Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1351–1360
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4051

CHARACTERIZATIONS OF GENERALIZED SUBMODULES OF QTAG-MODULES

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Received 28 December, 2021

Abstract. We characterize the α-large submodules of α-modules in terms of certain sequences of ordinals, and give some of their interesting properties. Also, we deal with α-large submodules of the closure of an unbounded direct sum of uniserial modules, and the α-large submodules of the smallest α-pure fully invariant submodules of the closure containing a given element.

2010 *Mathematics Subject Classification:* 16K20

Keywords: α-modules, α-large submodules, fully invariant submodules

1. Introduction and Fundamentals

Let R be any ring with unity. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module M is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length.

In 1976 Singh [16] introduced a class of modules called *TAG*-modules, defined by satisfying two properties relating to uniserial modules.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

It was shown that the theory of these modules very closely paralleled the theory of torsion abelian groups; for this reason they were referred to as *TAG*-modules. In 1987 Singh showed that the second property, with minimal additional hypotheses, can be deduced from the first and studied the modules satisfying only the first property and called them *QTAG*-modules. The study of *QTAG*-modules and their structure began with work of Singh in [17]. Since then, many authors have written about the structure of *QTAG*-modules. Not surprisingly, many of the developments parallel the earlier development of the structure of torsion abelian groups. Our main goal here is to study α-modules, a class of the *QTAG*-modules and further advance the study of the structure of *QTAG*-modules and the parallels with torsion abelian groups. The present paper is a natural extension of work already done in this field and certainly contributes to the overall knowledge of the structure of *QTAG*-modules.

All rings below are assumed to be associative and with nonzero identity element; all modules are assumed to be unital *QTAG*-modules. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. Let us denote by M^1, the submodule of M, containing elements of infinite height. The module M is h-divisible [9] if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded [16], if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$. A submodule N of M is h-pure [8] in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule $B \subseteq M$ is a basic submodule [9] of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A submodule F of M is said to be fully invariant [1] if each endomorphism of M sends F into itself. A fully invariant submodule $L \subseteq M$ is large [2], if $L + B = M$, for every basic submodule B in M.

For a module M and an ordinal α, $H_\alpha(M)$ is defined as $H_\alpha(M) = \bigcap_{\beta<\alpha} H_\beta(M)$. For an ordinal α, a submodule N of M is said to be α-pure [14], if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \alpha$. A submodule $N \subset M$ is nice [11] in M, if $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ for all ordinals α, i.e. every coset of M modulo N may be represented by an element of the same height. The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-*Ulm* invariant of M and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$. An ordinal α is said to be confinal with ω, if α is the limit of a countable ascending sequence of ordinals.

Imitating [12], for any uniform element $x \in M$, there exist uniform elements x_1, $x_2, \ldots$ such that $xR \supseteq x_1R \supseteq x_2R \supseteq \ldots$ and $d\left(\frac{x_iR}{x_{i+1}R}\right) = 1$. Now the *Ulm*-sequence of x is defined as $U_M(x) = (H_M(x), H_M(x_1), H_M(x_2), \ldots)$. These sequences are partially ordered because $U_M(x) \leq U_M(y)$ if $H_M(x_i) \leq H_M(y_i)$ for every i.

An h-reduced module M is totally projective [10] if it has a collection $\mathcal{N}$ of nice submodules such that (i) $0 \in \mathcal{N}$ (ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\sum_{i\in I} N_i \in \mathcal{N}$ (iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated.

It is interesting to note that almost all the results which hold for *TAG*-modules are also valid for *QTAG*-modules [14]. Many results, stated in the present paper, are clearly motivated from the paper [3]. Most of our notations and terminology will be standard being in agreement with [4] and [5].

2. α-MODULES

These α-modules were originally defined and carefully explored in [13]. Recall that a *QTAG*-module M is an α-module, where α is a limit ordinal, if $M/H_\beta(M)$ is totally projective for every ordinal $\beta < \alpha$. But there are some other closely related concepts which have been of interest: recall that a submodule $B \subseteq M$ is an α-basic submodule of an α-module M if B is totally projective of length at most α, B is α-pure submodule of M, and M/B is h-divisible; while a fully invariant submodule L of the α-module M is α-large if $M = B + L$, for all α-basic submodules B of M. A few recurring relationships between them, and certain related assertions in this direction were established in [6]. The same type of study was continued in [7] and a number of results have been obtained in terms of α-basic submodules and α-large submodules. Some new achievements in this theme for other important sorts of α-modules were established in [15]. The motivation for writing the present article is to promote in this direction some new concepts and to explore some structural consequences in the light of generalized submodules.

We start with the following subsection.

2.1. *α-large submodules*

The study of α-large submodules and its fascinating properties makes the theory of α-modules more interesting. We summarize only for information a few standard properties of this notion in the following: If L is an α-large submodule of M and $\beta <$ length of $L/H_\alpha(M)$, then $H_\beta(L)$ is α-large submodule of M. Likewise, an α-large submodule $L \subseteq M$ is totally projective, if M is totally projective.

Mimicking [12], for a sequence $n(L) = (n_0, n_1, n_2, \ldots)$ of non-negative, non-decreasing integers we may consider $L = \{x : x \in M, U_M(x) \geq n(L)\}$ as the submodule of M. This submodule is a large submodule of M. Since for any endomorphism f of M, $H_M(x) \leq H_M(f(x))$, L is fully invariant. Therefore with every large submodule L of M we may associate a sequence $n(L)$. In fact for every large submodule there is a sequence and, for every sequence, there is a large submodule.

If n is replaced by an arbitrary ordinal less than or equal to σ, then $n(L)$ may be extended to $\sigma(L)$, and all the definitions and results which hold for $n(L)$ may be extended for $\sigma(L)$. In $\sigma(L)$, for any large submodule L of M, the sequence of L as $\sigma(L)$ denoted by M^{σ}. It is fairly easy to see that M^{σ} is a fully invariant submodule of M.

We first give the following strengthening concept.

Definition 1. Let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be an increasing sequence of ordinals and symbol ∞; that is, for each n, either σ_n is an ordinal or $\sigma_n < \infty$ and $\infty < \infty$. With each such sequence σ we associate the fully invariant submodule M^{σ} of the *QTAG*-module M as

$$M^{\sigma} = \{x \in M : x \in H_{\sigma_n - n}(M), \forall\, n \in \mathbb{N}\}$$

Now we prove the following lemma.

Lemma 1. *Let α be an ordinal cofinal with ω and M a QTAG-module with a fully invariant submodule F such that $H_{\alpha}(M) = H_{\alpha}(F)$ and let $a \in Soc^n(M)$ for some natural n. Then there exists $b \in F$ such that $U_M(a-b) = U_M(a)$ and $(a-b)R \cap H_{\alpha}(M) = 0$.*

Proof. Let $a \in Soc^n(M)$ be any uniform element of exponent n and let $U_M(a) = (\sigma_0, \sigma_1 \ldots)$. Now $\sigma_{n+i} = \infty$ $i = 0,1,\ldots.$ If all $\sigma_j = \infty$, $j = 0,1,\ldots$, then $a \in H_{\alpha}(M)$ and since $H_{\alpha}(M) = H_{\alpha}(F)$, we get $a = b$. Similarly if $\sigma_j \neq \infty$, $0 \leq j \leq n$, then $aR \cap H_{\alpha}(M) = 0$, and we get $b = 0$. Thus the lemma is true for the trivial cases. Now suppose that $\sigma_k = \infty$ with $0 < k < n$, and k is the smallest index for which $\sigma_k = \infty$. Let $a' \in H_{\alpha}(M)$, $c \in F$ such that $a' = c'$ where $d\left(\dfrac{aR}{a'R}\right) = k$, $d\left(\dfrac{cR}{c'R}\right) = m+1$ and $m > \sigma_{k-1}$. Then $b = c'$ where $d\left(\dfrac{cR}{bR}\right) = m-k+1$. In this case $H_M(b') \geq m-k+i+1 > \sigma_i$ where $d\left(\dfrac{bR}{b'R}\right) = i$ and $0 \leq i \leq k$. This shows that $H_M(H_i((a-b)R)) = H_M(a') = \sigma_i$ where $d\left(\dfrac{aR}{a'R}\right) = i$ and $0 \leq i \leq k$. Therefore, $H_M(H_i((a-b)R)) = \infty$, for $i \geq k$, and we are done. □

We continue in this way by the following.

Lemma 2. *Let L be an α-large submodule of an α-module M, where α is cofinal with ω. Then $Soc^n(L)$ is a large submodule of $Soc^n(M)$ for some natural n.*

Proof. Let $a \in Soc^n(L)$ and let ψ be an endomorphism of $Soc^n(M)$. Then $H_{\alpha}(M) = H_{\alpha}(L)$ since $H_n(L)$ is α-large for every n and $H_{\alpha}(M)$ is contained in every α-large submodule of M. Thus by Lemma 1, one can easily deduce that $(a-b)R \cap H_{\alpha}(M) = 0 = (\psi(a)-c)R$, and $U_M(a-b) = U_M(a) \leq U_M(\psi(a)) = U_M(\psi(a)-c)$ for all $b, c \in L$. We are now in the hypothesis of [7, Lemma 2.3], there exists an endomorphism ϕ of M such that $\phi(a-b) = \psi(a)-c$. Therefore, $a-b \in L$ and L is fully invariant in M. Hence $\psi(a)-c \in L$, and it follows that $\psi(a) \in L \cap Soc^n(M) = Soc^n(L)$. The proof is over. □

And so, we prepare to prove the following.

Proposition 1. *Let L be an α-large submodule of an α-module M, where α is cofinal with ω. Then $L = M^{\sigma}$ where $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of ordinals.*

Proof. Clearly, $Soc^n(L)$ is a large submodule of $Soc^n(M)$ by Lemma 2. Since owing to the criterion from [7], there exists a strictly increasing sequence of ordinals such that $Soc^n(L) = Soc^n(M^{\sigma})$. Now $L + Soc^n(M) = M$ and $L/Soc^n(L)$ is h-divisible. Therefore, $U_M(x) \geq \sigma$, for every $x \in L$, and $L \subset M^{\sigma}$. But $Soc^n(M^{\sigma}) = M^{\sigma} \cap Soc^n(L)$. Furthermore, $M/Soc^n(L) = L/Soc^n(L) \oplus Soc^n(M)/Soc^n(L)$, therefore $M^{\sigma}/Soc^n(L) = L/Soc^n(L)$, and $L = M^{\sigma}$. □

The next corollary is a valuable consequence of the above proposition.

Corollary 1. *Let M be an α-module, where α is cofinal with ω. Then M^{σ} is an α-large submodule of M if and only if $M = M^{\sigma} + Soc^n(M)$, where $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ and $\sigma_n < \infty$ for each $n \in \mathbb{N}$.*

We also need the following simple but however useful

Definition 2. Let $\mu = \{\mu_n\}_{n\in\mathbb{N}}$ be an increasing sequence of ordinals and symbol ∞. We say that μ is larger than σ almost everywhere if there exists a non-negative integer n such that $\mu_i \geq \sigma_i$, for all $i > n$. With each such sequence σ we associate the fully invariant submodule $\dot{M}^{\sigma}$ of the $QTAG$-module M containing $Soc^n(M) + M^{\sigma}$ as

$$\dot{M}^{\sigma} = \left\{ x \in M : x' \in M^{\sigma},\ d\left(\frac{xR}{x'R}\right) = n \text{ for some } n \in \mathbb{N} \right\}$$

We are now in a position to proceed by proving the following theorem.

Theorem 1. *Let L be a submodule of an α-module M, where α is cofinal with ω. Then L is an α-large submodule of M if and only if $L = M^{\sigma}$ and $M = \dot{M}^{\sigma}$ for some strictly increasing sequence of ordinals.*

Proof. Let L be an α-large submodule of M. By Proposition 1, there exists a sequence σ such that $L = M^{\sigma}$. But, by virtue of Corollary 1, $\dot{M}^{\sigma} \supset Soc^n(M) + M^{\sigma} = M$, so that with Definition 2 at hand we are done.

Conversely, suppose $L = M^{\sigma}$ and $M = \dot{M}^{\sigma}$. Let x be any uniform element in M, then $U_M(x) \geq \sigma$, for some increasing sequence μ of ordinals with $\mu_i \geq \sigma_i$, for all $i > n$ and for each $n \in \mathbb{N}$. Therefore, $H_M(x') \geq \sigma_i$ where $d\left(\frac{xR}{x'R}\right) = i$. Now we choose $y \in M$ such that $x' = y'$ where $d\left(\frac{xR}{x'R}\right) = n+1$, $d\left(\frac{yR}{y'R}\right) = r$, and $r \geq \sigma_{n+1}$. Therefore there exists an element $z \in M$ such that $z = y'$ where $d\left(\frac{yR}{y'R}\right) = r - n - 1$.

Of course we claim that $z \in M^{\sigma}$. Indeed, $z' = x'$ where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{zR}{z'R}\right) = i$ for $i > n$. Thus, $H_M(z') = H_M(x') \geq \sigma_i$ where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{zR}{z'R}\right) = i$. But $H_M(z') \geq r - n - 1 + i \geq \sigma_{n+1} - (n+1) + i$ where $d\left(\frac{zR}{z'R}\right) = i$ for $0 \leq i \leq n$. Now, σ being a strictly increasing sequence of ordinals, we have $\sigma_{j+1} \geq \sigma_j + 1$, $j = 0, 1, \ldots$. Thus, by continuing the same process, we obtain $\sigma_{n+1} \geq \sigma_i + n - i + 1$ for $i \leq j \leq n$. Hence $H_M(z') \geq \sigma_i$ where $d\left(\frac{zR}{z'R}\right) = i$. It follows that $z \in M^{\sigma}$, $x - z \in Soc^n(M)$, and $x = z + (x - z) \in M^{\sigma} + Soc^n(M)$. This shows that $M = M^{\sigma} + Soc^n(M)$. Thus, by what we have just seen above, in view of the Corollary 1, $L = M^{\sigma}$ is an α-large in M, as desired. The proof of the theorem is completed. □

As immediate consequence, we yield the following.

Corollary 2. *If N is an α-pure submodule of an α-module M and L is an α-large submodule of M, then $N \cap L$ is an α-large submodule of N, where α is cofinal with ω.*

The next proposition shed some light about the relationships between generalized submodules.

Proposition 2. *Let B be an α-basic submodule of an α-module M and let L be an α-large submodule of M, where α is cofinal with ω. Then $B \cap L$ is an α-basic submodule of M and M/L is a direct sum of uniserial modules.*

Proof. Choose $B \cap L = B \cap Soc^n(L)$, for some n. Since $Soc^n(L)$ is large in $Soc^n(M)$, $B \cap L$ is an α-basic submodule of $Soc^n(L)$. But $Soc^n(L)$ is h-pure in L and $L/Soc^n(L)$ is h-divisible. Therefore $B \cap L$ is an α-basic submodule of L. Now $M/L \cong B/(B \cap L)$ and by Corollary 2, $B \cap L$ is fully invariant in B. Henceforth, it follows that $B/(B \cap L)$ is a direct sum of uniserial modules. □

We thus deduce the following statement.

Proposition 3. *Let L_1 be an α-large submodule of an α-module M and let L_2 be an α-large submodule of L_1, where α is cofinal with ω. Then L_2 is an α-large submodule of M.*

2.2. *α-pure fully invariant submodules.*

The notion of a fully invariant submodule of a module is, of course, a classical notion in the theory of *QTAG*-modules. There are numerous observations of fully invariant submodules more enlightening than the definition: If F is a fully invariant submodule of an α-module M of length α, and if α is an ordinal cofinal with ω, then M/F is also an α-module. In addition, if F is an unbounded of length μ, then M/F is a totally projective module, and F is a μ-module. Moreover, if F is an unbounded, fully invariant submodule of an α-module M, and if α is an ordinal cofinal with ω, then F is totally projective module only if M is totally projective module.

However, we now have the following.

Definition 3. Let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of ordinals. We say that σ has a gap at n if $\sigma_n + 1 < \sigma_{n+1}$ and finitely many gaps if there exists n such that $\sigma_n + i < \sigma_{n+i}$ for every $i = 1,2,\ldots$. If no such n exists we say that σ has infinitely many gaps. Now σ satisfies the gap condition for a *QTAG*-module M if the *Ulm*-invariant of M corresponding to σ_n is non-zero.

Now, we proceed by proving

Lemma 3. *Let α be an ordinal cofinal with ω and let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a sequence of ordinals. If M is an α-module and σ has finitely many gaps, then $M^\sigma \supset H_t(M)$ for some $t \geq 0$.*

Proof. By hypothesis, there exists n such that $\sigma_n + i = \sigma_{n+i}$, $i = 1,2,\ldots$. Let $t = \sigma_n$ and suppose $x \in H_t(M)$. Then $x = y'$ where $d\left(\frac{yR}{y'R}\right) = t$ for some $y \in M$. Thus, $H_M(x') \geq \sigma_n + i = \sigma_{n+i} \geq \sigma_i$ where $d\left(\frac{xR}{x'R}\right) = i$, for every $i = 0,1,\ldots$. Therefore $x \in M^\sigma$ and $H_t(M) \subset M^\sigma$, □

Definition 4. For a sequence σ and an element x of a *QTAG*-module M, we let

$$\Gamma_x^\sigma = \left\{ i : H_M(x') < \sigma_i,\ d\left(\frac{xR}{x'R}\right) = i \right\}.$$

The careful reader will observe that Theorem 1 can be restated in terms of Γ_x^σ; namely

Remark 1. For an α-module M and a sequence σ, M^σ is α-large in M if and only if Γ_x^σ is finite for every $x \in M$.

So, we come to the following lemma.

Lemma 4. *Let M be an α-module, where α is cofinal with ω. If the closure $\overline{M}$ of M is an unbounded direct sum of uniserial modules and suppose $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ is a strictly increasing sequence of ordinals. Then there exists $x \in \overline{M}$ such that Γ_x^ν is infinite.*

Proof. Write $M = \bigoplus_{i=1}^{\infty} M_i$ where M_i is a direct sum of uniserial modules of length i. Let t_1 be an integer such that $\sigma_{t_1} + 1 < \sigma_{t_1+1}$ and choose $r_1 \geq \sigma_{t_1} + 2$ such that $M_{r_1} \neq 0$. Let $x_{r_1} \in M_{r_1}$ be any uniform element such that $H_{M_{r_1}}(x_{r_1}) = \sigma_{t_1} - t_1$. Now we repeat this operation for an integer t_2, we get $t_2 \geq e(x_{r_1})$, $\sigma_{t_2} + 1 < \sigma_{t_2+1}$ and $r_2 \geq \sigma_{t_2} + 2$ with $M_{r_2} \neq 0$ and choose $x_{r_2} \in M_{r_2}$ be any uniform element such that $H_{M_{r_2}}(x_{r_2}) = \sigma_{t_2} - t_2$.

Thus, by continuing the same process, we obtain a sequence x_{r_i} of elements such that $H_{M_{r_i}}(x_{r_i}) < H_{M_{r_{(i+1)}}}(x_{r_{(i+1)}}) < \dots$ and $t_1+2 \leq e(x_{r_1}) \leq t_2 < t_2+2 \leq e(x_{r_2}) \leq \dots$. Now let $y = x_j \in \prod_{j=1}^{\infty} M_j$ where $x_j = 0$ unless $j = ti$ for some i, in which case $x_j = x_t i$. Since $H_{M_t}(x_t)$ increases as t increases, and it follows that $y \in \overline{M}$. Furthermore, Γ_x^y is infinite. Consequently, $H_M(y') = H_M(x'_{r_i})$ such that $d\left(\frac{x_{r_i}R}{x'_{r_i}R}\right) = d\left(\frac{yR}{y'R}\right) = ti$ and $H_M(y') = \sigma_{ti}+1 < \sigma_{ti+1}$ such that $d\left(\frac{yR}{y'R}\right) = ti+1$ for all $i = 1,2,\dots$. This ends the proof. □

Summarizing the above corresponding assertions, we immediately deduce

Proposition 4. *$\overline{M}^{\sigma}$ is α-large in $\overline{M}$ if and only if σ has finitely many gaps.*

Definition 5. Let M be a *QTAG*-module and $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is an increasing sequence of ordinals and symbol ∞. With each such sequence λ we associate the α-pure fully invariant submodule $\overline{M}^{(x)}$ of the closure $\overline{M}$ of M containing x as

$$\overline{M}^{(x)} = \left\{ y \in \overline{M} : y' \in H_{\lambda_n}(M),\ d\left(\frac{yR}{y'R}\right) = n,\ \text{for some } n \geq 0 \right\}$$

So, we shall now prove the following proposition.

Proposition 5. *Let α be an ordinal cofinal with ω and let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of ordinals with infinitely many gaps. If M is an α-module, then there exists an α-pure fully invariant submodule F of the closure $\overline{M}$ of M such that F^{σ} is α-large in F.*

Proof. Let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a given sequence. Now we construct the another sequence $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ form σ such that $\lambda_n = \sigma_{2n}$, for every $n = 0,1,\dots$. The sequence λ has a gap at each $i = 0,1,\dots$. Choose $M = \oplus M_n$ such that $M_{\lambda_n+1} \neq 0$ for every n. Then there exists $x \in \overline{M}$ such that $\lambda = H_{\overline{M}}(x)$.

We set $F = \overline{M}^{(x)}$. It is not difficult to show that Γ_y^{σ} is finite for every $y \in F$. Since $y' \in H_{\overline{M}}(x)$, where $d\left(\frac{yR}{y'R}\right) = t$, there exists an endomorphism of $\overline{M}$ which maps x onto y', and hence $H_F(y') \geq H_{\overline{M}}(x') = \sigma_{2i}$ where $d\left(\frac{yR}{y'R}\right) = t+i$ and $d\left(\frac{xR}{x'R}\right) = i$. But $H_F(y') = \lambda_{t+i}$, where $d\left(\frac{yR}{y'R}\right) = t+i$. Therefore $\lambda_{t+i} \geq \sigma_{2i} \geq \sigma_{t+i}$ for all $i \geq t$ and Γ_y^{σ} is finite. Thus, by Remark 1, F^{σ} is α-large in F. □

The following necessary and sufficient condition is of some interest.

Lemma 5. *Let $F = \overline{M}^{(x)}$ be an α-pure fully invariant submodule of the α-module M, where α is cofinal with ω, and let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of ordinals with infinitely many gaps satisfying the gap condition with respect to the closure $\overline{M}$ of M. Then F^{σ} is α-large in F if and only if for every $k \geq 0$, there exists $j_k \geq 0$, such that $\lambda_i \geq \sigma_{i+k}$, whenever $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ and $i \geq j_k$.*

Proof. "Necessity". We know that Γ_x^{σ} is finite for every $x \in F$. Thus $t = 0$ there exists j_0 such that $i > j_0$ implies $\lambda_i \geq \sigma_i$. Now $H_t(F^{\sigma})$ is also α-large in F. Define $\gamma = (\gamma_i)$ where $\gamma_i = \sigma_{i+k}$. We claim that $H_k(F^{\sigma}) = F^{\gamma}$. Clearly $H_k(F^{\sigma}) \subset F^{\gamma}$. Indeed, $F^{\sigma} = F \cap \overline{M}^{\sigma}$ since F is α-pure in $\overline{M}$.

After this, since σ satisfies the gap condition, there exists $a \in \overline{M}$ such that $\overline{M}^{\sigma} = \{\phi(a) : \phi \in \mathrm{End}\overline{M}\}$ and $H_{\overline{M}}(a) = \sigma$. Thus $H_{\overline{M}}(a') = \gamma$ such that $d\left(\frac{aR}{a'R}\right) = k$, and $\overline{M}^{\gamma} = H_k(\overline{M}^{\sigma})$.

Observe that

$$F^{\gamma} = F \cap \overline{M}^{\gamma} = F \cap H_k(\overline{M}^{\sigma}) \subset H_k(F) \cap H_k(\overline{M}^{\sigma}) \subset H_k(F \cap \overline{M}^{\sigma}) = H_k(F^{\sigma}).$$

This substantiates our claim.

"Sufficiency". We need only show that Γ_x^{σ} is finite for every $x \in F$. Now, there exists k such that $x' = \overline{M}^{\lambda}$ where $d\left(\frac{xR}{x'R}\right) = k$. Therefore, $H_F(x') \geq \lambda_i$ such that $d\left(\frac{xR}{x'R}\right) = i + k$, and for $i \geq j_k$ such that $\lambda_i \geq \sigma_{i+k}$. Hence Γ_x^{σ} is finite for every $x \in F$ and it follows that F^{σ} is α-large in F. □

ACKNOWLEDGMENTS

The author is very grateful to the referee for careful reading of the present manuscript, and to the Editor, for the professional editorial work.

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