AN EXISTENCE STUDY FOR A TRIPLED SYSTEM WITH 
\( p \)-LAPLACIAN INVOLVING \( \varphi \)-CAPUTO DERIVATIVES

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Abstract. In this paper, we study the existence and uniqueness of solutions for a tripled system of fractional differential equations with nonlocal integro multi point boundary conditions by using the \( p \)-Laplacian operator and the \( \varphi \)-Caputo derivatives. The presented results are obtained by the two fixed point theorems of Banach and Krasnoselskii. An illustrative example is presented at the end to show the applicability of the obtained results. To the best of our knowledge, this is the first time where such problem is considered.

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1. INTRODUCTION

The fractional calculus has many significant roles in various scientific fields of research, see for instance [10, 20]. As applied results, the fractional order differential equations have attired attention of several scientists in different fields of research [8, 19]. However, most of the published works have been achieved by using the fractional derivatives of type Riemann-Liouville, Hadamard, Katugampola, Atangana-Baleanu, Grunwald Letnikov and Caputo. The fractional derivatives of functions with respect to some other functions [14] are different from the others since their kernels appear in terms of other functions (called \( \varphi \)). Recently, some fractional differential results have been considered in [4–6, 11].

In most of the present articles, Schauder, Krasnoselskii, Darbo, or Monch theories have been used to prove existence of solutions of nonlinear fractional differential equations with some restrictive conditions [3, 18]. Some authors have worked on the solutions for fractional problems with \( p \)-Laplacian operators. We cite, for example [1, 2, 12, 16] where it has been studied nonlinear fractional equation with \( p \)-Laplacian operator for the solutions.

Here, we will mention some other research works for the reader. We begin by A. Devi, A. Kumar, D. Baleanu and A. Khan [2] where they worked on the stability results, for the following nonlinear FDEs involving Caputo derivatives of distinct
In the present research work, we study the existence and uniqueness of solutions for the following problem:

\[
\begin{align*}
\begin{cases}
\mathcal{D}^\alpha_p \Phi_p \left[ \mathcal{D}^\alpha_p \left( u(t) - \sum_{i=1}^{m} v_i(t) \right) \right] = -w(t,u(t)), t \in (0,1] \\
\Phi_p \left[ \mathcal{D}^\alpha_p \left( u(t) - \sum_{i=1}^{m} v_i(t) \right) \right]|_{t=0} = 0,
\end{cases}
\]

where \( 0 < r_1 \leq 1, n - 1 < r_2 \leq n, n \geq 4 \), and \( v_i, w \) are continuous functions. \( \mathcal{D}^\alpha_p \) denotes the \( \alpha \)-derivative of fractional order \( r_1 \) and \( r_2 \) in Caputo’s sense, respectively, and \( \Phi_p(z) = |z|^{p-2}z \) denotes the \( p \)-Laplacian operator and satisfies \( \frac{1}{p} + \frac{1}{q} = 1, (\Phi_p)^{-1} = \Phi_q \).

We can also cite the paper of A. Mahdjouba et al. [17] where they have investigated the existence and multiplicity of positive solutions of the following problem:

\[
\begin{align*}
\begin{cases}
\mathcal{D}^\alpha_p \Phi_p \left[ \mathcal{D}^\alpha_p \left( u(t) \right) \right] \Phi_p(x(t)) + a_1(t)f(u(\theta_1(t)),v(\theta_2(t))) = 0, 0 < t < 1, \\
\Phi_p \left[ \mathcal{D}^\alpha_p \left( v(t) \right) \right] \Phi_p(x(t)) + a_2(t)f(u(\theta_1(t)),v(\theta_2(t))) = 0, 0 < t < 1, \\
\mathcal{D}^\alpha_p u(0) = u(0) = u'(0), \quad \mathcal{D}^\alpha_p u(1) = \gamma \mathcal{D}^\alpha_p u(\eta), \\
\mathcal{D}^\alpha_p v(0) = v(0) = v'(0), \quad \mathcal{D}^\alpha_p v(1) = \gamma \mathcal{D}^\alpha_p v(\eta),
\end{cases}
\end{align*}
\]

where \( \eta \in (0,1), \gamma \in \left(0, \frac{1}{\eta^{n-m-1}}\right) \), \( \mathcal{D}^\alpha_p \), are the standard Riemann–Liouville fractional derivatives with \( r \in (2,3), m \in (1,2) \) such that \( r \geq m + 1 \), \( p \)-Laplacian operator is defined as \( \Phi_p(z) = |z|^{p-2}z \), \( p > 1 \), and the functions \( f, g \in C(\mathbb{R}, \mathbb{R}) \).

Then, S. Etemad with his coauthors [9] have been concerned with the existence study for the following tripled impulsive fractional problem

\[
\begin{align*}
\begin{cases}
\mathcal{D}^\kappa_m x_m(t) = f_m(t,x(t)), m = 1, 2, 3, \quad \text{and } t \in J' \\
x_m(a) = \Phi_m x_m(a) = \Theta_m x_m, \\
\Delta x_m|_{t=t_k} = I_{m,k}(x(t_k)), \Delta x'_m|_{t=t_k} = I_{m,k}(x(t_k)),
\end{cases}
\end{align*}
\]

where \( J = [a,b], J' = J - \{t_1, t_2, \ldots, t_p\}, a = t_0 < t_1 < \ldots < t_p = t_{p+1} = b \), \( \mathcal{D}^\kappa \), \( m = 1, 2, 3 \), are the Caputo fractional derivatives such that \( x_m \in (1,2] \), \( f_m : J \times \mathbb{R} \rightarrow \mathbb{R} \), \( x(t) = (x_1(t), x_2(t), x_3(t)), m, k, I_{m,k} : \mathbb{R}^3 \rightarrow \mathbb{R}, k = 1, 2, \ldots, p \), \( \Phi_m, \Theta_m \) are given functions, \( \Delta x_m|_{t=t_k} = x(t_k^+) - x(t_k^-) \), \( \Delta x'_m|_{t=t_k} = x'(t_k^+) - x'(t_k^-) \), and

\[
x(t_k^+) = \lim_{h \to 0^+} x_m(t_k + h), x(t_k^-) = \lim_{h \to 0^-} x_m(t_k + h).
\]

In the present research work, we study the existence and uniqueness of solutions for the following problem:

\[
\begin{align*}
\begin{cases}
\mathcal{D}^\gamma_0 \mathcal{D}^\gamma_0 \Phi_p \left[ \mathcal{D}^\gamma_0 \left( u(t) - \sum_{i=1}^{m} v_i(t) \right) \right] = H_m(t, u_1(t), u_2(t), u_3(t)), m = 1, 2, 3, \quad \text{and } t \in J = (0,1] \\
\mathcal{D}^\gamma_0 \Phi_p \left[ \mathcal{D}^\gamma_0 \left( u(t) - \sum_{i=1}^{m} v_i(t) \right) \right]|_{t=0} = 0, \\
u_m(0) = 0, \quad u_m(1) = \sum_{i=1}^{m} \lambda_i u_m \left( \zeta_i \right), \quad \zeta_i \in (0,1], \\
\Phi(1) - \Phi(0) = K > 0.
\end{cases}
\end{align*}
\]
Here, we take $D^m_{0^+}\phi, i = 1, 2, m = 1, 2, 3$ as the $\phi$–Caputo fractional derivatives of orders $r_{im}, 0 < r_{1m} < 1 < r_{2m} < 2$, and $I^{\sigma}\phi, 0 < \sigma < r_{2m}$, the fractional integral of order $\sigma, \lambda_{im} \in \mathbb{R}^+_+$, and $\phi: J \to \mathbb{R}$ is an increasing function such that $\phi'(t) \neq 0$, and $\psi_p(z) = |z|^{p-2}z$ denotes the $p$–Laplacian operator and satisfies $\frac{1}{p} + \frac{1}{q} = 1, (\psi_p)^{-1} = \psi_q$. For all $t \in J$, $G_{m}, H_{m}: J \times \mathbb{R}^3 \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later.

### 2. $\phi$–CAPUTO DERIVATIVES

In this section, we introduce some notations and definitions of $\phi$–Caputo approach, for details, see [6, 14, 19].

Let $\phi: J \to \mathbb{R}$ be an increasing function with $\phi'(t) \neq 0$, for all $t \in J$.

And throughout the paper, let $C = C(J, \mathbb{R})$ denotes the Banach space of all continuous mappings from $[0, 1]$ to $\mathbb{R}$ endowed with the norm $\|u\| = \sup_{t \in [0, 1]} u(t)$. It is clear that the space $C \times C \times C$ endowed with the norm $\|(u_1, u_2, u_3)\| = \|u_1\| + \|u_2\| + \|u_3\|$ is a Banach space. We pose for all $r > 0$, and $t \in [0, 1], (t > s)$

$$\phi_r(t, s) = \frac{\phi'(s) (\phi(t) - \phi(s))^{r-1}}{\Gamma(r)}.$$

**Definition 1.** For $\alpha > 0$, the left-sided $\phi$–Riemann Liouville fractional integral of order $\alpha$ for an integrable function $u: J \to \mathbb{R}$ with respect to another function $\phi: J \to \mathbb{R}$ that is an increasing differentiable function such that $\phi'(t) \neq 0$, for all $t \in J$ is defined as follows

$$I^\alpha_{a+}u(t) = \int_a^t \phi_\alpha(t, s) u(s) ds, \tag{2.1}$$

where $\Gamma$ is the gamma function. Note that equation (2.1) is reduced to the Riemann Liouville and Hadamard fractional integrals when $\phi(t) = t$ and $\phi(t) = \ln t$, respectively.

**Definition 2.** Let $n \in \mathbb{N}$ and let $\phi, u \in C^n(J)$ be two functions such that $\phi$ is increasing and $\phi'(t) \neq 0$, for all $t \in J$. The left-sided $\phi$–Riemann Liouville fractional derivative of a function $u$ of order $\alpha$ is defined by

$$D^\alpha_{a+}u(t) = \left(\frac{1}{\phi'(t)} \frac{d}{dt}\right)^n I^{n-\alpha}_{a+}\phi u(t)$$

$$= \left(\frac{1}{\phi'(t)} \frac{d}{dt}\right)^n \int_a^t \phi_{n-\alpha}(t, s) u(s) ds,$$

where $n = [\alpha] + 1$. 


Definition 3. Let \( n \in \mathbb{N} \) and let \( \varphi, u \in C^n(J) \) be two functions such that \( \varphi \) is increasing and \( \varphi'(t) \neq 0 \), for all \( t \in J \). The left-sided \( \varphi \)-Caputo fractional derivative of a function \( u \) of order \( \alpha \) is defined by

\[
^cD_{a^+}^{\alpha, \varphi} u(t) = I_{a^+}^{n-\alpha, \varphi} \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t),
\]

where \( n = [\alpha] + 1 \) for \( \alpha \notin \mathbb{N} \), \( n = \alpha \) for \( \alpha \in \mathbb{N} \).

To simplify notation, we will use the symbol

\[
i_{\alpha}^{[\varphi]}(t) = \left( \frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t).
\]

From the definition, it is clear that

\[
^cD_{a^+}^{\alpha, \varphi} u(t) = \begin{cases} 
\int_a^t \varphi_{n-\alpha}(t,s) u_{\alpha}(s)ds, & \text{if } \alpha \notin \mathbb{N}, \\
i_{\alpha}^{[\varphi]}(t), & \text{if } \alpha \in \mathbb{N}.
\end{cases}
\]

(2.2)

This generalization (2.2) yields to the Caputo fractional derivative operator when \( \varphi(t) = t \). Moreover, for \( \varphi(t) = \ln t \), it gives the Caputo Hadamard fractional derivative.

2.1. Auxiliary Lemma

Lemma 1. Let \( \alpha, \beta > 0 \), and \( u \in L^1(J) \). Then

\[
I_{a^+}^{\alpha, \varphi} I_{a^+}^{\beta, \varphi} u(t) = I_{a^+}^{\alpha+\beta, \varphi} u(t), \quad \text{a.e. } t \in J.
\]

In particular, if \( u \in C(J) \), then

\[
I_{a^+}^{\alpha, \varphi} I_{a^+}^{\beta, \varphi} u(t) = I_{a^+}^{\alpha+\beta, \varphi} u(t), \quad t \in J.
\]

Next, we recall the property describing the composition rules for fractional \( \varphi \)-integrals and \( \varphi \)-derivatives.

Lemma 2. Let \( \alpha > 0 \). The following holds:
If \( u \in C([a, b]) \), then

\[
^cD_{a^+}^{\alpha, \varphi} I_{a^+}^{\alpha, \varphi} u(t) = u(t), \quad t \in [a, b].
\]

If \( u \in C^n(J) \), \( n - 1 < \alpha < n \), then

\[
I_{a^+}^{\alpha, \varphi} D_{a^+}^{\alpha, \varphi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_k^{(k)}(a)}{k!} [\varphi(t) - \varphi(a)]^k,
\]

for all \( t \in [a, b] \). In particular, if \( 0 < \alpha < 1 \), we have

\[
I_{a^+}^{\alpha, \varphi} D_{a^+}^{\alpha, \varphi} u(t) = u(t) - u(a).
\]

Lemma 3. Let \( t > a \), \( \alpha \geq 0 \) and \( \beta > 0 \). Then

- \( I_{a^+}^{\alpha, \varphi} [\varphi(t) - \varphi(a)]^\beta = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} [\varphi(t) - \varphi(a)]^{\beta + \alpha - 1} \),
Then there exists \( z \in \text{convex and nonempty subset of a Banach space } U \). Let \( A : T \to C \) results. (We have (10) Lemma 8 (Krasnoselskii fixed point theorem, [15]).

Finally, we recall the fixed point theorems that will be used to prove the main results. (We have \( \mathcal{C} \) a Banach space in each theorem).

Lemma 4. Let \( \alpha > 0, n \in \mathbb{N} \) such that \( n - 1 < q \leq n \). Then

- \( c \mathcal{D}^\alpha_{a^+} \mathcal{M}^\alpha_{a^+} u(t) = c \mathcal{D}^{\alpha} u(t); \) if \( q > \alpha \).
- \( c \mathcal{D}^\alpha_{a^+} \mathcal{M}^\alpha_{a^+} u(t) = c \mathcal{D}^{\alpha} u(t); \) if \( q > \alpha \).

Lemma 5. Given a function \( u \in C^n [a, b] \) and \( 0 < q < 1 \), we have

\[
|I^q_{a^+} u(t_2) - I^q_{a^+} u(t_1)| \leq \frac{2 \|u\|}{\Gamma(q + 1)} (q(t_2) - q(t_1))^{q}.
\]

Lemma 6 (Banach fixed point theorem, [7]). Let \( U \) be a closed set in \( \mathcal{C} \) and \( T : U \to U \) satisfies

\[
|Tu - Tv| \leq \alpha |u - v|, \text{ for some } \alpha \in (0, 1), \text{ and for } u, v \in U.
\]

Then \( T \) admits one fixed point in \( U \).

Lemma 7 (Krasnoselskii fixed point theorem, [15]). Let \( M \) be a closed, bounded, convex and nonempty subset of a Banach space \( U \). Let \( A, B \) be operators such that

- \( A + By \in M \), where \( x, y \in M \),
- \( A \) is compact and continuous,
- \( B \) is a contraction mapping.

Then there exists \( z \in M \) such that \( z = Az + Bz \).

Lemma 8 ([13]). For the \( p \)-Laplacian operator \( \psi_p \), the following conditions hold true:

1. If \( |\varphi_1|, |\varphi_2| \geq \rho > 0, 1 < p \leq 2, \varphi_1 \varphi_2 > 0 \), then
   \[
   |\psi_p(\varphi_1) - \psi_p(\varphi_2)| \leq (p - 1) \rho^{p-2} |\varphi_1 - \varphi_2|.
   \]

2. If \( p > 2, |\varphi_1|, |\varphi_2| \leq \rho > 0 \), then
   \[
   |\psi_p(\varphi_1) - \psi_p(\varphi_2)| \leq (p - 1) \rho^{p-2} |\varphi_1 - \varphi_2|.
   \]

Lemma 9 ([10]). For nonnegative \( a_i, i = 1, \ldots, k \),

\[
\left( \sum_{i=1}^{k} a_i \right)^q \leq k^{q-1} \left( \sum_{i=1}^{k} a_i^q \right), q \geq 1.
\]

Now, we pass to prove the following result.
Lemma 10. For a given $h_m, g_m \in L^1(J, \mathbb{R}^3)$, the unique solution of the linear fractional initial value problem

$$\begin{align*}
D_{0+}^{\alpha \phi} \Psi_p \left[ D_{0+}^{\alpha \phi} (u_m(t) - I_{0+}^{\alpha \phi} g_m(t)) \right] &= h_m(t), \\
\text{for } m = 1, 2, 3, \text{ and } t \in J = (0, 1) \\
\Psi_p \left[ D_{0+}^{\alpha \phi} (u_m(t) - I_{0+}^{\alpha \phi} g_m(t)) \right] |_{t=0} &= 0, \\
u_m(0) &= 0, \quad u_m(1) = \sum_{i=1}^{n} \lambda_{im} u_m(\xi_{im}), \quad \xi_{im} \in (0, 1)
\end{align*}$$

is given by

$$\begin{align*}
u_m(t) &= \int_0^t \Phi_{2\alpha} (t,s) \Psi_q \left[ \int_0^s \Phi_{\alpha\eta} (s,e) h_m(e) de \right] ds + \int_0^t \Phi_\alpha (t,s) g_m(s) ds \\
&\quad - (\phi(t) - \phi(0)) \int_0^t \Phi_{2\alpha} (1,s) \Psi_q \left[ \int_0^s \Phi_{\alpha\eta} (s,e) h_m(e) de \right] ds \\
&\quad + (\phi(t) - \phi(0)) \left( \sum_{i=1}^{n} \lambda_{im} u_m(\xi_{i}) - \frac{g_m(0)}{K} \right).
\end{align*}$$

Proof. For $0 < r_{1m} < 1 < r_{2m} < 2$, Lemma 2 yields

$$\Psi_p \left[ D_{0+}^{r_{2m} \phi} (u_m(t) - I_{0+}^{r_{2m} \phi} g_m(t)) \right] = I_{0+}^{r_{1m} \phi} h_m(t) + c_{1m}$$

by conditions $\Psi_p \left[ D_{0+}^{r_{2m} \phi} (u_m(t) - g_m(t)) \right] |_{t=0} = 0$ we get $c_{1m} = 0$. Then

$$\left[ D_{0+}^{r_{2m} \phi} (u_m(t) - I_{0+}^{r_{2m} \phi} g_m(t)) \right] = \Psi_q \left[ I_{0+}^{r_{1m} \phi} h_m(t) \right]$$

so

$$\nu_m(t) = I_{0+}^{r_{2m} \phi} \left[ \Psi_q \left[ I_{0+}^{r_{1m} \phi} h_m(t) \right] \right] + I_{0+}^{r_{2m} \phi} g_m(t) + c_{2m} (\phi(t) - \phi(0)),$$

by conditions $\nu_m(0) = 0$ and $\nu_m(1) = \sum_{i=1}^{n} \lambda_{im} u_m(\xi_{i})$, we get

$$c_{2m} = \sum_{i=1}^{n} \lambda_{im} u_m(\xi_{i}) - \frac{g_m(0)}{K} - I_{0+}^{r_{2m} \phi} \left[ \Psi_q \left[ I_{0+}^{r_{1m} \phi} h_m(t) \right] \right] |_{t=1}.$$

\[\square\]

3. MAIN RESULTS

Taking into account Lemma 10, we define an operator $T : C \times C \times C \to C \times C \times C$

$$T (u_1, u_2, u_3) (t) = (T_1 (u_1, u_2, u_3) (t), T_2 (u_1, u_2, u_3) (t), T_3 (u_1, u_2, u_3) (t)), \quad (3.1)$$

where (for $m = 1, 3$)

$$T_m (u_1, u_2, u_3) (t) = \int_0^t \Phi_{2m} (t,s) \Psi_q \left[ \int_0^s \Phi_{\alpha\eta} (s,e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right] ds + \int_0^t \Phi_\alpha (t,s) G_m (s, u_1(s), u_2(s), u_3(s)) ds$$
\[ - (\varphi(t) - \varphi(0)) \int_0^1 \varphi_{r_{2m}}(1, s) \psi_q \left[ \int_0^s \varphi_{r_{1m}}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) \, de \right] \, ds \\
+ (\varphi(t) - \varphi(0)) \sum_{i=1}^n \frac{\lambda_{im}}{K} u_m(\xi_i) - \frac{G_m(0, 0, 0, 0)}{K}, \]  
(3.2)

and

\[ \varphi_{r_{2m}}(t, s) = \frac{\varphi'(s) (\varphi(t) - \varphi(s))^{r_{2m} - 1}}{\Gamma(r_{2m})}, \quad \varphi_{r_{1m}}(t, s) = \frac{\varphi'(s) (\varphi(t) - \varphi(s))^{r_{1m} - 1}}{\Gamma(r_{1m})}, \]
\[ \varphi_{r_{\sigma}}(t, s) = \frac{\varphi'(s) (\varphi(t) - \varphi(s))^{\sigma - 1}}{\Gamma(\sigma)}. \]

For the sake of convenience, we use the following notations (for \(m = 1, 3\)):

\[ \mathcal{K}_{1m} = \frac{2^{q-2} (K + 1)^{K_{r_{2m}}}}{\Gamma(1 + r_{2m})} \left( \frac{k_m K_{r_{1m}}}{\Gamma(1 + r_{1m})} \right)^{\sigma - 1}, \]
\[ \mathcal{K}_{2m} = \left( \frac{K^{\sigma} I_m}{\Gamma(1 + \sigma)} + \sum_{i=1}^n |\lambda_{im}| \right), \]
\[ \mathcal{K}_{3m} = \frac{2^{q-2} (K + 1)^{K_{r_{2m}}}}{\Gamma(1 + r_{2m})} \left( \frac{\mathcal{N}(K_{r_{1m}})}{\Gamma(1 + r_{1m})} \right)^{\sigma - 1} + \frac{K^{\sigma} M}{\Gamma(1 + \sigma)} + \mathcal{M}, \]
\[ \mathcal{K}_{4m} = \left( \frac{K_{r_{1m}} (p A_m + \mathcal{N})}{\Gamma(1 + r_{1m})} \right)^{\sigma - 2}, \]
\[ \mathcal{K}_{5m} = \frac{(q - 1) (1 + K)^{K_{r_{2m}} + r_{1m}}}{\Gamma(1 + r_{2m})} \left( \frac{\mathcal{K}_{r_{1m}}}{\Gamma(1 + r_{1m})} \right)^{\sigma - 2} + \frac{K^{\sigma} B_m}{\Gamma(1 + \sigma)} + \sum_{i=1}^n |\lambda_{im}|. \]

3.1. An Existence and Uniqueness Result

Here, by using the Banach contraction mapping principle, we prove an existence and uniqueness result.

**Theorem 1.** Let \(H_m, G_m : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}\) two continuous functions which satisfy the condition

\((\mathcal{A}_1)\) there exist positive real constants \( \mathcal{A}_m, B_m\) such that, for all \( t \in [0, 1] \) and \( u_1, v_1 \in \mathbb{R}, i, m = 1, 3\), we have

\[ |H_m(t, u_1, u_2, u_3) - H_m(e, v_1, v_2, v_3)| \leq \mathcal{A}_m (|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|), \]
\[ |G_m(t, u_1, u_2, u_3) - G_m(e, v_1, v_2, v_3)| \leq B_m (|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|). \]

Then, system (1.1) admits a unique solution on \([0, 1]\) provided that

\[ 3 (\mathcal{K}_{11} + \mathcal{K}_{22} + \mathcal{K}_{33}) < 1, \text{ and } \mathcal{K}_{51} + \mathcal{K}_{52} + \mathcal{K}_{53} < 1 \]
(3.3)

is valid.
Proof. We transform system (1.1) into a fixed point problem, \((u, v, w)(z) = \mathcal{T}(u, v, w)(z)\), where the operator \(\mathcal{T}\) is defined as in (3.1). Applying the Banach contraction mapping principle (Lemma 6), we show that the operator \(\mathcal{T}\) has a unique fixed point, which is the unique solution of system (1.1).

Let \(\sup_{t \in [0, 1]} H_m(t, 0, 0, 0) = N < \infty\) and \(\sup_{t \in [0, 1]} G_m(t, 0, 0, 0) = M < \infty\). Next, we set

\[\mathbb{U} \rho = \{(u_1, u_2, u_3) \in C \times C \times C, \|(u_1, u_2, u_3)\| \leq \rho\}\], in which

\[\rho \geq \max \left\{ \frac{2^{\sqrt{3N_0}}, 3N_0, m = 1, 3}{\sqrt{3}} \right\} \]

Observe that \(\mathbb{U} \rho\) is a bounded, closed, and convex subset of \(C\). First, we show that \(\mathcal{T} \mathbb{U} \rho \subseteq \mathbb{U} \rho\). For any \((u, v, w) \in \mathbb{U} \rho, t \in [0, 1]\), using the condition \((\mathcal{A}_1)\), we have

\[|H_m(t, u, v, w)| \leq |H_m(t, u, v, w) - H_m(e, 0, 0, 0)| + |H_m(e, 0, 0, 0)| \leq k_m(|u| + |v| + |w|) + N \leq \rho \mathcal{A}_m + N \]

and

\[|G_m(t, u, v, w)| \leq \rho \mathcal{B}_m + M.\]

Then, we obtain

\[
\begin{align*}
|\mathcal{T}_m(u_1, u_2, u_3)(t)| &\leq \int_0^t \varphi_{r_m}(t, s) \psi_q \left[ \int_0^s \varphi_{r_m}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right] ds \\
&\quad + \int_0^t \varphi_{*}(t, s) G_m(s, u_1(s), u_2(s), u_3(s)) ds \\
&\quad + |(\varphi(t) - \varphi(0))| \int_0^1 \varphi_{r_m}(1, s) \psi_q \left[ \int_0^s \varphi_{r_m}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right] ds \\
&\quad + |(\varphi(t) - \varphi(0))| \left( \sum_{i=1}^n \frac{|\lambda_{im}|}{K} |u_m(\zeta)| \right) + |G_m(0, 0, 0, 0)|
\end{align*}
\]

by Lemma 5 we get

\[
|\mathcal{T}_m(u_1, u_2, u_3)(t)| \leq \frac{(K + 1) K_{r_m}}{\Gamma(1 + r_{2m})} \left| \psi_q \left[ \int_0^s \varphi_{r_m}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right] \right| \\
+ \frac{K^\sigma (\rho \mathcal{B}_m + M)}{\Gamma(1 + \sigma)} \left( \sum_{i=1}^n \rho |\lambda_{im}| + M \right),
\]

and by \(\psi_q(z) = |z|^{q-2} z\), we have

\[
|\mathcal{T}_m(u_1, u_2, u_3)(t)| \leq \frac{(K + 1) K_{r_m}}{\Gamma(1 + r_{2m})} \left| \psi_q \left[ \int_0^s \varphi_{r_m}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right] \right|^{q-1} \\
+ \frac{K^\sigma (\rho \mathcal{B}_m + M)}{\Gamma(1 + \sigma)} \left( \sum_{i=1}^n \rho |\lambda_{im}| + M \right).
\]
Thanks to Lemma 9, for all $m = 1, 3$ we get

$$
|\mathcal{T}_m(u_1, u_2, u_3)(t)| \leq \frac{2^{q-2} (K + 1) K^{q-2n}}{q-1} \left( \frac{A_m K^{q-1}}{1 + r_{1m}} \right) \left( \frac{K^{q-1}}{1 + r_{1m}} \right) \left( \frac{A_m K^{q-1}}{1 + r_{1m}} \right) + \frac{K^q M}{1 + r_{2m}} + M
$$

Hence,

$$
\|\mathcal{T}(u_1, u_2, u_3)\| \leq \sum_{m=1}^{3} \left( \mathcal{K}_m p^{q-1} + \mathcal{K}_2 p + \mathcal{K}_3 \right) \leq \rho,
$$

which gives us $\mathcal{T} \cup \sigma \subset U \sigma$.

Next, we show that $\mathcal{T} : C \times C \times C \rightarrow C \times C \times C$ is a contraction.

Using condition ($\mathcal{A}_2$), for any $(u_1, u_2, u_3), (v_1, v_2, v_3) \in C \times C \times C$ and for each $t \in [0, 1]$, we have

$$
|\mathcal{T}_m(u_1, u_2, u_3) - \mathcal{T}_m(v_1, v_2, v_3)|
\leq \int_0^t \Phi_{r_{2m}}(t, s) \psi_q \left( \int_0^s \Phi_{r_{1m}}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right) ds
\quad - \int_0^t \Phi_{r_{2m}}(t, s) \psi_q \left( \int_0^s \Phi_{r_{1m}}(s, e) H_m(e, v_1(e), v_2(e), v_3(e)) de \right) ds
$$
Hence, by Lemma 5 and Lemma 8, we get

\[
|\mathcal{T}_m(u_1, u_2, u_3) - \mathcal{T}_m(v_1, v_2, v_3)|
\leq \frac{(1 + K) K^r_{2m}}{\Gamma(1 + r_{2m})} \left| \int_0^s \Phi_{r_{1m}}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right|
\leq \frac{(1 + K) K^r_{2m}}{\Gamma(1 + r_{2m})} \left[ \frac{(q - 1) \mathcal{A}_m (1 + K) K^r_{2m} + \mathcal{K}_m \mathcal{K}_m \mathcal{K}_m}{\Gamma(1 + r_{2m}) \Gamma(1 + r_{1m})} + \frac{K^\sigma B_m}{\Gamma(1 + \sigma)} + \sum_{i=1}^n |\lambda_{i,m}| \right]
\cdot (|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|)
\leq \mathcal{K}_m (|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|).
\]

Hence,

\[
|\mathcal{T}(u_1, u_2, u_3) - \mathcal{T}(v_1, v_2, v_3)| \leq (\mathcal{K}_1 + \mathcal{K}_1 + \mathcal{K}_3) (|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|).
\]

Since \( \mathcal{K}_1 + \mathcal{K}_1 + \mathcal{K}_3 < 1 \), by (3.3), the operator \( \mathcal{T} \) is a contraction. Therefore, using the Banach contraction mapping principle (Lemma 6), the operator \( \mathcal{T} \) has a unique fixed point. Hence, system (1.1) has a unique solution on \([0, 1]\). The proof is completed. \( \square \)
3.2. An Existence Result

Now we apply Krasnoselskii fixed point theorem (Lemma 7) to prove our second existence result. So, consider the following operator
\[
T(u_1, u_2, u_3)(t) = \sum_{m=1}^{3} (T_m(u_1, u_2, u_3))(t)
\]
\[
= \sum_{m=1}^{3} (P_{1m}(u_1, u_2, u_3)(t) + \sum_{m=1}^{3} (P_{2m}(u_1, u_2, u_3)(t)
\]
\[
= P_1(u_1, u_2, u_3)(t) + P_2(u_1, u_2, u_3)(t),
\]
where
\[
P_1(u_1, u_2, u_3)(t) = \sum_{m=1}^{3} (P_{1m}(u_1, u_2, u_3)(t),
\]
\[
P_2(u_1, u_2, u_3)(t) = \sum_{m=1}^{3} (P_{2m}(u_1, u_2, u_3)(t),
\]
and
\[
(P_{1m}(u_1, u_2, u_3)(t) = \int_{0}^{t} \Phi_{r_{2m}}(t, s) \psi_q \left[ \int_{0}^{s} \Phi_{r_{1m}}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right] ds
\]
\[- (\psi(t) - \psi(0)) \int_{0}^{1} \Phi_{r_{2m}}(1, s) \psi_q \left[ \int_{0}^{s} \Phi_{r_{1m}}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right] ds
\]
and
\[
(P_{2m}(u_1, u_2, u_3)(t) = \int_{0}^{t} \Phi_{q}(t, s) G_m(s, u_1(s), u_2(s), u_3(s)) ds
\]
\[+ (\psi(t) - \psi(0)) \left( \sum_{i=1}^{n} \lambda_{im} K^{-u_m(\xi_i)} - \frac{G_m(0, u_1(0), u_2(0), u_3(0))}{K} \right).
\]

**Theorem 2.** Let \(H_m, G_m : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}\) be continuous functions which satisfy condition (\(\mathcal{A}_1\)) in Theorem 1.

In addition, we assume that there exist two positive constants \(\Upsilon_{1m}, \Upsilon_{2m}\) such that, for all \(t \in [0, 1]\) and \(u_i, v_i \in \mathbb{R}, i, m = 1, 3,\) we have
\[
|H_m(t, u_1, u_2, u_3)| \leq \Upsilon_{1m},
\]
\[
|G_m(t, u_1, u_2, u_3)| \leq \Upsilon_{2m}.
\]

Moreover, assume that
\[
\sum_{i=1}^{n} |\lambda_{im}| \leq \frac{1}{3}, \text{ and } \left( \sum_{m=1}^{3} \frac{(q-1) A_m (1 + K) K^{r_{2m} + r_{1m}} X_{im}}{\Gamma(1 + r_{2m}) \Gamma(1 + r_{1m})} \right) < 1.
\]

Then, problem (1.1) admits at least one solution on \([0, 1]\).
Proof. The proof will be given in several steps. Let
\[ \mathcal{U}_\delta = \{ (u_1, u_2, u_3) \in \mathcal{C} \times \mathcal{C} \times \mathcal{C} | \| (u_1, u_2, u_3) \| \leq \delta \}, \]
in which
\[ \delta \geq 3 \max_{m=1}^n \left( \frac{Y_{1m} \Gamma^{r_{1m}}}{\Gamma(1+r_{1m})} \right)^{q-1} \left( \frac{(K+1) \Gamma^{r_{2m}}}{\Gamma(1+r_{2m})} + \frac{K^{\sigma} \Gamma^{r_{2m}}}{\Gamma(1+\sigma)} + M \right) \frac{1}{3} - \sum_{m=1}^n | \lambda_{im} |. \]

**Step 1.** We prove that
\[ |(\mathcal{I}(u_1, u_2, u_3))(t)| \leq \delta. \]
Let \((u_1, u_2, u_3), (v_1, v_2, v_3), (w_1, w_2, w_3) \in \mathcal{U}_\delta.\) As in the proof of Theorem 1, we have
\[ |(P_{1m}(u_1, u_2, u_3))(t) + (P_{2m}(u_1, u_2, u_3))(t)| \]
\[ \leq \left( \frac{Y_{1m} \Gamma^{r_{1m}}}{\Gamma(1+r_{1m})} \right)^{q-1} \left( \frac{(K+1) \Gamma^{r_{2m}}}{\Gamma(1+r_{2m})} + \frac{K^{\sigma} \Gamma^{r_{2m}}}{\Gamma(1+\sigma)} + M + \left( \sum_{i=1}^n | \lambda_{im} | \right) \right) \delta \leq \frac{\delta}{3}. \]
Hence
\[ |(\mathcal{I}(u_1, u_2, u_3))(t)| = |(\mathcal{I}_1(u_1, u_2, u_3))(t) + (\mathcal{I}_2(u_1, u_2, u_3))(t) + (\mathcal{I}_3(u_1, u_2, u_3))(t)| \]
\[ \leq |(\mathcal{I}_1(u_1, u_2, u_3))(t)| + |(\mathcal{I}_2(u_1, u_2, u_3))(t)| + |(\mathcal{I}_3(u_1, u_2, u_3))(t)| \]
\[ \leq \sum_{m=1}^3 | P_{1m}(u_1, u_2, u_3)(t) + P_{2m}(u_1, u_2, u_3)(t) | \leq \delta. \]
Accordingly, \( \sum_{m=1}^3 | P_{1m}(u_1, u_2, u_3)(t) + P_{2m}(u_1, u_2, u_3)(t) | \in \mathcal{U}_\delta \) and the condition (i) of Lemma 7 is satisfied.

**Step 2.** \( P_1 \) is a contraction.
Let \((u_1, u_2, u_3), (v_1, v_2, v_3) \in \mathcal{U}_\delta.\) We have the following estimate
\[ | P_{1m}(u_1, u_2, u_3)(t) - P_{1m}(v_1, v_2, v_3)(t) | \]
\[ \leq \int_0^t \Phi_{r_{1m}}(t, s) \psi_q \left( \int_0^s \Phi_{r_{1m}}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right) ds \]
\[ - \int_0^t \Phi_{r_{2m}}(t, s) \psi_q \left( \int_0^s \Phi_{r_{1m}}(s, e) H_m(e, v_1(e), v_2(e), v_3(e)) de \right) ds \]
\[ + K \int_0^t \Phi_{r_{2m}}(1, s) \psi_q \left( \int_0^s \Phi_{r_{1m}}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right) ds \]
\[ - \int_0^t \Phi_{r_{2m}}(1, s) \psi_q \left( \int_0^s \Phi_{r_{1m}}(s, e) H_m(e, v_1(e), v_2(e), v_3(e)) de \right) ds \]
\[ \leq \left( 1 + K \right) \frac{\Gamma^{r_{2m}}}{\Gamma(1+r_{2m})} \psi_q \left( \int_0^s \Phi_{r_{1m}}(s, e) H_m(e, u_1(e), u_2(e), u_3(e)) de \right) \]
- \psi_q \left( \int_0^\epsilon \varphi \left( \epsilon, \frac{t}{\epsilon} \right) d\epsilon \right) \\
\leq \frac{(q-1) A_m (1+K) K^{r_2m+\epsilon r_{2m}} K_{\delta m}}{\Gamma(1+r_{2m}) \Gamma(1+r_{1m})} |u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|.

So

\begin{align*}
|P_1(u_1, u_2, u_3)(t) - P_1(v_1, v_2, v_3)(t)| \\
\leq \left( \sum_{m=1}^{3} \frac{(q-1) A_m (1+K) K^{r_2m+\epsilon r_{2m}} K_{\delta m}}{\Gamma(1+r_{2m}) \Gamma(1+r_{1m})} \right) |u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|.
\end{align*}

Since \( \sum_{m=1}^{3} \frac{(q-1) A_m (1+K) K^{r_2m+\epsilon r_{2m}} K_{\delta m}}{\Gamma(1+r_{2m}) \Gamma(1+r_{1m})} < 1 \), the operator \( P_1 \) is a contraction.

**Step 3.** \( P_2 \) is compact and continuous.

Since \( H_m, G_m \) are continuous functions, this implies that the operator \( P_2 \) is continuous on \( U_\delta \). Moreover, \( P_2(u_1, u, u_3) \) is uniformly bounded by (3.4). Next, we show equicontinuity. Let \( (u_1, u, u_3) \in U_\delta \), we have

\begin{align*}
|P_2(u_1, u_2, u_3)(t)| &\leq \left| \int_0^t \varphi_\sigma (t, s) G_m(s, u_1(s), u_2(s), u_3(s)) ds \right| + K \left( \sum_{i=1}^{n} \frac{|\lambda_{im}|}{K} |u_m(\zeta_i)| + \frac{|G_m(0, u_1(0), u_2(0), u_3(0))|}{K} \right) \\
&\leq \left( \frac{\Gamma_{2m} K^\sigma}{\Gamma(1+\sigma)} + \sum_{i=1}^{n} |\lambda_{im}| + \gamma_{2m} \right) \delta.
\end{align*}

So

\begin{equation}
|P_2(u_1, u_2, u_3)(t)| \leq \sum_{m=1}^{3} \left( \frac{\Gamma_{2m} K^\sigma}{\Gamma(1+\sigma)} + \sum_{i=1}^{n} |\lambda_{im}| + \gamma_{2m} \right) \delta. \tag{3.5}
\end{equation}

Moreover, \( P_2(u_1, u, u_3) \) is uniformly bounded by (3.5). Next, we show equicontinuity for \( t_1, t_2 \in [0, 1] \) such that \( t_1 < t_2 \) we have

\begin{align*}
|P_2(u_1, u_2, u_3)(t_2) - P_2(u_1, u_2, u_3)(t_1)| \\
&\leq \left| \int_0^{t_2} \varphi_\sigma (t_2, s) G_m(s, u_1(s), u_2(s), u_3(s)) ds - \int_0^{t_1} \varphi_\sigma (t_1, s) G_m(s, u_1(s), u_2(s), u_3(s)) ds \right| \\
&\leq \frac{\gamma_{2m}}{\Gamma(1+\sigma)} (\varphi(t_2) - \varphi(t_1))^\sigma.
\end{align*}

So

\begin{align*}
|P_2(u_1, u_2, u_3)(t_2) - P_2(u_1, u_2, u_3)(t_1)| \\
&\leq \int_0^{t_2} \varphi_\sigma (t_2, s) G_m(s, u_1(s), u_2(s), u_3(s)) ds - \int_0^{t_1} \varphi_\sigma (t_1, s) G_m(s, u_1(s), u_2(s), u_3(s)) ds
\end{align*}
\[
\left( \frac{Y_{2m}}{\Gamma(1 + \sigma)} \right) (\varphi(t_2) - \varphi(t_1))^\sigma.
\]

Consequently,
\[
| \langle \mathcal{P}_2(u_1, u_2, u_3)(t_2) - \mathcal{P}_2(u_1, u_2, u_3)(t_1) \rangle | \to 0, \quad \text{as } t_1 \to t_2.
\]

This shows that \( \mathcal{P}_2 \mathcal{U}_\delta \) is equicontinuous. Hence, by Arzelà-Ascoli theorem \( \mathcal{P}_2 \) is completely continuous on \( \mathcal{U}_\delta \). As a consequence of Krasnoselskii’s fixed point theorem, we conclude that it has a fixed point which is a solution of (1.1). The proof of Theorem 2 is thus completely achieved. \( \square \)

4. An Illustrative Example

Consider the following nonlinear equation for all \( t \in (0, 1] \):

\[
\begin{cases}
    c D_0^\alpha \psi_p \left[ c D_0^\alpha \varphi_p \left( u(t) - \Gamma_0^\varphi \left( \frac{t^\rho + 1}{1 + (u(t))^\rho} \right) \right) \right] = \frac{1}{1 + \rho} \left( \frac{u(t)}{1 + (u(t))^\rho} \right), \\
    c D_0^\alpha \psi_p \left[ c D_0^\alpha \varphi_p \left( v(t) - \Gamma_0^\varphi \left( \frac{t^r + 1}{1 + (v(t))^r} \right) \right) \right] = \frac{t}{1 + \rho} \left( \frac{v(t)}{1 + (v(t))^r} \right), \\
    c D_0^\alpha \psi_p \left[ c D_0^\alpha \varphi_p \left( w(t) - \Gamma_0^\varphi \left( \frac{t^\rho + 1}{1 + (w(t))^\rho} \right) \right) \right] = \frac{t^\rho}{1 + \rho} \left( \frac{w(t)}{1 + (w(t))^\rho} \right), \\
    \psi_p \left[ c D_0^\alpha \varphi_p \left( u(t) - \Gamma_0^\varphi \left( \frac{t^\rho + 1}{1 + (u(t))^\rho} \right) \right) \right] \bigg|_{t=0} = 0, \\
    \psi_p \left[ c D_0^\alpha \varphi_p \left( v(t) - \Gamma_0^\varphi \left( \frac{t^r + 1}{1 + (v(t))^r} \right) \right) \right] \bigg|_{t=0} = 0, \\
    \psi_p \left[ c D_0^\alpha \varphi_p \left( w(t) - \Gamma_0^\varphi \left( \frac{t^\rho + 1}{1 + (w(t))^\rho} \right) \right) \right] \bigg|_{t=0} = 0, \\
    u(1) = \sum_{i=1}^n \frac{1}{\eta(\zeta_i)} u(\zeta_i), v(1) = \sum_{i=1}^n \frac{1}{\varphi(\zeta_i)} v(\zeta_i), w(1) = \sum_{i=1}^n \frac{1}{\psi(\zeta_i)} w(\zeta_i), \quad \zeta_i \in (0, 1]
\end{cases}
\]

and

\[
K = 1,
\]

\[
Y_{11} = A_1 = \frac{1}{2}, Y_{12} = A_2 = \frac{1}{4}, Y_{13} = A_3 = \frac{e}{2},
\]

\[
Y_{21} = B_1 = \frac{1 + e}{2}, Y_{22} = B_2 = 1, Y_{23} = B_3 = \frac{e}{2}.
\]

Thus, the assumptions \( (A_1) \) are satisfied and Theorem 1-2 implies that (4.1) has a unique solution on \([0, 1]\).
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