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BOUNDARY VALUE PROBLEMS OF HIGHER ORDER FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS INVOLVING GRONWALL'S INEQUALITY IN BANACH SPACES

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Abstract. Using the Caputo derivative of order $q \in (\alpha - 1, \alpha)$, we examine boundary value problems for fractional integro-differential equations in Banach spaces. Using an a priori estimate technique, the Holder's inequality, a suitable singular Gronwall's inequality, and the fixed point theorem are utilised to prove the existence and uniqueness of solutions.

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1. INTRODUCTION

When representing natural events, differential and integro-differential equations and inclusions are more realistic, and they can be found in a variety of applications (refer[10]-[9], and the cited references). ODEs and integration to arbitrary non integer order are generalised to fractional differential equations(FDEs).

Differential equations with fractional order derivatives have recently proven to be effective tools for modelling a wide range of physical phenomena as well as in a variety of scientific and technical domains. In ordinary and partial FDEs with fractional order, there has been a lot of advancement in recent years; look into the monographs of Abbas *et al.*[2], Baleanu *et al.* [8], the papers by Abbas *et al.* (refer [1]-[3]). Many applications exist in control, electrochemistry, porous media, electromagnetism, viscoelastic, and other fields (see [15]-[21]).

Miller and Ross have a comprehensive bibliography on this subject. [18]. As a result, the theory of FDEs has been extensively developed. The monographs of Kilbas et al. [16], Lakshmikantham et al. [17]. Particularly, Agarwal et al. [4] provide necessary criteria for the existence and uniqueness of solutions for many classes of starting and boundary value problems utilising the Caputo fractional derivative for FDEs and inclusions in $\mathbb{R}$. Some FDEs and optimum controllers in abstract Banach spaces have recently been investigated by Balachandran et al. [6, 7], Dong et al.[12], El-Borai [13], Henderson and Ouahab [14], Hernández [15], Wang et al. ([20]-[21]), and Zhou et al. ([23]-[24]). Chalishajar and Karthikeyan [11] have recently proved the existence of impulsive fractional order integro-differential equations in Banach spaces using a combination of generalised Grownwall's inequality, Caputo derivative, and the fixed point approach. The writers have looked at an abstract boundary condition in this paper. In this paper, we use the generalised boundary condition to generalise the conclusion found in [11].

The remainder of this work is laid out as follows: The problem is explicitly defined with motivations in Section 2. We make some notes and recall certain concepts and findings from the preparation. Theorems 2 and 3 are presented in Section 3, with the first based on the Banach contraction principle and the second based on Schaefer's fixed point theorem.

2. Preliminaries

The goal of this paper is to build on previous research [5,22] on FBVPs, for FDEs in $\mathbb{R}$ to U is an abstract Banach space of type

$$\left\{\begin{array}{rcl} {}^{c}D^{q}z(v) & = & \phi(v,z(v),(Az)(v)),\ v\in I=[0,T],\ q\in(\alpha-1,\alpha),\\ z(0) & = & z_0,\ z'(0)=z_0^1,\ z''(0)=z_0^2,\cdots,\ z^{(\alpha-1)}(0)=z_0^{\alpha-2},\\ z^{(\alpha-1)}(T) & = & z_T \end{array}\right. \tag{2.1}$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order q, $\phi: I\times U\times U\to U$ is a given function satisfying F(z) a few assumptions to be detailed later, and z_0, z_0^j $(j=1,2,\cdots,\alpha-2,\alpha\geq 4,\alpha$ is a integer), z_T are some elements of U and S is a nonlinear integral operator given by

$$(Az)(v)=\int_0^v \psi(v,s)z(s)ds,$$

with $\varphi_0=\max\{\int_0^v \psi(v,s)ds:(v,s)\in I\times I\}$ where $\psi\in C(I\times I,\mathbb{R}^+)$.

Some existence and uniqueness results for the fractional BVP (2.1) are shown using Holder's inequality, a suitable singular Gronwall's inequality, and the fixed point method. There are at least three discrepancies between the preceding results obtained in [5]: To establish the priori bounds, (i) the work space is not $\mathbb{R}$ but the abstract Banach space X; (ii) f is not necessarily jointly continuous and fulfils some weaker constraints; (iii) another singular Gronwall's inequality is presented.

The study of such problems has gotten a lot of attention in recent years, both theoretically and practically. The following recent works on this topic will be mentioned in [7, 14], without making an attempt to be exhaustive The following conditions are frequently used by authors: for the existence of solutions, the nonlinear term ϕ must satisfy F(z) that there exist functions.$u_1,u_2\in C([0,1],[0,\infty))$ as a result $1\geq v\geq 0$ and each $x\in\mathbb{R}$,

$$|\phi(v,x)|\leq u_1(v)|x|+u_2(v),$$

In terms of uniqueness, they believe that the nonlinear term ϕ satisfies the requirement that there exist functions. $u_1, u_2 \in C([0,1],[0,\infty))$ as if for each $1 \geq v \geq 0$ and any $x, z \in \mathbb{R}$,

$$|\phi(v,x) - \phi(v,y)| \leq u_1(v)|x-y|.$$

Now we'll go over the notation, terminology, and preliminary findings that will be used throughout the work. We denote $C(I,U)$ all continuous functions from the Banach space$I$ into U with the norm $\|z\|_\infty := \sup\{\|z(v)\| : v \in I\}$. For measurable functions $\mu : I \to \mathbb{R}$, define the norm $\|\mu\|_{L^p(I,\mathbb{R})} = \left(\int_I |\mu(v)|^p dt\right)^{\frac{1}{p}}$, $1 \leq p < \infty$. We denote $L^p(I,\mathbb{R})$ the Banach space of all Lebesgue measurable functions μ with $\|\mu\|_{L^p(I,\ \mathbb{R})} < \infty$.

To follow the contents of this paper, we'll require the following essential definitions and properties of fractional calculus theory. For more information, see, for instance, [16].

Definition 1. ([19]) The Caputo fractional order derivative of order q of h for a suitable function h given on the interval $[c,d]$ is defined by

$$({}^cD^q_{a+}h)(v) = \frac{1}{\Gamma(\alpha - q)} \int_a^v (v-s)^{\alpha-q-1} h^{(\alpha)}(s)ds,$$

Lemma 1. *Let $q > 0$, then the differential equation ${}^cD^q h(v) = 0$ has the following general solution*

$$h(v) = d_0 + d_1 v + d_2 v^2 + \cdots + d_{\alpha-1} v^{\alpha-1},$$

where $d_j \in \mathbb{R}$, $j = 0,1,2,\cdots,\alpha-1,\alpha$, with $\alpha = -[-q]$.

Lemma 2. *Let $q > 0$, then $I^q({}^cD^q h)(v) = h(v) + d_0 + d_1 v + d_2 v^2 + \cdots + d_{\alpha-1} v^{\alpha-1}$, for some $d_j \in \mathbb{R}$, $j = 0,1,2,\cdots,\alpha-1$, $\alpha = -[-q]$.*

We introduce the concept of a solution of the fractional BVP (2.1), which is similar to Definition 3.7 in [4].

Definition 2. A function $z \in C(I,U)$ with its q-derivative existing on I is said to be a solution of the fractional BVP (2.1) if y satisfies the equation ${}^cD^q z(v) = \phi(v, z(v), (Az)(v))$ a.e. on I, and the conditions $z(0) = z_0$, $z'(0) = z_0^1$, $z''(0) = z_0^2$, $\cdots$, $z^{(\alpha-1)}(0) = z_0^{\alpha-2}$, $z^{(\alpha-1)}(T) = z_T$.

The following auxiliary lemma is required for the existence of solutions for the fractional BVP (2.1).

Lemma 3. *Let $\bar{\phi} : I \to U$ be continuous. A function $z \in C(I,U)$ is a solution of the fractional integral equation*

$$z(v) = \frac{1}{\Gamma(q)} \int_0^v (v-s)^{q-1} \bar{\phi}(s)ds$$

$$-\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha}\bar{\phi}(s)ds$$
$$+z_0+z_0^1 v+\frac{z_0^2}{2!}v^2+\cdots+\frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2}+\frac{z_T}{(\alpha-1)!}v^{\alpha-1},$$

if and only if z is a solution of the following fractional BVP

$$\begin{cases} {}^cD^q z(v) &= \bar{\phi}(v),\ v\in I=[0,T],\ q\in(\alpha-1,\alpha), \\ z(0) &= z_0,\ z'(0)=z_0^1,\ z''(0)=z_0^2,\cdots,\ z^{(\alpha-1)}(0)=z_0^{\alpha-2}, \\ z^{(\alpha-1)}(T) &= z_T. \end{cases} \tag{2.2}$$

Proof. The proof can be completed quickly using Lemma 3.8 in [4] and mathematical induction. □

We get the following result as a result of Lemma 3, which is relevant in the next section.

Lemma 4. *Let $\phi: I\times U\times U\to U$ be continuous function $z\in C(I,U)$ is a solution of the fractional integral equation*

$$z(v)=\frac{1}{\Gamma(q)}\int_0^v (v-s)^{q-1}\phi(s,z(s),(Sz)(s))ds$$
$$-\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))ds$$
$$+z_0+z_0^1 v+\frac{z_0^2}{2!}v^2+\cdots+\frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2}+\frac{z_T}{(\alpha-1)!}v^{\alpha-1},$$

if and only if z is a solution of the fractional BVP (2.1)

Lemma 5. (Bochner theorem) *A measurable function ϕ: $I\to U$ is Bochner integrable if $\|\phi\|$ is Lebesgue integrable.*

Lemma 6. (Mazur lemma) *If $\mathcal{U}$ is a compact subset of U, then its convex closure $\overline{conv}\ \mathcal{U}$ is compact.*

Lemma 7. (Ascoli-Arzela theorem) *Let $\mathcal{S}=\{s(v)\}$ is a function family of continuous mappings $s:[c,d]\to X$. If $\mathcal{S}$ is uniformly bounded and equicontinuous, and for any $v^*\in[c,d]$, the set $\{s(v^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_\alpha(v)\}$*
$(\alpha=1,2,\cdots,v\in[c,d])$ in $\mathcal{S}$.

Theorem 1. (Schaefer's fixed point theorem) *Let $F:U\to U$ completely continuous operator. If the set $E(F)=\{z\in U: z=\rho^*F(z)\ for\ some\ \rho^*\in[0,1]\}$ is bounded, then F has fixed points.*

3. MAIN RESULTS

We introduce the following hypotheses, before stating and proving the main results. Our first result is based on the principle of Banach contraction. The following assumptions are made:

(H1) The function $\phi : I \times U \times U \to U$ is measurable with respect to v on I.

(H2) There exists a constant $q_1 \in (0, q-\alpha+1)$ and real-valued functions $m_1(v)$, $m_2(v) \in L^{\frac{1}{q_1}}(I,U)$ such that

$$\|\phi(v,z(v),(Sz)(v)) - \phi(v,z(v),(Az)(v))\| \le m_1(v)\|y-z\| + m_2(v)\|Sy-Sz\|,$$

for each $v \in I$, and all $y, z \in U$.

(H3) There exists a constant $q_2 \in (0, q-\alpha+1)$ and real-valued function $h(v) \in L^{\frac{1}{q_2}}(I,U)$ such that $\|\phi(v,z,Sz)\| \le h(v)$, for each $v \in I$, and all $z \in U$.

For brevity, let $M = \|m_1 + \gamma_0 m_2\|_{L^{\frac{1}{q_1}}(I,U)}$ and $H = \|h\|_{L^{\frac{1}{q_2}}(I,U)}$.

Our second finding is based on the well known Schaefer's fixed point theorem. The following assumptions are made:

(H4) There exist constants $\rho \in [0, 1-\frac{1}{p})$ for some $1 < p < \frac{1}{\alpha-q}$ and $N > 0$ such that

$$\|\phi(v,x,Sx)\| \le N(1+\varphi_0\|x\|^{\rho}) \; for\ each\ v \in I\ and\ all\ x \in U.$$

(H5) For every $v \in I$, the set

$$K_1 = \left\{(v-s)^{q-1}\phi(s,z(s),(Sz)(s)) : z \in C(I,U),\ s \in [0,v]\right\}$$

and

$$K_2 = \left\{(v-s)^{q-\alpha}\phi(s,z(s),(Sz)(s)) : z \in C(I,X),\ s \in [0,v]\right\}$$

are relatively compact.

Theorem 2. *Assume that* (H1)-(H3) *hold. If*

$$\Phi_{q,T,\alpha} = \frac{M}{\Gamma(q)} \frac{T^{q-q_1}}{(\frac{q-q_1}{1-q_1})^{1-q_1}} + \frac{M}{(\alpha-1)!\Gamma(q-\alpha+1)} \frac{T^{q-q_1}}{(\frac{q-q_1-\alpha+1}{1-q_1})^{1-q_1}} < 1, \tag{3.1}$$

then the fractional BVP (2.1) *has a unique solution on* I.

Proof. For each $v \in I$, we have

$$\begin{aligned}
\int_0^v \left\|(v-s)^{q-1}\phi(s,z(s),(Sz)(s))\right\| ds &\le \left(\int_0^v (v-s)^{\frac{q-1}{1-q_2}} ds\right)^{1-q_2} \left(\int_0^v (h(s))^{\frac{1}{q_2}} ds\right)^{q_2} \\
&\le \left(\int_0^v (v-s)^{\frac{q-1}{1-q_2}} ds\right)^{1-q_2} \left(\int_0^T (h(s))^{\frac{1}{q_2}} ds\right)^{q_2} \\
&\le \frac{T^{q-q_2} H}{\left(\frac{q-q_2}{1-q_2}\right)^{1-q_2}}.
\end{aligned}$$

Thus $\|(v-s)^{q-1}\phi(s,z(s),(Sz)(s))\|$ is Lebesgue integrable with respect to $s\in[0,v]$ for all $v\in I$ and $z\in C(I,U)$. Then $(v-s)^{q-1}\phi(s,z(s),(Sz)(s))$ is Bochner integrable with respect to $s\in[0,v]$ for all $v\in I$ due to Lemma 5,

$$\begin{aligned}\int_0^T \left\|(v-s)^{q-1}\phi(s,z(s),(Sz)(s))\right\|ds &\leq \left(\int_0^T (T-s)^{\frac{q-\alpha}{1-q_2}}ds\right)^{1-q_2}\left(\int_0^T (h(s))^{\frac{1}{q_2}}ds\right)^{q_2}\\ &\leq \frac{T^{q-q_2-\alpha+1}H}{(\frac{q-q_2-\alpha+1}{1-q_2})^{1-q_2}}.\end{aligned}$$

Thus, $\|(T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))\|$ is Lebesgue integrable with respect to $s\in[0,T]$ for all $v\in I$ and $z\in C(I,X)$. Then $(T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))$ is Bochner integrable with respect to $s\in[0,T]$ for all $v\in I$ due to Lemma 5.

Hence, the FBVP (1) is equivalent to the following fractional integral equation

$$\begin{aligned}z(v) = {}&\frac{1}{\Gamma(q)}\int_0^v (v-s)^{q-1}\phi(s,z(s),(Sz)(s))ds\\ &-\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))ds\\ &+z_0+y_0^1v+\frac{z_0^2}{2!}v^2+\cdots+\frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2}+\frac{z_T}{(\alpha-1)!}v^{\alpha-1},\quad v\in I\end{aligned}$$

Let

$$\begin{aligned}r \geq {}&\frac{HT^{q-q_2}}{\Gamma(q)(\frac{q-q_2}{1-q_2})^{1-q_2}}+\frac{HT^{q-q_1}}{(\alpha-1)!\Gamma(q-\alpha+1)}(\frac{q-q_2-\alpha+1}{1-q_2})^{1-q_2}\\ &+\|z_0\|+\|z_0^1\|T+\frac{\|z_0^2\|}{2!}T^2+\cdots+\frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!}T^{\alpha-2}+\frac{\|y_T\|}{(\alpha-1)!}T^{\alpha-1}.\end{aligned}$$

Now we define the operator $\aleph$ on $B_r:=\{z\in C(I,U):\|z\|\leq r\}$ as follows

$$\begin{aligned}(\aleph(z))(v) = {}&\frac{1}{\Gamma(q)}\int_0^v (v-s)^{q-1}\phi(s,z(s),(Sz)(s))ds\\ &-\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))ds\\ &+z_0+z_0^1v+\frac{v_0^2}{2!}t^2+\cdots+\frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2}+\frac{z_T}{(\alpha-1)!}v^{\alpha-1},\quad v\in I.\end{aligned}\tag{3.2}$$

As a result, the existence of a fractional BVP (2.1) solution equates to the existence of a fixed point for the operator $\aleph$ on Br. The Banach contraction principle will be used to show that $\aleph$ has a fixed point. There are two parts to the proof.

Part 1. $\aleph(z)\in B_r$ for every $y\in B_r$.

For every $z\in B_r$ and any $\theta>0$, by (H3) and Holder's inequality, we get

$$\|(\aleph(z))(v+\theta)-(\aleph(z))(v)\|$$

$$
\begin{aligned}
&\leq \Big\| \frac{1}{\Gamma(q)} \int_0^{v+\theta} (v+\theta-s)^{q-1} \phi(s,z(s),(Sz)(s))ds \\
&\quad - \frac{1}{\Gamma(q)} \int_0^{v} (v-s)^{q-1} \phi(s,z(s),(Sz)(s))ds \Big\| \\
&\quad + \Big\| \frac{(v+\theta)^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \phi(s,z(s),(Sz)(s))ds \\
&\quad + \frac{(v+\theta)^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \phi(s,z(s),(Sz)(s))ds \Big\| \\
&\quad + \Big\| z_0^1 (v+\theta-t) + \frac{y_0^2}{2!}[(t+\theta)^2 - v^2] + \dots \\
&\quad + \frac{z_0^{\alpha-2}}{(\alpha-2)!}[(v+\theta)^{\alpha-2} - v^{\alpha-2}] \\
&\quad + \frac{z_T}{(\alpha-1)!}[(v+\theta)^{\alpha-1} - v^{\alpha-1}] \Big\| \\
&\leq \frac{H}{\Gamma(q)} \left(\frac{(v+\theta)^{\frac{q-q_2}{1-q_2}}}{\frac{q-q_2}{1-q_2}} - \frac{\theta^{\frac{q-q_2}{1-q_2}}}{\frac{q-q_2}{1-q_2}} - \frac{v^{\frac{q-q_2}{1-q_2}}}{\frac{q-q_2}{1-q_2}} \right)^{1-q_2} \\
&\quad + \frac{H}{\Gamma(q)} \left(\frac{\theta^{\frac{q-q_2}{1-q_2}}}{\frac{q-q_2}{1-q_2}} \right)^{1-q_2} + \frac{[(v+\theta)^{\alpha-1} - v^{\alpha-1}]}{(\alpha-1)!\Gamma(q-\alpha+1)} \frac{T^{q-q_2-\alpha+1} H}{(\frac{q-q_2-\alpha+1}{1-q_2})^{1-q_2}} \\
&\quad + \|z_0^1\|(v+\theta-v) + \frac{\|z_0^2\|}{2!}[(v+\theta)^2 - v^2] + \dots \\
&\quad + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!}[(v+\theta)^{\alpha-2} - v^{\alpha-2}] \\
&\quad + \frac{zT\|}{(\alpha-1)!}[(v+\theta)^{\alpha-1} - v^{\alpha-1}].
\end{aligned}
$$

It is self-evident that the right-hand side of the inequality above tends to zero.$\theta \to 0$. Therefore, $\aleph$ is continuous on I, i.e., $\aleph(z) \in C(I,X)$. Moreover, for $z \in B_r$ and all $t \in I$, we get

$$
\begin{aligned}
\|(\aleph(z))(v)\| &\leq \frac{1}{\Gamma(q)} \int_0^{v} (v-s)^{q-1} \|\phi(s,z(s),(Sz)(s))\| ds \\
&\quad + \frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \|\phi(s,z(s),(Sz)(s))\| ds \\
&\quad + \|z_0\| + \|z_0^1\| T + \frac{\|z_0^2\|}{2!} T^2 + \dots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!} T^{\alpha-2} + \frac{\|z_T\|}{(\alpha-1)!} T^{\alpha-1} \\
&\leq r,
\end{aligned}
$$

which implies that $\|\aleph(z)\|_\infty \leq r$, Thus, we can conclude that for all $z \in B_r$, $\aleph(z) \in B_r$, i.e., $\aleph : B_r \to B_r$.

Step 2. F is a contraction mapping on B_r.

For $y, z \in B_r$ and any $v \in I$, using (H2) and Holder's inequality, we get

$$
\begin{aligned}
&\|(\aleph(z))(v) - (\aleph(z))(v)\| \\
&\quad \leq \frac{1}{\Gamma(q)} \int_0^v (v-s)^{q-1} \|\phi(s, x(s), (Sx)(s)) - \phi(s, z(s), (Sz)(s))\| ds \\
&\qquad + \frac{T^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \|\phi(s, y(s), (Sy)(s)) \\
&\qquad\quad - \phi(s, z(s), (Sz)(s))\| ds \\
&\quad \leq \frac{\|y-z\|_\infty}{\Gamma(q)} \left(\int_0^v (v-s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} \left(\int_0^v (m_1(s) + \gamma_0 m_2(s))^{\frac{1}{q_1}} ds \right)^{q_1} \\
&\qquad + \frac{[(v+\theta)^{\alpha-1} - v^{\alpha-1}]}{(\alpha-1)!\Gamma(q-\alpha+1)} \left(\int_0^T (T-s)^{\frac{q-\alpha}{1-q_1}} ds \right)^{1-q_1} \\
&\qquad\quad \times \left(\int_0^T (m_1(s) + \gamma_0 m_2(s))^{\frac{1}{q_1}} ds \right)^{q_1}
\end{aligned}
$$

So we obtain

$$\|\aleph(y) - \aleph(z)\|_\infty \leq \Phi_{q,T,\alpha} \|y-z\|_\infty.$$

Thus, $\aleph$ is contraction due to the condition (3.1). By Banach contraction principle, we can deduce that $\aleph$ has an unique fixed point which is the unique solution of the fractional BVP (2.1). □

Our second results is based on the well known Schaefer's fixed point theorem.

We adopt following assumptions:

Theorem 3. *Let* (H1), (H4), *and* (H5) *assumptions are satisfied. Then the fractional* BVP (1) *has at least one solution on* I.

Proof. Let us transform the FBVP (2.1) into a fixed point problem. Consider the operator $\aleph : C(I, U) \to C(I, U)$ defined as (.). Because of (3.2), Holder's inequality, and Lemma 5, it is clear that F is clearly defined.

We've divided the proof into multiple steps for ease of understanding.

Step 1. $\aleph$ is a continuous operator.

Let $\{z_\alpha\}$ be a sequence such that $z_\alpha \to z$ in $C(I, U)$. Then for each $v \in I$, using the continuity of f, we have

$$
\begin{aligned}
&\|\aleph(z)_\alpha - \aleph(z)\|_\infty \\
&\quad \leq \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^q}{(\alpha-1)!\Gamma(q-\alpha+2)} \right) \|\phi(\cdot, z_\alpha(\cdot), (Sz_\alpha)(\cdot)) \\
&\qquad - \phi(\cdot, z(\cdot), (Sz)(\cdot))\|_\infty \to 0
\end{aligned}
$$

as $\alpha \to \infty$.

Step 2. $\aleph$ maps bounded sets into bounded sets in $C(I,U)$.

Indeed, it is enough to show that for any $\xi^* > 0$, there exists a $l > 0$ such that for each $z \in B_{\xi^*} = \{z \in C(I,U) : \|z\|_\infty \le \xi^*\}$, we have $\|z\|_\infty \le l$.
For each $v \in I$, by (H4), we get

$$\|(\aleph(z))(v)\| \le \left(\frac{1}{\Gamma(q+1)} + \frac{1}{(\alpha-1)!\Gamma(q-\alpha+2)}\right)T^q\gamma_0 N(1+(\xi^*)^\rho)$$
$$+\|z_0\| + \|z_0^1\|T + \frac{\|z_0^2\|}{2!}T^2 + \cdots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!}T^{\alpha-2} + \frac{\|z_T\|}{(\alpha-1)!}T^{\alpha-1} := l$$

which implies that $\|\aleph(z)\|_\infty \le l$.

Step 3. $\aleph$ maps bounded sets into equicontinuous sets of $C(I,U)$.

Let $0 \le v_1 < v_2 \le T,\ z \in B_{\xi^*}$. Using (H4) again, we have

$$\|(\aleph(z))(v_2) - (\aleph(z))(v_1)\|$$
$$\le \frac{\gamma_0 N(1+(\xi^*)^\rho)}{\Gamma(q+1)}(v_2^q - v_1^q) + \frac{T^{q-\alpha+1}\gamma_0 N(1+(\xi^*)^\rho)}{(\alpha-1)!\Gamma(q-\alpha+2)}(v_2^2 - v_1^2)$$
$$+\|z_0^1\|(v_2 - v_1) + \frac{\|z_0^2\|}{2!}(v_2^2 - v_1^2) + \cdots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!}(v_2^{\alpha-2} - v_1^{\alpha-2})$$
$$+\frac{\|z_T\|}{(\alpha-1)!}(v_2^{\alpha-1} - v_1^{\alpha-1}).$$

As $v_2 \to v_1$, the right-hand side of the above inequality tends to zero, therefore $\aleph$ is equicontinuous.

Now, let $\{z_\alpha\},\ \alpha = 1,2,\cdots$ be a sequence on B_{ξ^*}, and

$$(\aleph(z)_\alpha)(v) = (\aleph_1 z_\alpha)(v) + (\aleph_2 z_\alpha)(v) + (\aleph_3 z)(v), \quad v \in I,$$

where

$$(\aleph_1 z_\alpha)(v) = \frac{1}{\Gamma(q)}\int_0^v (v-s)^{q-1}\phi(s, z_\alpha(s), (Sz_\alpha)(s))ds, \quad v \in I,$$
$$(\aleph_2 z_\alpha)(v) = -\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha} f(s, z_\alpha(s), (Sz_\alpha)(s))ds, \quad v \in I,$$
$$(\aleph_3 z)(v) = z_0 + z_0^1 v + \frac{z_0^2}{2!}v^2 + \cdots + \frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2} + \frac{z_T}{(\alpha-1)!}v^{\alpha-1}, \quad v \in I.$$

In view of the condition (H5) and Lemma 6, we known that $\overline{conv}\, K_1$ is compact. For any $v^* \in I$,

$$(\aleph_1 z_\alpha)(v^*) = \frac{1}{\Gamma(q)}\lim_{k\to\infty}\sum_{j=1}^{k}\frac{v^*}{k}\left(v^* - \frac{it^*}{k}\right)^{q-1}\phi\left(\frac{it^*}{k}, z_\alpha\left(\frac{it^*}{k}\right), (Sz_\alpha)\left(\frac{it^*}{k}\right)\right)$$

$$= \frac{z^*}{\Gamma(q)}\omega_{\alpha 1},$$

where $$\omega_{\alpha 1} = \lim_{k\to\infty}\sum_{j=1}^{k}\frac{1}{k}\left(v^* - \frac{it^*}{k}\right)^{q-1} f\left(\frac{it^*}{k}, z_\alpha\left(\frac{it^*}{k}\right), (Sz_\alpha)\left(\frac{it^*}{k}\right)\right).$$

Since $\overline{conv}\, K_1$ is convex and compact, we known that $\omega_{\alpha 1} \in \overline{conv}\, K_1$. Hence, for any $v^* \in I$, the set $\{(\aleph_1 z_\alpha)(v^*)\}$ is relatively compact. From Lemma 7, every $\{(\aleph_1 z_\alpha)(v)\}$ contains a uniformly convergent subsequence $\{(\aleph_1 z_{\alpha_k})(v)\},\ k = 1,2,\cdots$ on I. Thus, the set $\{\aleph_1 z : z \in B_{\xi^*}\}$ is relatively compact.

Set

$$(\overline{\aleph_2} z_\alpha)(v) = -\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^v (v-s)^{q-\alpha}\phi(s, z_\alpha(s), (Sz_\alpha)(s))ds, \quad v \in I.$$

For any $v^* \in I$,

$$\begin{aligned}(\overline{\aleph_2} z_\alpha)(v^*) &= -\frac{(v^*)^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \\ &\quad \times \lim_{k\to\infty}\sum_{j=1}^{k}\frac{v^*}{k}\left(v^* - \frac{it^*}{k}\right)^{q-\alpha}\phi\left(\frac{it^*}{k}, y_\alpha\left(\frac{it^*}{k}\right), (Sy_\alpha)\left(\frac{it^*}{k}\right)\right) \\ &= -\frac{(v^*)^{\alpha}}{(\alpha-1)!\Gamma(q-\alpha+1)}\omega_{\alpha 2},\end{aligned}$$

where

$$\omega_{\alpha 2} = \lim_{k\to\infty}\sum_{j=1}^{k}\frac{1}{k}\left(v^* - \frac{it^*}{k}\right)^{q-\alpha}\phi\left(\frac{it^*}{k}, z_\alpha\left(\frac{it^*}{k}\right), (Sz_\alpha)\left(\frac{it^*}{k}\right)\right).$$

Since $\overline{conv}\, K_2$ is convex and compact, we known that $\omega_{\alpha 2} \in \overline{conv}\, K_2$. Hence, for any $v^* \in I$, the set $\{(\overline{\aleph_2} z_\alpha)(v^*)\}$ is relatively compact. From Lemma 7, every $\{(\overline{\aleph_2} z_\alpha)(v)\}$ contains a uniformly convergent subsequence
$\{(\overline{\aleph_2} z_{\alpha_k})(v)\},\ k = 1,2,\cdots$ on I. Particularly, $\{(\aleph_2 z_\alpha)(v)\}$ contains a uniformly convergent subsequence $\{(\aleph_2 y_{\alpha_k})(v)\},\ k = 1,2,\cdots$ on I. Thus, the set $\{\aleph_2 z : z \in B_{\xi^*}\}$ is relatively compact.

Obviously, the set $\{\aleph_3 z : z \in B_{\xi^*}\}$ is relatively compact. As a result, the set $\{\aleph(z) : z \in B_{\xi^*}\}$ is relatively compact.

As a consequence of Step 1-3, we conclude that F is continuous and hence completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(\aleph) = \{z \in C(I,U) : z = \rho^* \aleph(z),\ \text{for some}\ \rho^* \in [0,1]\}$$

is bounded.

Let $z \in E(\aleph)$, then $z = \rho^* \aleph(z)$ for some $\rho^* \in [0,1]$. Thus, for each $v \in I$, we have

$$\begin{aligned} z(v) = \rho^* \Bigg(& \frac{1}{\Gamma(q)} \int_0^v (v-s)^{q-1} \phi(s, z_\alpha(s), (Sz_\alpha)(s)) ds \\ & - \frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \phi(s, z_\alpha(s), (Sz_\alpha)(s)) ds \\ & + z_0 + z_0^1 v + \frac{z_0^2}{2!} v^2 + \cdots + \frac{z_0^{\alpha-2}}{(\alpha-2)!} v^{\alpha-2} + \frac{z_T}{(\alpha-1)!} v^{\alpha-1} \Bigg). \end{aligned}$$

For each $v \in I$, we have

$$\begin{aligned} \|z(v)\| &\leq \|(\aleph(z))(v)\| \\ &\leq \frac{\gamma_0 N T^q}{\Gamma(q+1)} + \frac{\gamma_0 N T^q}{(\alpha-1)!\Gamma(q-\alpha+1)} + \|z_0\| + \|z_0^1\| T \\ &\quad + \frac{\|z_0^2\|}{2!} T^2 + \cdots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!} T^{\alpha-2} + \frac{\|z_T\|}{(\alpha-1)!} T^{\alpha-1} \\ &\quad + \frac{\gamma_0 N}{\Gamma(q)} \int_0^v (v-s)^{q-1} \|z(s)\|^\rho ds \\ &\quad + \frac{\gamma_0 N T^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \|z(s)\|^\rho ds. \end{aligned}$$

By Lemma 2.9 in [9], there exists a $M^* > 0$ such that $\|z(v)\| \leq M^*, \;\; v \in I$.

Thus for every $v \in I$, we have $\|z\|_\infty \leq M^*$.

This establishes the boundedness of the set $E(\aleph)$. We derive that $\aleph$ has a fixed point that is a solution of the fractional BVP as a result of Schaefer's fixed point theorem. □

4. EXAMPLES

We provide two examples in this part to demonstrate the utility of our main results.

Example 1. Let us consider the following FBVPs,

$$\begin{cases} {}^cD^q z(v) &= \frac{e^{-pv}|z(v)|}{(1+ke^v)(1+|z(v)|)} + \int_0^v \frac{e^{-pv}}{16} s \frac{|z(s)|}{1+|z(s)|} ds, \\ & \qquad y \in [0,1],\ v \in I = [0,T],\ q \in (3,4),\ k > 0, \\ z(0,y) &= 0,\ z'(0,y) = 0,\ z''(0,y) = 0\ z'''(T,y) = 0,\ \ y \in [0,1], \\ z(v,0) &= z(v,1) = 0 \quad v > 0. \end{cases} \tag{4.1}$$

where $z > 0$ is a constant. Set

$$\phi(v,y) = \frac{e^{-pv} y}{(1+e^v)(1+y)},\ (v,y) \in I_1 \times [0,\infty),\ k(v,s) = \frac{e^{-pv}}{16} s.$$

Let $z_1, z_2 \in [0,\infty)$ and $v \in I_1$. Then we have

$$|\phi(v,z_1,Sz_1)-\phi(v,z_2,Sz_2)| \leq \frac{9e^{-pv}}{16}|z_1-z_2|.$$

Obviously, for all $z \in [0,\infty)$ and each $v \in I_1$,

$$|f(v,z,Sz)| \leq \frac{9e^{-pt}}{16}.$$

For $v \in I_1$, $\zeta \in (0,q-3)$, let $m_1(v) = m_2(v) = h(v) = \frac{e^{-pv-s}}{32} \in L^{\frac{1}{\zeta}}(I_1,\mathbb{R})$, $M = \left\| \frac{9e^{-pv}}{16} \right\|_{L^{\frac{1}{\zeta}}(I_1,\mathbb{R})}$. Choosing some $z > 0$ large enough and suitable $\zeta \in (0,q-3)$, one can arrive at the following inequality

$$\Omega_{q,T} = \frac{M}{\Gamma(q)} \frac{T^{q-\zeta}}{(\frac{q-\zeta}{1-\zeta})^{1-\zeta}} + \frac{M}{2!\Gamma(q-2)} \frac{T^{q-\zeta}}{(\frac{q-\zeta-2}{1-\zeta})^{1-\zeta}} < 1,$$

Clearly, all of the assumptions in Theorem 2 are met. Our findings can be used to solve the issue (4.1).

Example 2. Let us consider the another FBVPs,

$$\begin{cases} {}^cD^q z(v) &= \frac{v^{p+3}|z(v)|^\rho}{(1+e^v)(1+z(v)|)} + \int_0^v \frac{v^{p+3}}{16} s \frac{|z(s)|^\rho}{1+|z(s)|} ds, \\ & \qquad y \in [0,1],\ q \in I = [0,T],\ q \in (3,4), \\ z &> -q,\ \ \rho \in [0,-\frac{1}{p}),\ \ 1 < p < \frac{1}{4-q}, \\ z(0,y) &= 0,\ \ z'(0,y) = 0,\ \ z''(0,y) = 0\ \ z'''(T,y) = 0,\ \ y \in [0,1], \\ z(v,0) &= z(v,1) = 0\ \ \ t > 0. \end{cases} \tag{4.2}$$

Set

$$\phi(v,u) = \frac{v^{p+3}{}^\rho}{u}(1+e^v)(1+z),\ \ v \in I \times [0,\infty),\ \ k(v,s) = \frac{v^{p+3}}{16}s.$$

Obviously, for all $z \in U$ and each $v \in I$.

$$|\phi(v,z)| \leq \frac{T^{p+3}}{16}|z|^\rho.$$

Since $z > -q$ and $p+3 > -(q-3)$. As a result of satisfying all of Theorem 3 assumptions, we may apply our findings to the Problem (4.2)

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