying ϕ in this formula implies

$$(2(n-1)-n\mu)\phi V + (2(n-1)+\mu)\phi hV = 2n\Big(\phi D\lambda + \frac{r-\lambda}{m}\phi V\Big), \tag{4.7}$$

which, combining with (4.5), gives

$$\Big\{(2(n-1)-n\mu)\mu - (2(n-1)+\mu)\Big(\kappa - \frac{r-\lambda}{m}\Big) - 2n\mu\frac{r-\lambda}{m}\Big\}\phi V$$

$$= \Big(2n\mu - 2(n-1) - \mu\Big)\phi D\lambda, \tag{4.8}$$

implying

$$\Big\{(2(n-1)-n\mu)\mu - (2(n-1)+\mu)\Big(\kappa - \frac{r-\lambda}{m}\Big) - 2n\mu\frac{r-\lambda}{m}\Big\}V$$
$$-\Big(2n\mu - 2(n-1) - \mu\Big)D\lambda \in \mathbb{R}\xi.$$

Case I. If $d := 2n\mu - 2(n-1) - \mu = 0$, i.e. $\mu = \frac{2(n-1)}{2n-1}$, then from (4.8) we know

$$(2(n-1)-n\mu)\mu - (2(n-1)+\mu)\Big(\kappa - \frac{r-\lambda}{m}\Big) - 2n\mu\frac{r-\lambda}{m} = 0.$$

That is, $\kappa = \frac{(n-1)^2}{n(2n-1)}$ for $n > 1$. Clearly, in this case we have

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}} = \frac{1-\frac{2(n-1)}{2n-1}}{\sqrt{1-\kappa}} = \frac{1}{(2n-1)\sqrt{1-\kappa}} > -1.$$

For $n = 1$, we have $\mu = 0$. Equation (4.6) is simplified as

$$\kappa\eta(V)\xi = D\lambda + \frac{r-\lambda}{m}V,$$

which, combining with (4.5), yields $\kappa = 0$. Here we have used the conclusion that $\phi V \neq 0$, which is obtained by Theorem 1.

Case II. If $d = 2n\mu - 2(n-1) - \mu \neq 0$ then we can write

$$D\lambda = cV + s\xi, \tag{4.9}$$

where

$$c = \frac{1}{d}\Big\{(2(n-1)-n\mu)\mu - (2(n-1)+\mu)\Big(\kappa - \frac{r-\lambda}{m}\Big) - 2n\mu\frac{r-\lambda}{m}\Big\}$$

and s is a smooth function. By (4.6), we have

$$(2(n-1)+\mu)hV + (n(2\kappa+\mu) - 2(n-1))\eta(V)\xi$$
$$= \Big\{2n\Big(c + \frac{r-\lambda}{m}\Big) - (2(n-1)-n\mu)\Big\}V + 2ns\xi. \tag{4.10}$$

Now applying h in this formula and recalling (4.2) imply

$$(2(n-1)+\mu)(\kappa-1)\phi^2 V = \Big\{2n\Big(c + \frac{r-\lambda}{m}\Big) - (2(n-1)-n\mu)\Big\}hV. \tag{4.11}$$

Combining (4.10) with (4.11) we get

$$\Big[\Big\{2n\Big(c + \frac{r-\lambda}{m}\Big) - (2(n-1)-n\mu)\Big\}\Big(n(2\kappa+\mu) - 2(n-1)\Big)$$
$$+ (2(n-1)+\mu)^2(\kappa-1)\Big]\eta(V)\xi - 2ns\Big\{2n\Big(c + \frac{r-\lambda}{m}\Big) - (2(n-1)-n\mu)\Big\}\xi$$

$$= \Big[\Big\{2n\Big(-c+\frac{r-\lambda}{m}\Big)-(2(n-1)-n\mu)\Big\}^2+(2(n-1)+\mu)^2(\kappa-1)\Big]V. \tag{4.12}$$

Since $\phi V\neq 0$, this implies that

$$\Big\{2n\Big(c+\frac{r-\lambda}{m}\Big)-(2(n-1)-n\mu)\Big\}^2+(2(n-1)+\mu)^2(\kappa-1)=0, \tag{4.13}$$

then λ is constant. Hence equation (4.5) becomes

$$\Big(\kappa-\frac{r-\lambda}{m}\Big)\phi V+\mu\phi hV=0. \tag{4.14}$$

Furthermore, from (4.9) we find $s=c=0$ since $V\notin\mathbb{R}\xi$. Thus it follows from (4.12) that

$$\Big\{2n\Big(\frac{r-\lambda}{m}\Big)-(2(n-1)-n\mu)\Big\}\Big(\kappa-\frac{r-\lambda}{m}\Big)\eta(V)=0. \tag{4.15}$$

II(a). When $2(n-1)+\mu\neq 0$, it follows from (4.13) and (4.15) that

$$\Big(\kappa-\frac{r-\lambda}{m}\Big)\eta(V)=0. \tag{4.16}$$

Consequently, either $\kappa-\frac{r-\lambda}{m}=0$ or $\eta(V)=0$. If $\kappa-\frac{r-\lambda}{m}=0$ then (4.14) yields $\mu=0$. This implies $n>1$ and equation (4.13) becomes

$$n^2\kappa^2-(n^2-1)\kappa=0,$$

i.e. $\kappa=0$ or $\kappa=\frac{n^2-1}{n^2}$. Since $\kappa<1$, relation $\kappa=\frac{n^2-1}{n^2}$ does not hold.

If $\eta(V)=0$, we differentiate this along ξ and obtain $r=\lambda$ by (4.3). Hence equations (4.14) and (4.7) respectively become

$$\kappa\phi V+\mu\phi hV=0 \quad \text{and} \quad (2(n-1)-n\mu)\phi V+(2(n-1)+\mu)\phi hV=0.$$

Using ϕ to act on the above relations yields

$$\begin{aligned}&\kappa V+\mu hV=0,\\ &(2(n-1)-n\mu)V+(2(n-1)+\mu)hV=0.\end{aligned}$$

Thus

$$\begin{cases}\kappa+\mu\sqrt{1-\kappa}=0,\\ (2(n-1)-n\mu)+(2(n-1)+\mu)\sqrt{1-\kappa}=0.\end{cases}$$

or

$$\begin{cases}\kappa-\mu\sqrt{1-\kappa}=0,\\ (2(n-1)-n\mu)-(2(n-1)+\mu)\sqrt{1-\kappa}=0.\end{cases}$$

Here $\sqrt{1-\kappa}$ is an eigenvalue of h. The above two cases shall lead to

$$\frac{2(n-1)(1+\sqrt{1-\kappa})}{\sqrt{1-\kappa}-n}=\frac{\kappa}{\sqrt{1-\kappa}},$$

i.e.

$$x^3+(n-2)x^2+(2n-3)x+n=0,$$

where $x=\sqrt{1-\kappa}$. Clearly, the above relation does not hold for $n>1$ since $x>0$.

II(b). When $2(n-1)+\mu=0$ then equation (4.13) yields

$$n^2-1=n\frac{r-\lambda}{m}. \tag{4.17}$$

Substituting this into (4.14) yields

$$-(1-\kappa)-\frac{n^2-n-1}{n}=2(n-1)\sqrt{1-\kappa}.$$

If $n=1$ then $\kappa=\mu=0$. For $n>1$, the above relation is impossible since $\kappa<1$. In this case, M is flat.

Summing up the above discussion, we proved that $\kappa=\mu=0$ for $n=1$ and for $n>1$, either $\kappa=\mu=0$ or $I_M>-1$. Therefore we complete of the proof by Theorem 4 and Theorem 5. □

When $V=Df$, it is clear that $V^\flat$ is closed, thus we have

Corollary 1. *A non-Sasakian contact metric* (κ,μ)*-manifold* M^{2n+1}*, admitting a nontrivial quasi-Yamabe gradient soliton* (g,f,m,λ)*, is locally isomorphic to* $E^{n+1}\times S^n(4)$ *for* $n>1$ *and flat for* $n=1$.

Proof. Since $r=2n(2(n-1)+\kappa-n\mu)$ and λ are constants, by (2.9) we have

$$0=\frac{2n+1}{m}(r-\lambda)^2-\frac{1}{2n}(r-\lambda)r.$$

If $\lambda\neq r$ then

$$0=2n(2n+1)(r-\lambda)-mr.$$

Equation (4.5) becomes

$$\left(\kappa-\frac{r}{2n(2n+1)}\right)\phi V+\mu\phi hV=0. \tag{4.18}$$

For Case I in the proof of Theorem 6, $\kappa=\frac{(n-1)^2}{n(2n-1)}$ and $\mu=\frac{2(n-1)}{2n-1}$. A direct computation yields $\kappa-\frac{r}{2n(2n+1)}=0$. This implies from (4.18) that $\kappa=\mu=0$ and $n=1$. For Case II(a), since $r\neq\lambda$ we see $\eta(V)\neq 0$. Thus in this case we also have $\kappa=\mu=0$.

When $r=\lambda$, for Case I in the proof of Theorem 6, equation (4.7) is simplified as

$$(n-1)V+2nhV=0.$$

Using h to act on this and recalling (4.2), we obtain

$$(n-1)hV+2n\left(1-\frac{(n-1)^2}{n(2n-1)}\right)V=0.$$

Thus the previous two formulas yields

$$(n-1)^2-4n^2\left(1-\frac{(n-1)^2}{n(2n-1)}\right)=0,\text{ i.e. }2n^3+9n^2-8n+1=0.$$

Obviously, it is impossible.

For Case II, it follows from (4.5) that if $\kappa = 0$ then $\mu = 0$. If $\kappa \neq 0$ then $\eta(V) = 0$ by (4.16). Thus from Case II(a) in the proof of Theorem 6, we know that is impossible.

Summarizing the above discussion, we know $\kappa = \mu = 0$. Thus the desired conclusion is proved by Theorem 4. □

5. K-CONTACT MANIFOLDS WITH CLOSED ALMOST QUASI-YAMABE SOLITONS

In this section we study a K-contact manifold admitting a closed almost quasi-Yamabe soliton.

Theorem 7. *Let (g,V,m,λ) be a closed almost quasi-Yamabe soliton on a K-contact manifold M^{2n+1}. If $r-\lambda$ is nonnegative and attains a maximum in M then either the soliton is trivial or $r-\lambda = m$. Moreover, if M is compact, then (g,V,m,λ) is trivial under the condition that $r-\lambda$ is nonnegative.*

Proof. As $V^\flat$ is closed, equation (1.3) is equivalent to

$$\nabla_Y V = (r-\lambda)Y + \frac{1}{m} g(V,Y)V. \tag{5.1}$$

Via this formula one derives easily

$$\begin{aligned} R(X,Y)V &= \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V \\ &= X(r-\lambda)Y + \frac{1}{m}g(V,Y)(r-\lambda)X - Y(r-\lambda)X - \frac{1}{m}g(V,X)(r-\lambda)Y. \end{aligned} \tag{5.2}$$

By (2.5), taking an inner product of (5.2) with ξ gives

$$X(r-\lambda) + \frac{r-\lambda}{m}\eta(V)\eta(X) - \xi(r-\lambda)\eta(X) - \frac{r-\lambda}{m}g(V,X) = g(\phi^2 X, V).$$

Now replacing X by ϕX yields

$$\phi D(r-\lambda) - \frac{r-\lambda-m}{m}\phi V = 0, \quad \text{i.e.} \quad D(r-\lambda) - \frac{r-\lambda-m}{m}V \in \mathbb{R}\xi.$$

We write $D(r-\lambda) - \frac{r-\lambda-m}{m}V = c\xi$ for some function c on M.

On the other hand, contracting (5.2) over X yields

$$2n\Big(-D(r-\lambda) + \frac{r-\lambda}{m}V\Big) = QV.$$

Since $Q\xi = 2n\xi$, the previous two formulas imply $c = g(D(r-\lambda) - \frac{r-\lambda-m}{m}V, \xi) = 0$, i.e.

$$D(r-\lambda) = \frac{r-\lambda-m}{m}V. \tag{5.3}$$

Differentiating (5.3) along X yields

$$\frac{X(r-\lambda)}{m}V + \frac{r-\lambda-m}{m}\nabla_X V = \nabla_X D(r-\lambda).$$

Further, contracting this over X gives

$$\frac{V(r-\lambda)}{m}+\frac{r-\lambda-m}{m}\mathrm{div}V=\Delta(r-\lambda). \tag{5.4}$$

Since $\Delta(r-\lambda-m)^2=2|D(r-\lambda-m)|^2+2(r-\lambda-m)\Delta(r-\lambda-m)$ and

$$\mathrm{div}V=(2n+1)(r-\lambda)+\frac{1}{m}|V|^2 \tag{5.5}$$

obtained from (5.1), it follows from (5.3) and (5.4) that

$$\Delta(r-\lambda-m)^2=6|D(r-\lambda-m)|^2+2(r-\lambda-m)^2\frac{(2n+1)(r-\lambda)}{m}.$$

If $r-\lambda$ is nonnegative and attains a maximum in M then $r-\lambda=m$ or $r-\lambda=0$. For $r-\lambda=0$, equation (5.3) yields $V=0$, i.e. the soliton is trivial.

If M is compact, it is easy to get from (5.5) that $V=0$ under the assumption that $r-\lambda$ is nonnegative. □

If $V=Df$, equation (5.3) becomes

$$\frac{r-\lambda-m}{m}Df=D(r-\lambda). \tag{5.6}$$

In view of (2.8), we know

$$\Delta f=\frac{1}{m}|Df|^2+(2n+1)(r-\lambda).$$

Inserting this into equation (5.4), we get

$$\frac{1}{m}g(Df,D(r-\lambda))+\frac{r-\lambda-m}{m}\left[(2n+1)(r-\lambda)+\frac{1}{m}|Df|^2\right]=\Delta(r-\lambda). \tag{5.7}$$

Making use of (2.9) and (5.6), it follows from (5.7) that

$$(4n(2n+1)-2r)(r-\lambda)=g(Df,Dr). \tag{5.8}$$

Thus the following conclusion is clear from (5.8) and (5.6).

Corollary 2. *Let M^{2n+1} be a K-contact manifold with an almost quasi-Yamabe gradient soliton (g,f,m,λ). If the scalar curvature r is constant then either (g,f,m,λ) is trivial, or $r=2n(2n+1)$.*

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