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THE DETERMINANTS OF CERTAIN DOUBLE BANDED $(0,1)$ TOEPLITZ MATRICES

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Abstract. We prove a conjecture proposed recently in Linear Algebra and its Applications about an exact formula for certain double banded $(0,1)$ Toeplitz matrices. Moreover, we extend the result to a more general setting.

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1. INTRODUCTION

For any integers $n > t \geqslant 2$, let $\Delta(n,t)$ be the determinant of the $n \times n$ matrix whose (i,j)-entry is 1, if

$$j - i \in \{-2, -1, 0, 1, t\},$$

and 0, otherwise. In [7], Shitov proves that $\Delta(n, n-k)$ is of period 4, for $n \geqslant 2k$. The procedure is quite ingenious and, as an immediate consequence, two conjectures stated in [2] by Anđelić and da Fonseca on $\Delta(n, n-1)$ and $\Delta(n, n-2)$ are proved. For alternative approaches, the reader is referred to [1, 5, 6]. A more general double banded $(0,1)$ Toeplitz matrix has been considered by the authors in [4], where an explicit formula to the period of the determinants of those matrices is determined.

Based on numerical experiments, Shitov [7] proposes several conjectures towards two directions. One direction is the following:

Conjecture 1. *For any given integer $c \geqslant 0$, the sequence $\Delta(2k+c, k+c)$ admits an exact formula.*

As mentioned in [7, Statements 9 and 10], two straightforward instances of this conjecture are $\Delta(8k, 4k) = k^2 + 1$ and $\Delta(8k+2, 4k+2) = k^2$.

The aim of this note is to prove the above conjecture. Indeed, we will obtain a more general result than what Shitov conjectured. A preparatory case will be considered

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in the next section. In Section 3, we prove Conjecture 1. In the last section, the conjecture is extended to a more general setting.

2. THE PREPARATORY CASE

We start this section with the definition of a special double banded $(0,1)$ Toeplitz matrix.

Definition 1. For nonnegative integers n,c,s, let $A_s\left(n,\frac{n+c}{2}\right)$ be the $n\times n$ matrix whose (i,j)-entry is equal to 1, if

$$j-i\in\left\{-s,-s+1,-s+2,-s+3,\frac{n+c}{2}\right\},$$

and 0, otherwise. In particular, we set $A_1(n)$ for the $n\times n$ matrix whose (i,j)-entry is equal to 1, if $j-i=-1,0,1,2$, and 0, otherwise.

Remark 1. Notice that we require

$$n\geqslant\max\{c+2,10-c-2s\},$$

i.e., $n\geqslant c+2$ and $n\geqslant 10-c-2s$. The reason why $n\geqslant c+2$ is because we want to guarantee that the upper band (with entries 1) satisfying $j-i=\frac{n+c}{2}$ contains at least one 1, i.e., $\frac{n-c}{2}\geqslant 1$. On the other hand, we consider $n\geqslant 10-c-2s$ because the matrices should have two disjoint bands (following the type of matrices in Conjecture 1).

We denote $\Delta_s(n,\frac{n+c}{2})=\det A_s\left(n,\frac{n+c}{2}\right)$ and $\Delta_1(n)=\det A_1(n)$. It is worth mentioning that $\Delta_2(2k+c,k+c)$ coincides with the notation $\Delta(2k+c,k+c)$ that Shitov used in Conjecture 1.

Our first goal is to establish a formula for $\Delta_1\left(n,\frac{n+c}{2}\right)$.

Theorem 1. *Assume that* $n\geqslant\max\{c+2,8-c\}$.

(i) *Suppose that c is even.*

- *Assume that* $c\equiv 0\,(\bmod 4)$. *Then*

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\begin{cases}1 & \text{if } n\equiv c\,(\bmod 8),\\ -\frac{n-c+6}{8} & \text{if } n\equiv c+2\,(\bmod 8),\\ \frac{n-c+8}{4} & \text{if } n\equiv c+4\,(\bmod 8),\\ -\frac{n-c+2}{8} & \text{if } n\equiv c+6\,(\bmod 8).\end{cases}$$

- *Assume that* $c\equiv 2\,(\bmod 4)$. *Then*

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\begin{cases}0 & \text{if } n\equiv c\,(\bmod 8),\\ -\frac{n-c-2}{8} & \text{if } n\equiv c+2\,(\bmod 8),\\ \frac{n-c+4}{4} & \text{if } n\equiv c+4\,(\bmod 8),\\ -\frac{n-c-6}{8} & \text{if } n\equiv c+6\,(\bmod 8).\end{cases}$$

(ii) *Suppose that c is odd.*

- *Assume that $c \equiv 1 \pmod 4$. Then*

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\begin{cases}1 & \text{if } n\equiv c \pmod 8,\\ \frac{n-c+6}{8} & \text{if } n\equiv c+2 \pmod 8,\\ -\frac{n-c}{4} & \text{if } n\equiv c+4 \pmod 8,\\ \frac{n-c+2}{8} & \text{if } n\equiv c+6 \pmod 8).\end{cases}$$

- *Assume that $c \equiv 3 \pmod 4$. Then*

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\begin{cases}0 & \text{if } n\equiv c \pmod 8,\\ \frac{n-c+14}{8} & \text{if } n\equiv c+2 \pmod 8,\\ -\frac{n-c+4}{4} & \text{if } n\equiv c+4 \pmod 8,\\ \frac{n-c+10}{8} & \text{if } n\equiv c+6 \pmod 8).\end{cases}$$

The proof of Theorem 1 is based on the following lemmas. They reveal the recurrence relations and initial conditions of $\Delta_1\left(n,\frac{n+c}{2}\right)$.

Notice that, as a special case of [4, Lemma 3.2], the recurrence relations of $\Delta_1\left(n,\frac{n+c}{2}\right)$ follow immediately.

Lemma 1 ([4, Lemma 3.2])**.** *For any integers n,c satisfying $n \geqslant \max\{c+2,8-c\}$ and $c \geqslant 0$, we have*

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\Delta_1\left(n-4,\frac{n+c}{2}\right)+\begin{cases}0 & \text{if } n\equiv c \pmod 8,\\ (-1)^{c+1} & \text{if } n\equiv c+2,c+6 \pmod 8,\\ 2(-1)^c & \text{if } n\equiv c+4 \pmod 8).\end{cases}$$

Next, we determine the initial conditions of $\Delta_1\left(n,\frac{n+c}{2}\right)$.

Lemma 2. *When $c \geqslant 3$,*

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\begin{cases}\Delta_1(c-2)+(-1)^{c+1} & \text{if } n=c+2,\\ \Delta_1(c)+2(-1)^c & \text{if } n=c+4,\\ \Delta_1(c+2)+(-1)^{c+1} & \text{if } n=c+6,\\ \Delta_1(c+4) & \text{if } n=c+8.\end{cases}$$

Proof. The proofs of the four cases are similar, so we only show the first case, i.e., when $n=c+2$. From Lemma 1, we have

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\Delta_1\left(n-4,\frac{n+c}{2}\right)+(-1)^{c+1}.$$

Since $n=c+2$, the above equation is in fact

$$\Delta_1(c+2,c+1)=\Delta_1(c-2,c+1)+(-1)^{c+1}.$$

The result follows now by observing that $A_1(c-2,c+1) = A_1(c-2)$. $\square$

The values of $\Delta_1(n)$ are known from [4, Theorem 3.3].

Lemma 3 ([4, Theorem 3.3])**.** *When $n \geqslant 1$,*

$$\Delta_1(n) = \begin{cases} (-1)^n & \text{if } n \equiv 0 \pmod 4, \\ (-1)^{n+1} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2,3 \pmod 4. \end{cases}$$

From Remark 1, we require that $n \geqslant \max\{c+2, 8-c\}$. But in order to make the recurrence relations shown in Lemma 1 applicable, we slightly adjust it into $n \geqslant \max\{c+2, 10-c\}$. Keeping this requirement in mind, we calculate $\Delta_1\left(n, \frac{n+c}{2}\right)$ with small values of n, for each c, which can be regarded as the initial conditions of the sequence $\Delta_1\left(n, \frac{n+c}{2}\right)$.

Lemma 4.

(i) *When $c = 0$,*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 1 & \text{if } n = 8, \\ -2 & \text{if } n = 10, 14, \\ 5 & \text{if } n = 12. \end{cases}$$

(ii) *When $c = 1$,*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 1 & \text{if } n = 7, 9, \\ 2 & \text{if } n = 11, \\ -3 & \text{if } n = 13. \end{cases}$$

(iii) *When $c = 2$,*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 2 & \text{if } n = 6, \\ 0 & \text{if } n = 8, 10, \\ -1 & \text{if } n = 12. \end{cases}$$

(iv) *Assume that $c \geqslant 3$ is odd.*

- *If $c \equiv 1 \pmod 4$, then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 1 & \text{if } n = c+2, c+6, c+8, \\ -1 & \text{if } n = c+4. \end{cases}$$

- *If $c \equiv 3 \pmod 4$, then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 2 & \text{if } n = c+2, c+6, \\ -2 & \text{if } n = c+4, \\ 0 & \text{if } n = c+8. \end{cases}$$

(v) *Assume that $c \geqslant 4$ is even.*

– *If $c \equiv 0 \pmod 4$, then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} -1 & \text{if } n = c+2, c+6, \\ 3 & \text{if } n = c+4, \\ 1 & \text{if } n = c+8. \end{cases}$$

– *If $c \equiv 2 \pmod 4$, then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 0 & \text{if } n = c+2, c+6, c+8, \\ 2 & \text{if } n = c+4. \end{cases}$$

Proof. The items (i)-(iii) can be obtained directly. When $c \geqslant 3$, (iv) and (v) follow from Lemmas 2 and 3. □

Finally, when $n \geqslant \max\{c+2, 8-c\}$, we can obtain $\Delta_1\left(n, \frac{n+c}{2}\right)$ as it is shown in Theorem 1, by combining the recurrence relations of Lemma 1 and the initial conditions found in Lemma 4.

3. A CONJECTURE OF SHITOV

In this section, we present the explicit expressions for $\Delta_2\left(n, \frac{n+c}{2}\right)$, which solve Conjecture 1. We remark that, with $n = 2k+c$,

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \Delta_2(2k+c, k+c) = \Delta(2k+c, k+c),$$

according to the notation of Shitov.

Theorem 2. *Assume that $n \geqslant \max\{c+2, 6-c\}$ and $c \geqslant 0$.*

(i) *Suppose that c is even.*

- *Assume that $c \equiv 0 \pmod 4$. Then*

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \begin{cases} \left(\frac{n-c}{8}\right)^2 + 1 & \text{if } n \equiv c \pmod 8, \\ -\left(\frac{n-c+6}{8}\right)^2 & \text{if } n \equiv c+2 \pmod 8, \\ \left(\frac{n-c-4}{8}\right)^2 & \text{if } n \equiv c+4 \pmod 8, \\ \frac{(3n-3c-2)(n-c+2)}{64} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

- *Assume that $c \equiv 2 \pmod 4$. Then*

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \begin{cases} \left(\frac{n-c}{8}\right)^2 & \text{if } n \equiv c \pmod 8, \\ -\frac{(n-c)^2+12(n-c)-92}{64} & \text{if } n \equiv c+2 \pmod 8, \\ \frac{(n-c+20)(n-c+4)}{64} & \text{if } n \equiv c+4 \pmod 8, \\ \frac{3(n-c)^2+36(n-c)+124}{64} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

(ii) *Suppose that c is odd.*

- *Assume that* $c \equiv 1 \pmod 4$. *Then*

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \begin{cases} \left(\frac{n-c+8}{8}\right)^2 & \text{if } n \equiv c \pmod 8, \\ -\left(\frac{n-c-2}{8}\right)^2 & \text{if } n \equiv c+2 \pmod 8, \\ \frac{(n-c)^2+8(n-c)+80}{64} & \text{if } n \equiv c+4 \pmod 8, \\ \frac{(3n-3c+14)(n-c+2)}{64} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

- *Assume that* $c \equiv 3 \pmod 4$. *Then*

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \begin{cases} \frac{(n-c)(n-c-16)}{64} & \text{if } n \equiv c \pmod 8, \\ -\frac{(n-c)^2-4(n-c)-124}{64} & \text{if } n \equiv c+2 \pmod 8, \\ \left(\frac{n-c+4}{8}\right)^2 & \text{if } n \equiv c+4 \pmod 8, \\ \frac{3(n-c)^2-12(n-c)+28}{64} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

In order to provide the extension from $s = 1$ (Theorem 1) to $s = 2$ (Theorem 2), we recall the classical Dodgson's determinant-evaluation rule [3].

Lemma 5 ([3]). *For any* $n \times n$ *matrix* A, $n \geqslant 2$, *we have*

$$\det A \det A_2 = \det A_{11} \det A_{nn} - \det A_{1n} \det A_{n1},$$

where A_{ij} *is the submatrix obtained from* A *by deleting the ith row and jth column, and* A_2 *is the principal submatrix of* A *induced by* $\{2, \ldots, n-1\}$.

Applying Dodgson's determinant-evaluation rule to $A_1\left(n+1, \frac{n+c+2}{2}\right)$, we obtain

$$\Delta_1\left(n+1, \frac{n+c+2}{2}\right)\Delta_1\left(n-1, \frac{n+c+2}{2}\right) = \left(\Delta_1\left(n, \frac{n+c+2}{2}\right)\right)^2 - \Delta_0\left(n, \frac{n+c+4}{2}\right)\Delta_2\left(n, \frac{n+c}{2}\right). \tag{3.1}$$

Observe that $A_0\left(n, \frac{n+c+4}{2}\right)$ is an upper triangular matrix whose main diagonal entries are all equal to 1. This means that $\Delta_0\left(n, \frac{n+c+4}{2}\right) = 1$. An equivalent form of (3.1) can be obtained immediately as follows:

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \left(\Delta_1\left(n, \frac{n+c+2}{2}\right)\right)^2 - \Delta_1\left(n+1, \frac{n+c+2}{2}\right)\Delta_1\left(n-1, \frac{n+c+2}{2}\right). \tag{3.2}$$

The three determinants $\Delta_1(*,*)$ on the right-hand side of (3.2) are known from Theorem 1. It is worth indicating that here we need the precondition that $n \geqslant c+6$, or equivalently, $n \geqslant \max\{c+6, 6-c\}$, which guarantees the using of Theorem 1 to the three determinants $\Delta_1(*,*)$. Then the expression of $\Delta_2\left(n, \frac{n+c}{2}\right)$ can be obtained after some straightforward calculations, proving Theorem 2.

As to the remaining two cases $n=c+2$ and $n=c+4$, they just correspond to the two conjectures proposed in [2], which were confirmed in [6] (see also [1, 4, 5, 7]. Following the notations in the paper, they claim that

$$\Delta_2\left(c+2,c+1\right)=\begin{cases}-1 & \text{if } c\equiv 0\,(\bmod 4),\\ 0 & \text{if } c\equiv 1\,(\bmod 4),\\ 1 & \text{if } c\equiv 2\,(\bmod 4),\\ 2 & \text{if } c\equiv 3\,(\bmod 4),\end{cases}$$

and

$$\Delta_2\left(c+4,c+2\right)=\begin{cases}0 & \text{if } c\equiv 0\,(\bmod 4),\\ 2 & \text{if } c\equiv 1\,(\bmod 4),\\ 3 & \text{if } c\equiv 2\,(\bmod 4),\\ 1 & \text{if } c\equiv 3\,(\bmod 4),\end{cases}$$

which agree with Theorem 2 as well.

Therefore, setting $n=2k+c$, Theorem 2 presents the exact formula for $\Delta_2(2k+c,k+c)$, for any fixed integers $k\geqslant 1$ and $k+c\geqslant 3$ (other integers k,c are against the requirement $n\geqslant\max\{c+2,6-c\}$ mentioned in Remark 1), confirming the conjecture of Shitov.

As examples, we can extend Statements 9 and 10 in [7], claiming that $\Delta_2(8k,4k)=k^2+1$ and $\Delta_2(8k+2,4k+2)=k^2$. In fact, from Theorem 2, we have:

Proposition 1. *For all integers $k\geqslant 1$ and $c\geqslant 0$,*

$$\Delta_2(8k+c,4k+c)=\begin{cases}k^2+1 & \text{if } c\equiv 0\,(\bmod 4),\\ (k+1)^2 & \text{if } c\equiv 1\,(\bmod 4),\\ k^2 & \text{if } c\equiv 2\,(\bmod 4),\\ k(k-2) & \text{if } c\equiv 3\,(\bmod 4).\end{cases}$$

4. An extension

It is worth mentioning that the conjecture of Shitov is not only true for $\Delta(2k+c,k+c)$, but also valid for the determinants of a much larger family of matrices. First we introduce this family of matrices, which is a generalization of the matrix $A_s\left(n,\frac{n+c}{2}\right)$ defined in Definition 1.

Definition 2. For nonnegative integers n,r,s,t, let $A_{s,t}(n,r)$ be the $n\times n$ matrix whose (i,j)-entry is equal to 1, if

$$j-i\in\{-s,-s+1,-s+2,-s+3,r,r+1,\dots,r+t-1\},$$

and 0, otherwise.

Remark 2. In particular, when $t=1$ and $r=\frac{n+c}{2}$, $A_{s,t}(n,r)$ would be reduced to the matrix $A_s\left(n,\frac{n+c}{2}\right)$ investigated in previous sections.

Set $\Delta_{s,t}(n,r) = \det A_{s,t}(n,r)$. Now we recall several formulae obtained in our previous publication [4, Theorems 3.3 and 3.4], which lead to an explicit expression of $\Delta_{1,t}(n,r)$, in terms of the determinants of form $\Delta_1(*)$.

Assume that $n \equiv p \,(\bmod 4)$ and $n-r-\ell \equiv q_\ell \,(\bmod 4)$, for any nonnegative integer ℓ. When $n > r$, set

$$\nu = \max\{s \in \mathbb{Z} : n-r-4s > 0\},$$

where $\mathbb{Z}$ represents the set of the integers. Clearly $n-r-4\nu = q_0$, if $q_0 > 0$, and $n-r = 4(\nu+1)$, if $q_0 = 0$.

Theorem 3 ([4, Theorems 3.3 and 3.4])**.**

(i) *Assume that* $r = 0,1,2,3$. *For* $n \geqslant 1$, *we have*

$$\Delta_{1,t}(n,r) = \begin{cases} (-1)^n & \text{if } n \equiv 0 \,(\bmod r+t+1), \\ (-1)^{n+1} & \text{if } n \equiv 1 \,(\bmod r+t+1), \\ 0 & \text{if } n \equiv 2,3,\ldots,r+t \,(\bmod r+t+1). \end{cases}$$

(ii) *Assume that* $4 \leqslant r \leqslant n-1$. *If* $r \geqslant \frac{n-1}{2}$, *we have*

$$\Delta_{1,t}(n,r) = (-1)^{n-p}\Delta_1(p) + (-1)^n \sum_{j=0}^{\nu} \big((-1)^{q_{\min\{t,n-r-4j\}+1}}\Delta_1(q_{\min\{t,n-r-4j\}+1}) + (-1)^{q_1+1}\Delta_1(q_1)\big).$$

(iii) *Assume that* $r \geqslant n$. *For* $n \geqslant 1$, *we have*

$$\Delta_{1,t}(n,r) = \begin{cases} (-1)^n & \text{if } n \equiv 0 \,(\bmod 4), \\ (-1)^{n+1} & \text{if } n \equiv 1 \,(\bmod 4), \\ 0 & \text{if } n \equiv 2,3 \,(\bmod 4). \end{cases}$$

Remark 3. We divided the result into three parts. This is due to the fact that $A_{1,t}(n,r)$ has one band when $r = 0,1,2,3$ or $r \geqslant n$, and two (disjoint) bands when $4 \leqslant r \leqslant n-1$.

Recall that the formulae about determinants of form $\Delta_1(*)$ are known (see Lemma 3). Thus we can get the following corollary immediately.

Corollary 1. *The sequence* $\Delta_{1,t}(n,r)$ *admits an exact formula under the conditions mentioned in Theorem 3.*

As in the previous section, based on the formulae of $\Delta_{1,t}(n,r)$ (Theorem 3), we can obtain a formula of $\Delta_{2,t}(n,r)$:

$$\Delta_{2,t}(n,r) = \big(\Delta_{1,\min\{t,n-r-1\}}(n,r+1)\big)^2 - \Delta_{1,t}(n+1,r+1)\Delta_{1,\min\{t,n-r-2\}}(n-1,r+1),$$

with the help of Lemma 5 (Dodgson's determinant-evaluation rule).

Corollary 2. *The sequence $\Delta_{2,t}(n,r)$ admits an exact formula, when $r=0,1,2$ with $n \geqslant 2$, or $3 \leqslant r \leqslant n-1$ and $r \geqslant \frac{n-2}{2}$, or $r \geqslant n$ with $n \geqslant 2$.*

The above corollary extends the conjecture of Shitov to a much larger family of matrices.

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