Miskolc Mathematical Notes
Vol. 24 (2023), No. 2, pp. 635–651

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4032

QUALITATIVE STUDY FOR IMPULSIVE PANTOGRAPH FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION VIA ψ-HILFER DERIVATIVE

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Received 13 December, 2021

Abstract. In this paper, we study the existence and stability of solutions for impulsive pantograph fractional integro-differential equation via ψ-Hilfer fractional derivative in a appropriate Banach space. Our approach is based on fixed point theorems of Darbo's and Mönch via Kuratowski measure of non-compactness. An example is given to illustrate our approach.

2010 *Mathematics Subject Classification:* 26A33; 34A60; 34A08; 34A37

Keywords: measure of non-compactness, fixed point theorem, ψ-Hilfer fractional derivative

1. INTRODUCTION

Fractional differential equations are one of the most successful and interesting branches of mathematics. Its results can be used to prove some important properties in many fields of science and engineering [18, 20, 22, 24].

Many effective theoretical studies published by several researchers which reside on the result of existence, uniqueness and the stability for differential equations involving a fractional derivative with various conditions, see [10, 11]. The class of impulsive fractional differential equations is distinguished from others by the modeling of phenomena which undergo distortions, in particular in the field of medicine and physics, see for example [6].

In the book [13], Fečkan et al. found a revised formula for the solutions of an impulsive differential equation involving the Caputo derivative. In the references [4, 5, 8, 12], the authors are interested in the study of impulsive differential equations involving the derivative of Riemann or that of Hilfer. One of the properties of solutions is the stability in the sense of Hyers which is introduced in [16] by the study of a question posed by Ulam.

M. Fečkan is partially supported by the Slovak Research and Development Agency under the Contract No. APVV-18-0308, and the Slovak Grant Agency VEGA No. 1/0358/20 and No. 2/0127/20.

Recently, the stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias have been studied by many researchers for certain differential problems considered, see [1, 2, 23].

In view of the above considerations, we consider the following impulsive pantograph fractional integro-differential equation

$$(\mathfrak{P})\quad \begin{cases} {}^H\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi}y(t)=f(t,y(t),\mathfrak{I}_{t_k^+}^{\delta,\psi}y(\sigma_k(t))), \quad t\in(t_k,t_{k+1}],\ k=0,\ldots,m,\\ \Delta_{\gamma,\psi}y|_{t_k}=J_k(y(t_k^-)),\ k=0,\ldots,m,\\ \mathfrak{I}_{c^+}^{1-\gamma,\psi}y(c^+)=M\Gamma(\gamma),\end{cases}$$

where ${}^H\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi}$ denote the left-sided $\psi-$Hilfer fractional derivative of order $0<\alpha<1$ and of type β, $0\leq\beta\leq 1$, $\gamma=\alpha+\beta(1-\alpha)$, $\delta>1-\gamma$. The operator $\mathfrak{I}_{t_k^+}^{1-\gamma,\psi}$ denotes the left-sided $\psi-$Riemann-Liouville fractional integral of order $\varsigma\in\{\delta,1-\gamma\}$, $f:(c,L]\times E^2\to E$ a function satisfying some specified conditions, t_k, $k=0,\ldots,m$ are pre-fixed points satisfying $t_0=c<t_1\leq\cdots\leq t_m<t_{m+1}=L$ and, $J_k:E\to E$, $\Delta_{\gamma,\psi}y|_{t_k}=\frac{\mathfrak{I}^{1-\gamma,\psi}y(t_k^+)}{\Gamma(\gamma)}-y(t_k^-)$, where $y(t_k^-)=\lim_{t\longrightarrow t_k^-}y(t)$, $k=1,\ldots,m$, $M\in E$, $\sigma_k(t)=t_k+\sigma(t-t_k), k=0,\ldots,m$ with $0<\sigma\leq 1$ and $\psi\in\mathcal{C}^1([c,L],\mathbb{R}^+)$ satisfies $\psi'(t)>0$, for all $t\in[c,L]$. The $\psi-$Hilfer fractional derivative is in a general form and in particular cases, it covers special cases. Pantograph equations arise in electrodynamics [21]. Our conditions are also related of other systems mentioned above, and in particular to the following recent papers. Initial value problems for two different classes of implicit ϕ-Hilfer fractional pantograph differential equations are considered in [3]. ϕ-Caputo differential inclusion boundary value problems are studied in [7] supplemented with mixed integro-derivative conditions in the frame of the ϕ-Riemann-Liouville operators. Nonlinear impulsive pantograph fractional BVPs under Caputo proportional fractional derivative are investigated in [17]. The significance of our impulsive and initial conditions in $(\mathfrak{P})$ compare with the above results relies on the fact that they are nonlocal.

This paper is organized in the following way. In Section 2 we give some general results and preliminaries, in Section 3, we show the existence results for the problem $(\mathfrak{P})$ based on fixed point theorems of Darbo's and Mönch and in Section 4, we present a result about the stability in the sense of Ulam-Hyers-Rassias of Problem $(\mathfrak{P})$. Finally an illustrative example will be presented in Section 5.

2. Preliminary results

In this section, we introduce some notation and technical results which are used throughout this paper. Let $I=[a,b]$, $b>a$ and $(E,\|\cdot\|)$ be a Banach space. $\mathcal{C}(I,E)$ be the space of continuous functions on I with the norm

$$\|u\|_\infty=\sup\{\|u(t)\|,\ t\in I\}.$$

$L^1(I,E)$ is the space of E-valued Bochner integrable functions on I with the norm

$$\|h\|_{L^1} = \int_a^b \|h(t)\|dt.$$

For all $\eta > -1$ and $s,r \in [0,L]$ with $r > s$, we pose $\mathrm{N}_{\eta,\psi}(r,s) = (\psi(r)-\psi(s))^{\eta}$. We consider the Banach spaces of functions

$$\mathcal{C}_{1-\gamma,\psi}([a,b]) = \{y \in \mathcal{C}((a,b],E): \lim_{t\to a^+} \mathrm{N}_{1-\gamma,\psi}(t,a)y(t)) \text{ exists and finite}\}.$$

A norm in this space is given by

$$\|y\|_{\gamma,\psi} = \sup_{t\in[a,b]} \mathrm{N}_{1-\gamma,\psi}(t,a)\|y(t)\|,$$

and

$$\begin{aligned}\mathbb{PC}_{1-\gamma,\psi}([c,L]) \quad &= \Big\{y:(c,L]\to E:\ y_k \in \mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}],E) \\ &\quad \text{with } y(t_k)=y(t_k^-),\ \text{for all } k=0,\dots,m, \Big\},\end{aligned}$$

with the norm

$$\|y\|_{\mathbb{PC}_{\gamma,\psi}} = \max_{k=0,\dots,m.} \|y\|_{\gamma,\psi},$$

where y_k is the restriction of y to $(t_k,t_{k+1}]$, $k=0,\dots,m$. Let us now give the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For all $G \subseteq E$, we denote by $S_b(G)$ the set of all bounded subsets of G.

Definition 1 ([9,15]). Let $D \in S_b(E)$. The Kuratowski measure of non-compactness ϑ of the subset D is defined as follows:

$$\vartheta(D) = \inf\{e > 0 : \Omega \text{ admits a finite cover by sets of diameter} \leq e\}.$$

Lemma 1 ([9, 15]). *Let $A,B \in S_b(E)$. The following properties hold:*

(i_1) $\vartheta(A) = 0$ *if and only if A is relatively compact,*
(i_2) $\vartheta(A) = \vartheta(\overline{A})$, *where $\overline{A}$ denotes the closure of A,*
(i_3) $\vartheta(A+B) \leq \vartheta(A)+\vartheta(B)$,
(i_4) $A \subset B$ *implies* $\vartheta(A) \leq \vartheta(B)$,
(i_5) $\vartheta(a.A) = |a|.\vartheta(A)$ *for all* $a \in \mathbb{R}$,
(i_6) $\vartheta(\{a\}\cup A) = \vartheta(A)$ *for all* $a \in E$,
(i_7) $\vartheta(A) = \vartheta(Conv(A))$, *where $Conv(A)$ is the smallest convex that contains A.*

Lemma 2 ([15]). *If D is a equicontinuous and bounded subset of $\mathcal{C}([a,b],E)$, then $\vartheta(D(.)) \in \mathcal{C}([a,b],\mathbb{R}^+)$*

$$\vartheta_{\mathcal{C}}(D) = \max_{t\in[a,b]} \vartheta(D(t)),\ \vartheta\left(\left\{\int_a^b w(t)dt : w \in D\right\}\right) \leq \int_a^b \vartheta(D(t))dt,$$

where $D(t) = \{w(t) : w \in D\}$ and $\vartheta_{\mathcal{C}}$ is the non-compactness measure on the space $\mathcal{C}([a,b],E)$.

We denote by $\vartheta^k_{\gamma,\psi}$ and $\vartheta_{\gamma,\psi}$ the Kuratowski measures of non-compactness defined respectively on $\mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}])$, $k=0,\ldots,m$ and $\mathbb{PC}_{1-\gamma,\psi}([c,L])$.

Lemma 3 ([15]). *For all bounded subset D of $\mathbb{PC}_{1-\gamma,\psi}([c,L])$, we have*

$$\vartheta_{\gamma,\psi}(D) = \max_{k=0,\ldots,m} \vartheta^k_{\gamma,\psi}(D_k),$$

where D_k is the restriction of D on $(t_k,t_{k+1}]$.

Theorem 1 ([15]). *Let ρ the Kuratowski measure of non-compactness on Banach space E and G a closed, bounded and convex subset of E which contains the 0. Let Δ be an operator from G to G, assume that Δ is continuous and satisfied, for every subset V of G, we have the following implication:*

$$V = \Delta(V)\cup\{0\} \ \text{or}\ V = \overline{conv}\,\Delta(V) \implies \rho(V)=0.$$

Then the set $\{w\in G : \Delta(w)=w\}$ is nonempty.

Theorem 2 ([14]). *Let ρ the Kuratowski measure of non-compactness on Banach space E, G a nonempty, closed, bounded and convex subset of E and Δ be an continuous operator from G to G such that, for all nonempty subset V of G:*

$$\rho(\Delta(V)) \le \varsigma\rho(V),$$

where $0\le\varsigma<1$. Then Δ has a fixed point in G.

We begin with some definitions from the theory of fractional calculus.

Definition 2 ([18, 25]). Let ℓ be an integrable function defined on $(a,b]$,

(i) the ψ-Riemann- Liouville fractional integral of order $\alpha>0$ of the function ℓ is defined by

$$\mathfrak{I}^{\alpha,\psi}_{a^+}\ell(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\ell(s)ds,$$

(ii) the ψ-Riemann- Liouville fractional derivative of order $\alpha>0$ of the function ℓ is defined by

$${}^{RL}\mathcal{D}^{\alpha,\psi}_{a^+}h(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\left(\int_a^t \psi'(s)\mathrm{N}_{n-\alpha-1,\psi}(t,s)\ell(s)ds\right),$$

where Γ is the gamma function and $n=[\alpha]+1$ ($[\alpha]$ represents the integer part of the real number α).

Definition 3 ([18, 25]). Let $\psi\in\mathcal{C}^1([a,b],E)$ a functions such that $\psi'(t)>0$, for all $t\in[a,b]$. The ψ-Hilfer fractional derivative of a function ℓ of order $0<\alpha<1$ and type $0\le\beta\le1$ is given by

$${}^{H}\mathcal{D}^{\alpha,\beta,\psi}_{a^+}\ell(t) = \mathfrak{I}^{\beta(1-\alpha),\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathfrak{I}^{(1-\beta)(1-\alpha),\psi}\ell(t) = \mathfrak{I}^{1-\gamma,\psi RL}D^{\gamma,\psi}\ell(t),$$

where $\gamma=\alpha+\beta(1-\alpha)$.

Lemma 4 ([18]). *Let* $\alpha, \rho \in \mathbb{R}_+^*$ *and* $t > a$. *We have then*

(i_1) $\mathfrak{I}_{a^+}^{\alpha,\psi} \mathrm{N}_{\rho-1,\psi}(t,a) = \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)} \mathrm{N}_{\alpha+\rho-1,\psi}(t,a)$.

(i_2) ${}^H\mathcal{D}_{a^+}^{\alpha,\rho,\psi} \mathrm{N}_{\rho-1,\psi}(t,a) = \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)} \mathrm{N}_{\rho-\alpha-1,\psi}(t,a), 0 < \alpha < 1,\ \rho > 1$.

We consider the spaces

$$\mathcal{C}_{1-\gamma,\psi}^{\gamma}([a,b]) = \Big\{u \in \mathcal{C}_{1-\gamma,\psi}([a,b]),\ {}^{RL}\mathcal{D}_{a^+}^{\gamma} u \in \mathcal{C}_{1-\gamma,\psi}([a,b])\Big\},$$

$$\mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L]) = \Big\{u \in \mathbb{PC}_{1-\gamma,\psi}([c,L]) :\ {}^{RL}\mathcal{D}_{t_k^+}^{\gamma,\psi} u_k \in \mathcal{C}_{1-\gamma}([t_k,t_{k+1}]),\ k = 0,\dots,m\Big\},$$

and

$$\mathbb{PC}_{1-\gamma,\psi}^{\alpha,\beta}([c,L]) = \Big\{u \in \mathbb{PC}_{1-\gamma,\psi}([c,L]) :\ {}^{H}\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi} u_k \in \mathcal{C}_{1-\gamma}([t_k,t_{k+1}]),\ k = 0,\dots,m\Big\}.$$

Lemma 5 ([19]). *Let* $0 < \alpha < 1,\ 0 \le \beta \le 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $f \in \mathcal{C}_{1-\gamma}^{\gamma}([a,b])$, *then*

$$\mathfrak{I}_{a^+}^{\gamma,\psi} \mathcal{D}_{a^+}^{\gamma,\psi} f = \mathfrak{I}_{a^+}^{\alpha,\psi} \mathcal{D}_{a^+}^{\alpha,\beta,\psi} f \text{ and } \mathcal{D}_{a^+}^{\gamma,\psi} \mathfrak{I}_{a^+}^{\alpha,\psi} f = \mathcal{D}_{a^+}^{\beta(1-\alpha)} f.$$

Lemma 6 ([19]). *Let* $\omega : (a,b] \to E$ *be a function such that* $\omega(.) \in \mathcal{C}_{1-\gamma,\psi}([a,b])$. *Then, a function* $y \in \mathcal{C}_{1-\gamma,\psi}^{\gamma}([a,b])$ *is a solution of linear fractional differential problem:*

$$\begin{cases} {}^H\mathcal{D}_{a^+}^{\alpha,\beta,\psi} y(t) = \omega(t), & 0 < \alpha < 1,\ 0 \le \beta \le 1; \\ \mathfrak{I}_{a^+}^{1-\gamma,\psi} y(a^+) = \omega_0, & \gamma = \alpha + \beta - \alpha\beta. \end{cases}$$

if and only if y satisfies the following integral equation:

$$y(t) = \frac{\omega_0 \mathrm{N}_{\gamma-1,\psi}(t,a)}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \mathrm{N}_{\alpha-1,\psi}(t,s) \omega(s) ds.$$

For any $k \in \{1,\dots,m\}$, we define the constants $\Xi_{j=1}^{k-i} N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})$, $i = 1,\dots,k$ by

$$\Xi_{j=1}^{k-i} N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j}) = \begin{cases} 1, & i = k; \\ \prod_{j=1}^{k-i} \mathrm{N}_{\gamma-1,\psi}(t_{k-j+1},t_{k-j}), & i = 1,\dots,k-1, \end{cases}$$

we also put

$$T^* = \frac{\Gamma(\gamma)}{\Gamma(\delta+\gamma)} \max\Bigg\{ \sup_{t\in(t_k,t_{k+1}]} \Big(\frac{\mathrm{N}_{\gamma-1,\psi}(t,t_k)}{\mathrm{N}_{\gamma-1,\psi}(\sigma_k(t),t_k)} \Big), k = 0,\dots,m \Bigg\} \text{ and}$$

$$T_\eta = \max\Big\{1, \mathrm{N}_{\eta,\psi}(t_{k+1},t_k), k = k = 0,\dots,m\Big\},\ \eta > -1.$$

Lemma 7. *Let* $f : (c,L] \times E^2 \to E$ *be a function such that* $f(.,y(.),\mathfrak{I}_{t_k}^{\delta,\psi} y(\sigma_k(.))) \in \mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}])$ $k = 0,\dots,m$, *for all* $y \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$. *If* $y \in \mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L])$. *Then,*

y is a solution of Problem $(\mathfrak{P})$ if and only if y satisfies the following integral equation:

$$y(t)=\begin{cases} M\mathrm{N}_{\gamma-1,\psi}(t,c)+\frac{1}{\Gamma(\alpha)}\int_c^t \psi'(s)\mathrm{N}_{\gamma-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds, & \text{if } t\in I_0,\\ \mathrm{N}_{\gamma-1,\psi}(t,t_k)\Big[M\prod_{i=1}^{k}\mathrm{N}_{\gamma-1,\psi}(t_i,t_{i-1}) & \\ +\sum\limits_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})J_i(y(t_i^-)) & \\ +\sum\limits_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})\mathfrak{I}_{t_{i-1}^+}^{\alpha,\psi}f\big(t_i,y(t_i),\mathfrak{I}_{t_{i-1}}^{\delta,\psi}y(\sigma_{t_{i-1}}(t_i))\big)\Big] & \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds, & \text{if } t\in I_k,\end{cases} \tag{2.1}$$

where $I_k=(t_k,t_{k+1}]$, $k=1,\dots,m$.

Proof. First, we prove the necessity. Let $y\in\mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L])$ be a solution of $(\mathfrak{P})$. If $t\in(a,t_1]$, we have ${}^{H}\mathcal{D}_{t_0}^{\alpha,\beta,\psi}y(t)=f\big(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t))\big)$, from Lemma 6, we get

$$y(t)=M\mathrm{N}_{\gamma-1,\psi}(t,c)+\frac{1}{\Gamma(\alpha)}\int_c^t\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds.$$

If $t\in(t_1,t_2]$, then, from Lemma 6, we get

$$y(t)=\frac{\mathfrak{I}_{t_1^+}^{1-\gamma,\psi}y(t_1^+)}{\Gamma(\gamma)}\mathrm{N}_{\gamma-1,\psi}(t,t_1)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_1}^{\delta,\psi}y(\sigma_1(s))\big)ds.$$

By using the condition $\frac{\mathfrak{I}_{t_1^+}^{1-\gamma,\psi}y(t_1^+)}{\Gamma(\gamma)}=y(t_1^-)+J_1(y(t_1^-))$, we obtain

$$\begin{aligned} y(t)&=\big(y(t_1)+J_1(y(t_1^-))\big)\mathrm{N}_{\gamma-1,\psi}(t,t_1)\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f(s,y(s),\mathfrak{I}_{t_1}^{\delta,\psi}y(\sigma_1(s)))ds\\ &=\Big[J_1(y(t_1^-))+M\mathrm{N}_{\gamma-1,\psi}(t_1,a)+\mathfrak{I}_{c^+}^{\alpha,\psi}f\big(t_1^-,y(t_1^-),\mathfrak{I}_{t_0}^{\delta,\psi}y(\sigma_1(t_1))\big)\Big]\mathrm{N}_{\gamma-1,\psi}(t,t_1)\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_1}^{\delta,\psi}y(\sigma_1(s))\big)ds. \end{aligned}$$

If $t\in(t_2,t_3]$, by utilizing Lemma 6 and the condition $\frac{\mathfrak{I}_{t_2^+}^{1-\gamma,\psi}y(t_2^+)}{\Gamma(\gamma)}=y(t_2^-)+J_2(y(t_2^-))$, we arrive

$$\begin{aligned} y(t)&=\frac{\mathfrak{I}_{t_2^+}^{1-\gamma,\psi}y(t_2^+)}{\Gamma(\gamma)}\mathrm{N}_{\gamma-1,\psi}(t,t_2)\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_2}^{\delta,\psi}y(\sigma_2(s))\big)ds\\ &=\big(J_2(y(t_2^-))+y(t_2^-)\big)\mathrm{N}_{\gamma-1,\psi}(t,t_2) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_2}^{\delta,\psi}y(\sigma_2(s))\big)ds\\
=\ &\mathrm{N}_{\gamma-1,\psi}(t,t_2)\big[M\mathrm{N}_{\gamma-1,\psi}(t_1,t_0)\mathrm{N}_{\gamma-1,\psi}(t_2,t_1)+\mathrm{N}_{\gamma-1,\psi}(t_2,t_1)J_1(y(t_1^-))+J_2(y(t_2^-))\\
&+\mathrm{N}_{\gamma-1,\psi}(t_2,t_1)\mathfrak{I}_{c^+}^{\alpha}f\big(t_1^-,y(t_1^-),\mathfrak{I}_{t_0}^{\delta,\psi}y(\sigma_0(t_1))\big)+\mathfrak{I}_{t_1}^{\alpha,\psi}f\big(t_2^-,y(t_2^-),\mathfrak{I}_{t_1}^{\delta,\psi}y(\sigma_1(t_2))\big)\big]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_2}^{\delta,\psi}y(\sigma_2(s))\big)ds.
\end{aligned}
$$

If $t\in(t_k,t_{k+1}]$, we continue the procedure and again using Lemmas 6 and the condition $\frac{\mathfrak{I}_{t_k^+}^{1-\gamma,\psi}y(t_k^+)}{\Gamma(\gamma)}=y(t_k^-)+J_k(y(t_k^-))$, we find

$$
\begin{aligned}
y(t)=\mathrm{N}_{\gamma-1,\psi}(t,t_k)&\Bigg[M\prod_{i=1}^{k}\mathrm{N}_{\gamma-1,\psi}(t_i,t_{i-1})+\sum_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})J_i(y(t_i^-))\\
&+\sum_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})\mathfrak{I}_{t_{i-1}^+}^{\alpha,\psi}f\big(t_i,y(t_i),\mathfrak{I}_{t_{i-1}}^{\delta,\psi}y(\sigma_{t_{i-1}}(t_i))\big)\Bigg]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds.
\end{aligned}
$$

Conversely, assume that y satisfies the impulsive equation (2.1). If $t\in(t_0,t_1]$, by using 6, we get

$$\mathfrak{I}_{c^+}^{1-\gamma,\psi}y(t)=M \text{ and } {}^{H}\mathcal{D}_{t_0}^{\alpha,\beta,\psi}y(t)=f\big(t,y(t),\mathfrak{I}_{t_0}^{\delta,\psi}y(\sigma_{t_0}(t))\big),\quad \text{for each } t\in(t_0,t_1].$$

By recurrence, if $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$ and according to Lemma 6, we get

$${}^{H}\mathcal{D}_{t_k}^{\alpha,\beta,\psi}y(t)=f\big(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_{t_k}(t))\big),\quad \text{for each } t\in(t_k,t_{k+1}].$$

And, we can easily show that

$$\Delta_{\alpha,\psi}y|_{t=t_k}=J_k(y(t_k^+)).$$

□

3. Existence of the solution

Suppose that the function $f:(c,L]\times E^2\to E$ verifies

$$f(.,u(.),v(.))\in\mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L]),$$

for all $u(.),v(.)\in\mathbb{PC}_{1-\gamma}([c,L])$, $f(.,0,0)\in\mathcal{C}([c,L],E)$ and there exists $A,B\in\mathbb{R}^+$ and $\lambda\geq 1-\alpha$ such that

($\mathbf{H_1}$) For any $u,v,\overline{u},\overline{v}\in E$ and for all $t\in I_k,\ k=1,\dots,m:$

$$\|f(t,u,v)-f(t,\overline{u},\overline{v})\|\leq A\mathrm{N}_{\lambda,\psi}(t,t_k)\|u-\overline{u}\|+B\|v-\overline{v}\|.$$

($\mathbf{H_2}$) For each nonempty, bounded set $\Omega \subset \mathbb{PC}_{1-\gamma,\psi}([c,L])$, for all $t \in I_k, k = 0,\dots,m$, we have

$$\vartheta(f(t,\Omega(t),\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t))) \le A\mathrm{N}_{\lambda,\psi}(t,t_k)\vartheta(\Omega(t))) + B\vartheta(\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t))),$$

where $\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t)) = \{\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t)), y \in \mathbb{PC}_{1-\gamma,\psi}([c,L])\}$, $k = 0,\dots,m$.

Suppose that the functions $J_k : E \to E$, $k = 1,\dots,m$, are continuous and there exists $C \in \mathbb{R}^+$ such that

($\mathbf{H_3}$) For any $u \in E$:

$$\|J_k(u)\| \le C\|u\|,\ k = 1,\dots,m.$$

($\mathbf{H_4}$) For each nonempty, bounded set $\Omega \subset \mathbb{PC}_{1-\gamma,\psi}([c,L])$, we have

$$\vartheta(J_k(\Omega(t))) \le C\vartheta(\Omega(t)),\ k = 0,\dots,m.$$

($\mathbf{H_5}$) $\overline{T}_{\gamma-1}^m T_\alpha\Big(mC\Gamma(\alpha+1) + (m+1)(AT_\lambda + BT^*T_\delta)\Big) < \Gamma(\alpha+1)$,
where $\overline{T}_{\gamma-1} = \max\left\{T_{1-\gamma}, T_{\gamma-1}\right\}$.

Our first result concerning the existence of solutions of the problem ($\mathfrak{P}$) for which we have used the fixed point theorem of Mönch's is as follows:

Theorem 3. *We assume that the hypotheses from* ($\mathbf{H_1}$) *to* ($\mathbf{H_5}$) *are satisfied, then problem* ($\mathfrak{P}$) *has at least one solution in* $\mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L])$.

Proof. Consider the operator $\Lambda : \mathbb{PC}_{1-\gamma,\psi}([c,L]) \to \mathbb{PC}_{1-\gamma,\psi}([c,L])$ defined by

$$\begin{aligned}\Lambda y(t) = \mathrm{N}_{\gamma-1,\psi}(t,t_k)\Bigg[&M\prod_{i=1}^{k}\mathrm{N}_{\gamma-1,\psi}(t_i,t_{i-1}) + \sum_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})J_i(y(t_i^-))\\ &+\sum_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})\mathfrak{I}_{t_{i-1}^+}^{\alpha,\psi}f\big(t_i,y(t_i),\mathfrak{I}_{t_{i-1}}^{\delta,\psi}y(\sigma_{t_{i-1}}(t_i))\big)\Bigg]\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds,\end{aligned}$$

for any $t \in I_k, k = 1,\dots,m$. From the definition of the operator Λ and Lemma 7, we see that the fixed points of Λ are solutions of problem ($\mathfrak{P}$). For this reason, it suffices to verify the axioms of Theorem 1, it is done in four steps.

First step. We start to prove that Λ is continuous. Let $\varepsilon > 0$ and $\{y_n\}_{n\in\mathbb{N}} \to y$ in $\mathbb{PC}_{1-\gamma,\psi}([c,L])$. The hypothesis ($\mathbf{H_1}$) and ($\mathbf{H_3}$) confirm the existence of an integer $n_1 \in \mathbb{N}$ such that, for all $n \ge n_1$ and $t \in I_k$, $k = 0,\dots,m$, we have

$$\|f\big(t,y_n(t),\mathfrak{I}_{t_k}^{\delta,\psi}y_n(\sigma_k(t))\big) - f\big(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t))\big)\| < \frac{\Gamma(\alpha+1)\varepsilon}{(2m+1)(AT_\lambda + BT^*T_\delta)T_\alpha\overline{T}_{\gamma-1}^m} \tag{3.1}$$

and

$$\|J_k(y_n(t_k^-)) - J_k(y(t_k^-))\| < \frac{\varepsilon}{2m\overline{T}^m_{\gamma-1}}. \tag{3.2}$$

Thus, for all $t \in I_k,\ k = 0,\dots,m$, we have

$$\begin{aligned}
&\mathrm{N}_{\gamma-1,\psi}(t,t_k)\|\Lambda y_n(t) - \Lambda y(t)\| \le \sum_{i=1}^{k} T^m_{\gamma-1}\|J_i(y_n(t_i^-)) - J_i(y(t_i^-))\| \\
&+ \sum_{i=1}^{k} T^m_{\gamma-1}\mathfrak{I}^{\gamma,\psi}_{t_i}\|f\big(t_{i+1}, y_n(t_{i+1}), \mathfrak{I}^{\delta,\psi}_{t_i} y_n(\sigma_i(t_{i+1}))\big) - f\big(t_{i+1}, y(t_{i+1}), \mathfrak{I}^{\delta,\psi}_{t_i} y(\sigma_i(t_{i+1}))\big)\| \\
&+ \frac{T_{1-\gamma}}{\Gamma(\alpha)}\int_{t_k}^{t} \psi'(s)\mathrm{N}_{1-\alpha}(t,s)\|f\big(s, y_n(s), \mathfrak{I}^{\delta,\psi}_{t_k} y_n(\sigma_k(s))\big) - f\big(s, y(s), \mathfrak{I}^{\delta,\psi}_{t_k} y(\sigma_k(s))\big)\|ds.
\end{aligned}$$

By equations (3.1) and (3.2), we get

$$\|\Lambda y_n - \Lambda y\|_{\mathbb{PC}_{1-\gamma,\psi}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, Λ is continuous on $\mathbb{PC}_{1-\gamma,\psi}([c,L])$.

Second step. Now we will prove that Λ is bounded. Let $y \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$, from ($\mathbf{H_1}$) it is easy to deduce that $Ny \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$. Using ($\mathbf{H_1}$) and ($\mathbf{H_3}$), for all $y \in D_\kappa = \{y \in \mathbb{PC}_{1-\gamma,\psi}([c,L]) : \|y\|_{\mathbb{PC}_{\gamma,\psi}} < \kappa\}$ and $t \in I_k$, we get

$$\begin{aligned}
\|\mathrm{N}_{1-\gamma,\psi}(t,t_k)\Lambda y(t)\| &\le \|M\|T^m_{\gamma-1} + T^m_{\gamma-1}\sum_{i=1}^{k}\|\mathrm{N}_{1-\gamma,\psi}(t_i,t_{i-1})J_i(y(t_i^-))\| \\
&+ T^m_{\gamma-1}\sum_{i=1}^{k}\mathrm{N}_{1-\gamma,\psi}(t_i,t_{i-1})\mathfrak{I}^{\alpha,\psi}_{t_{i-1}^+}\|f\big(t_i, y(t_i), \mathfrak{I}^{\delta,\psi}_{t_{i-1}} y(\sigma_{t_{i-1}}(t_i))\big)\| \\
&+ \frac{\mathrm{N}_{1-\gamma,\psi}(t,t_k)}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\|f\big(s, y(s), \mathfrak{I}^{\delta,\psi}_{t_k} y(\sigma_{t_k}(s))\big)\|ds \\
&\le T^m_{\gamma-1}\big(\|M\| + Cm\kappa\big) + \frac{(m+1)T_\alpha\overline{T}^m_{\gamma-1}}{\Gamma(\alpha+1)}\Big(f^* + \kappa\big(AT_\lambda + BT^*T_\delta\big)\Big),
\end{aligned}$$

where $f^* = \sup_{t\in[c,L]}\big(\|f(t,0,0)\|\big)$.

Third step. We prove that $(\Lambda D)_k$ is equicontinuous for all bounded subset D of $\mathbb{PC}_{1-\gamma,\psi}([c,L])$, $k = 1,\dots,m$, where $(\Lambda D)_k$ the restriction of ΛD on the interval I_k, let D_κ be the subset which was previously defined. It suffices to prove that $(\Lambda D_\kappa)_k$ is equicontinuous in $\mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}])$. Let $y \in (D_\kappa)_k$ and $t_1,t_2 \in I_k$ with $t_1 < t_2$, from ($\mathbf{H_1}$), we have

$$\begin{aligned}
&\|\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)\Lambda y(t_2) - \mathrm{N}_{1-\gamma,\psi}(t_1,t_k)\Lambda y(t_1)\| \\
&\le \frac{\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)[\mathrm{N}_{\alpha-1,\psi}(t_1,s) - \mathrm{N}_{\alpha-1,\psi}(t_2,s)]\|f\big(s, y(s), \mathfrak{I}^{\delta,\psi}_{t_k} y(\sigma_k(s))\big)\|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{[\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)-\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)]}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)\|f(s,y(s,\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s)))\|ds\\
&+\frac{\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)\|f(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s)))\|ds\\
\leq&\frac{f^*+A\kappa T_\lambda+B\kappa T^*T_\delta}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)[\mathrm{N}_{\alpha-1,\psi}(t_1,s)-\mathrm{N}_{\alpha-1,\psi}(t_2,s)]ds\\
&+\frac{f^*[\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)-\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)]}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)ds\\
&+\frac{A\kappa[\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)-\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)]}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)\mathrm{N}_{\lambda+\alpha-1,\psi}(s,t_k)ds\\
&+\frac{B\kappa T^*[\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)-\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)]}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)\mathrm{N}_{\delta+\gamma-1,\psi}(s,t_k)ds\\
&+\frac{f^*+A\kappa T_\lambda+B\kappa T^*T_\delta}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)ds\\
\leq&\frac{f^*+A\kappa\Psi_\lambda^*}{\Gamma(\alpha+1)}[2\mathrm{N}_{\alpha,\psi}(t_2,t_1)+\mathrm{N}_{\alpha,\psi}(t_1,t_k)-\mathrm{N}_{\alpha,\psi}(t_2,t_k)]\\
&+\frac{f^*T_\alpha+A\kappa T_{2\alpha+\lambda-1}+B\kappa T^*T_{\alpha+\delta+\lambda-1}}{\Gamma(\alpha+1)}[\Psi_{1-\alpha}(t_2,t_k)-\Psi_{1-\alpha}(t_1,t_k)],
\end{aligned}
$$

Taking t_2 tends towards t_1, we get that, the last inequality tends to zero. Then $(\Lambda D_\kappa)_k$ is equicontinuous in $\mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}])$, $k=0,\dots,m$.

Final step. We verify that Λ satisfies the assumptions of theorem 1. We pose

$$D=\{y\in\mathbb{PC}_{1-\gamma,\psi}([c,L]):\|y\|_{\mathbb{PC}_{\gamma,\psi}}\leq R\},$$

where R is a real number verifies the following equality

$$R>\frac{\overline{T}_{\gamma-1}^m\Big(\Gamma(\alpha+1)\|M\|+(m+1)T_\alpha f^*\Big)}{\Gamma(\alpha+1)-\overline{T}_{\gamma-1}^m T_\alpha\Big(mC\Gamma(\alpha+1)+(m+1)(AT_\lambda+BT^*T_\delta)\Big)}. \tag{3.3}$$

First, we now show that Λ is defined from D to D, Indeed, for any $y\in D$, by above conditions $(\mathbf{H_2}),(\mathbf{H_5})$ and by according to a little calculation, for all $t\in I_k$, we have

$$
\begin{aligned}
\|\mathrm{N}_{1-\gamma,\psi}\Lambda y(t)\leq&\frac{\overline{T}_{\gamma-1}^m}{\Gamma(\alpha+1)}\Big(\Gamma(\alpha+1)\|M\|+(m+1)T_\alpha f^*\Big)\\
&+\frac{\overline{T}_{\gamma-1}^m T_\alpha}{\Gamma(\alpha+1)}\Big(mC\Gamma(\alpha+1)+(m+1)(AT_\lambda+BT^*T_\delta)\Big)R.
\end{aligned}
$$

From inequality (3.3), we obtain

$$\forall y\in D:\|\Lambda y\|_{\mathbb{PC}_{\gamma,\psi}}<R.$$

Then Λ remains defined from D to D. Note that D is bounded, convex and closed subset of $\mathbb{PC}_{\gamma,\psi}([a,L])$ and Λ is continuous on D, we can easily show the following equality

$$\vartheta^k_{\gamma,\psi}((NV)_k) = \sup\left\{\vartheta\left(\mathrm{N}_{1-\gamma,\psi}(t,t_k)\Lambda V(t)\right),\ t\in I_k\right\},$$

for all $V\subset D,\ k=0,\ldots,m$. Next, we need to prove the following implication

$$V\subset \overline{conv}\{\Lambda(V)\cup\{0\}\} \Longrightarrow \vartheta_{\gamma,\psi}(V)=0,\ \text{for any } V\subset D.$$

Let $V\subset D$ such that $V\subset \overline{conv}\{\Lambda(V)\cup\{0\}\}$. From $(\mathbf{H_2})$, $(\mathbf{H_4})$, Lemmas 1-2 and the previous steps, for all $t\in I_k$, we have

$$\begin{aligned}\vartheta\left(\mathrm{N}_{1-\gamma,\psi}(\Lambda V)(t)\right) &\le \frac{AT_\lambda T^m_{\gamma-1}}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_{i+1},s)\vartheta^i_{\gamma,\psi}((\Lambda V)_i)ds\\
&+\frac{BT^*T_\delta T^m_{\gamma-1}}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_{i+1},s)\vartheta^i_{\gamma,\psi}((\Lambda V)_i)ds + CT^m_{\gamma-1}\sum_{i=0}^{k-1}\vartheta^i_{\alpha,\psi}((\Lambda V)_i)\\
&+\frac{AT_\lambda+BT^*T_\delta}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\vartheta^k_{\gamma,\psi}((\Lambda V)_k)ds.\end{aligned}$$

Thus,

$$\vartheta_{\gamma,\psi}(\Lambda V)\le \frac{T^m_{\gamma-1}\left[(m+1)AT_{\lambda+\alpha}+(m+1)BT^*T_{\delta+\alpha}+mC\Gamma(\alpha+1)\right]}{\Gamma(\alpha+1)}\vartheta_{\gamma,\psi}(\Lambda V).$$

By condition $(\mathbf{H_5})$, we get $\vartheta_{\gamma,\psi}(\Lambda V)=0$, that is $\vartheta_{\gamma,\psi}(V)=0$. From Theorem 1, Λ has a fixed point $\bar{y}\in D$ which is a solution of Problem $(\mathfrak{P})$. Let us now show that the fixed point of Λ is included in $\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L])$, Let $t\in I_k,\ k=0,\ldots,m$. So, we have

$$\begin{aligned}\bar{y}(t) &= \mathrm{N}_{\gamma-1,\psi}(t,t_k)\Bigg[M\prod_{i=1}^{k}\mathrm{N}_{\gamma-1,\psi}(t_i,t_{i-1})+\sum_{i=1}^{k}\Xi^{k-i}_{j=1}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})J_i(\bar{y}(t_i^-))\\
&+\sum_{i=1}^{k}\Xi^{k-i}_{j=1}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})\mathfrak{I}^{\alpha,\psi}_{t^+_{i-1}}f\left(t_i,\bar{y}(t_i),\mathfrak{I}^{\delta,\psi}_{t_{i-1}}\bar{y}(\sigma_{t_{i-1}}(t_i))\right)\Bigg]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\left(s,\bar{y}(s),\mathfrak{I}^{\delta,\psi}_{t_k}\bar{y}(\sigma_k(s))\right)ds.\end{aligned}$$

By entering ${}^{RL}\mathcal{D}^{\gamma}_{t_k^+}$ on both sides and utilizing Lemma 6, we find

$$\begin{aligned}{}^{RL}\mathcal{D}^{\gamma}_{t_k^+}\bar{y}(t) &={}^{RL}\mathcal{D}^{\gamma}_{t_k^+}\mathfrak{I}^{\alpha,\psi}_{t_k^+}f\left(t,\bar{y}(t),\mathfrak{I}^{\delta,\psi}_{t_k}\bar{y}(\sigma_{t_k}(t))\right)\\
&={}^{RL}\mathcal{D}^{\beta(1-\alpha)}_{t_k^+}f\left(t,\bar{y}(t),\mathfrak{I}^{\delta,\psi}_{t_k}\bar{y}(\sigma_{t_k}(t))\right).\end{aligned}$$

Thus, according to the hypotheses on f, we deduce that ${}^{RL}\mathcal{D}^{\gamma}_{t_k^+}\bar{y}(t) \in \mathcal{C}^{\gamma}_{1-\gamma}(I_k)$, $k = 0,\dots,m$, from the definition of $\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L])$, we conclude that the fixed point $\bar{y}$ of Λ is an element of such space. □

Our present result is based on the Darbo's fixed point theorem.

Theorem 4. *Suppose that the conditions* $(\mathbf{H_1})-(\mathbf{H_5})$ *are valid. Then, the problem* $(\mathfrak{P})$ *has at least one solution. Moreover its solutions belong to* $\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L]) \subset \mathbb{PC}^{\alpha,\beta}_{1-\gamma,\psi}([c,L])$.

Proof. By Lemma 7, the solutions of Problem $(\mathfrak{P})$ and fixed points of operator Λ are coincident. We will prove that Λ satisfies the conditions of Darbo's fixed point Theorem 2. According to what precedes the operator Λ is defined from D to D, continuous, bounded and that ΛD is equicontinuous, it suffices to prove that there exists a real $0<\xi<1$ such that

$$\vartheta_{\gamma,\psi}(\Lambda V) \le \xi\vartheta_{\gamma,\psi}(V), \text{ for all } V \subset D.$$

Let $V\subset D$ and $t\in I_k$, $k=0,\dots,m$. From $(\mathbf{H_2})$, $(\mathbf{H_4})$ and by using Lemmas 1-3, we have

$$\begin{aligned}\vartheta\left(\mathrm{N}_{1-\gamma,\psi}(t,t_k)(\Lambda V)(t)\right) &= \vartheta\left(\left\{\mathrm{N}_{1-\gamma,\psi}(t,t_k)\Lambda u(t),\ u\in V\right\}\right)\\ &\le \frac{AT_\lambda T^m_{\gamma-1}}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_{i+1},s)\vartheta^i_{\gamma,\psi}((\Lambda V)_i)ds\\ &\quad+\frac{BT^*T_\delta T^m_{\gamma-1}}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_{i+1},s)\vartheta^i_{\gamma,\psi}((\Lambda V)_i)ds\\ &\quad+CT^m_{\gamma-1}\sum_{i=0}^{k-1}\vartheta^i_{\alpha,\psi}((\Lambda V)_i)\\ &\quad+\frac{AT_\lambda+BT^*T_\delta}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\vartheta^k_{\gamma,\psi}((\Lambda V)_k)ds.\end{aligned}$$

So, from $(\mathbf{H_5})$ there exists a real $0<\xi<1$ such that

$$\vartheta_{\gamma,\psi}(\Lambda V) \le \xi\vartheta_{\gamma,\psi}(V), \text{ for all } V \subset D,$$

where $\xi = \frac{T^m_{\gamma-1}}{\Gamma(\alpha+1)}\left[(m+1)AT_{\lambda+\alpha}+(m+1)BT^*T_{\delta+\alpha}+mC\Gamma(\alpha+1)\right]$. So, Theorem 2 assures us that the operator has at least one fixed point $\bar{y}$. Method similar to that of the last step of our first result, we find that its fixed points belong to $\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L]) \subset \mathbb{PC}^{\alpha,\beta}_{1-\gamma,\psi}([c,L])$. □

4. Stability

For any $\varepsilon > 0$, $\zeta > 0$ and $\theta : (c, L] \to \mathbb{R}^+$ be a continuous function, we consider the following system of inequalities

$$(\mathfrak{S}) \quad \begin{cases} \|{}^H\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi} y(t) - f(t, y(t), \mathfrak{I}_{t_k^+}^{\delta,\psi} y(\sigma_k(t)))\| \le \varepsilon\theta(t), & t \in I_k,\ k = 0, \ldots, m, \\ \|\Delta_{\gamma,\psi} y|_{t_k} - J_k(y(t_k^-))\| \le \varepsilon\zeta, k = 1, \ldots, m. \end{cases}$$

Definition 4. Problem $(\mathfrak{P})$ is said to be stable in the sense of Ulam-Hyers-Rassias according to (θ, ζ) if there is a real number $\chi_{(\theta,\zeta)}$, for all solution $\varpi \in \mathbb{PC}_{1-\gamma,\psi}([c, L])$ of problem $(\mathfrak{S})$ there exists a solution $y \in \mathbb{PC}_{1-\gamma,\psi}([c, L])$ of problem $(\mathfrak{P})$ such that

$$\|y(t) - \varpi(t)\| \le \varepsilon\chi_{(\theta,\zeta)}\big(\theta(t) + \zeta\big), \text{ for all } t \in (c, L].$$

Remark 1. A function ϖ of $\mathbb{PC}_{1-\gamma,\psi}([c, L])$ is called solution of problem $(\mathfrak{S})$ if there exists a function $\varphi \in \mathcal{C}([c, L))$ and constants $\rho_k \in E$, $k = 0, \ldots, m$ satisfies $\|\varphi(t)\| \le \varepsilon\theta(t)$ and $\|\rho_k\| \le \varepsilon\zeta$, $t \in I_k$, $k = 0, \ldots, m$, such that ϖ is a solution of the following problem

$$\begin{cases} {}^H\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi} y(t) = f(t, y(t), \mathfrak{I}_{t_k^+}^{\delta,\psi} y(\sigma_k(t))) + \varphi(t), & t \in I_k,\ k = 0, \ldots, m, \\ \Delta_{\gamma,\psi} y|_{t_k} = J_k(y(t_k^-)) + \rho_k,\ k = 1, \ldots, m. \end{cases}$$

In the following we give a result about the stability in the sense of Ulam-Hyers-Rassias of Problem $(\mathfrak{P})$. We are interested in studying the case where $\theta : (c, L] \to \mathbb{R}^+$ is a constant function.

Theorem 5. *Suppose that the conditions* $(\mathbf{H_1}) - (\mathbf{H_5})$ *are valid. Then, the problem* $(\mathfrak{P})$ *is stable in the sense of Ulam-Hyers-Rassias according to* (θ, ζ).

Proof. Let $\varepsilon > 0$, $\varpi \in \mathbb{PC}_{1-\gamma,\psi}([c, L])$ be any solution of Problem $(\mathfrak{S})$ and y be the solution of the following problem

$$(\mathfrak{P}) \quad \begin{cases} {}^H\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi} y(t) = f(t, y(t), \mathfrak{I}_{t_k^+}^{\delta,\psi} y(\sigma_k(t))), & t \in I_k,\ k = 0, \ldots, m, \\ \Delta_{\gamma,\psi} y|_{t_k} = J_k(y(t_k^-)), k = 1, \ldots, m, \\ \mathfrak{I}_{t_k^+}^{1-\gamma,\psi} y(t_k^+) = \mathfrak{I}_{t_k^+}^{1-\gamma,\psi} \varpi(t_k^+), k = 0, \ldots, m. \end{cases}$$

From Lemma 6 the solution y of the previous problem is written in the following form

$$y(t) = \frac{\mathfrak{I}_{t_k^+}^{1-\gamma,\psi} y(t_k^+)}{\Gamma(\gamma)} \mathrm{N}_{\gamma-1,\psi}(t, t_k) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \psi'(s) \mathrm{N}_{\alpha-1,\psi}(t, s) f\big(s, y(s), \mathfrak{I}_{t_k}^{\delta,\psi} y(\sigma_k(s))\big) ds.$$

Since ϖ is a solution of Problem $(\mathfrak{S})$ and by utilizing the remark 1, we have

$$\begin{cases} {}^H\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi} \varpi(t) = f(t, \varpi(t), \mathfrak{I}_{t_k^+}^{\delta,\psi} \varpi(\sigma_k(t))) + \varphi(t), & t \in I_k = (t_k, t_{k+1}],\ k = 0, \ldots, m, \\ \Delta_{\gamma,\psi} \varpi|_{t_k} = J_k(\varpi(t_k^-)) + \rho_k,\ k = 0, \ldots, m. \end{cases}$$

Thus, for all $t \in I_k,\ k = 0, \ldots, m$, ϖ is given by

$$\varpi(t) = \frac{I_{t_k^+}^{1-\gamma,\psi}\varpi(t_k^+)}{\Gamma(\gamma)} \mathrm{N}_{\gamma-1,\psi}(t,t_k) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\Big(\varphi(t) + f\big(s,\varpi(s),\mathfrak{I}_{t_k}^{\delta,\psi}\varpi(\sigma_k(s))\big)\Big)ds.$$

Since $\mathfrak{I}_{t_k^+}^{1-\gamma,\psi}y(t_k^+) = \mathfrak{I}_{t_k^+}^{1-\gamma,\psi}\varpi(t_k^+)$. So, for all $t \in (t_k,t_{k+1}],\ k = 0,\ldots,m$, we have

$$\|\varpi(t) - y(t)\| \le \mathfrak{I}_{t_k^+}^{\alpha,\psi}\|f\big(t,\varpi(t),\mathfrak{I}_{t_k}^{\delta,\psi}\varpi(\sigma_k(t))\big) - f\big(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t))\big)\| + \mathfrak{I}_{t_k^+}^{\alpha,\psi}\|\varphi(t)\|.$$

From $(\mathbf{H_2})$, we get

$$\left(1 - \frac{T_\alpha(AT_\lambda + BT^*T_\delta)}{\Gamma(\alpha+1)}\right)\|\varpi - y\|_{\mathbb{PC}_{1-\gamma,\psi}} \le \frac{\varepsilon T_\alpha}{\Gamma(\alpha+1)}\theta.$$

Thus, for all $t \in (c,L]$, we obtain

$$\|\varpi(t) - y(t)\| \le \frac{T_{\gamma-1}T_\alpha}{\Gamma(\alpha+1) - T_\alpha(AT_\lambda + BT^*T_\delta)}(\theta+\zeta)\varepsilon.$$

Thus, the proof is completed. □

5. Example

We pose $\psi(t) = t$, $m = 1$, $t_0 = 0,\ t_1 = 0.5,\ t_2 = 1, \sigma = 1,\ \alpha = \beta = \lambda = \delta = 0.5$ and

$$E = \{(y_1,y_2,\ldots,y_n,\ldots) : \sup_n |y_n| < \infty\},\ \text{with } \|y\| = \sup_n |y_n|.$$

We take the following problem

$$^{H}\mathcal{D}_{t_k}^{\alpha,\beta,\psi}y(t) = \Big(f_n(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t)))\Big)_{n=1}^{\infty},\ t \in (t_k,t_{k+1}] \subset (0,1],\ k = 0,1 \quad (5.1)$$

$$\mathfrak{I}_{0^+}^{1-\gamma,\psi}y(0^+) = (1,0,\ldots,0,\ldots). \quad (5.2)$$

$$\Delta_{\gamma,\psi}y|_{t=\frac{1}{2}} = J_1(y(\tfrac{1}{2}^-)), \quad (5.3)$$

with

$$f_n(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t))) = \frac{\mathfrak{I}_{t_k}^{\delta,\psi}y_n(\sigma_k(t))}{10+nt^2} + \frac{\sqrt{t-t_i}}{10+t+t^2}y_n(t), k = 0,1, n \in \mathbb{N}^* \text{ and}$$

$$J_1(u) = \frac{1}{10}u, \text{ for all } u \in E.$$

We can easily see that $f : (t_k,t_{k+1}] \times E \to E,\ k = 0,1$ and $J_1 : E \to E$ are continuous and there exists $A = B = C = \frac{1}{10}$ such that

$$\|f(t,u,v) - f(t,\overline{u},\overline{v})\| \le A\sqrt{t-t_k}\|u-\overline{u}\| + B\|v-\overline{v}\|, \text{ for all } t \in I_k \text{ and } u,v,\overline{u},\overline{v} \in E \text{ and}$$

$$\|J_1(u)\| = C\|u\|, \text{ for all } u \in E.$$

So, $(\mathbf{H_2})$ and $(\mathbf{H_4})$ are valid. Next, let Ω be a bounded subset of $\mathbb{PC}_{1\gamma,\psi}([0,1])$, we have

$$\vartheta\left(f(t,\Omega(t),\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t)))\right) \leq \frac{1}{10}\left(\sqrt{t-t_k}\vartheta\left(\Omega(t)\right)+\vartheta\left(\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t))\right), t \in I_k\right.$$

and

$$\vartheta\left(J_1(\Omega(t)\right) \leq \frac{1}{10}\left(\Omega(t)\right).$$

Thus, $(\mathbf{H_1})$ and $(\mathbf{H_3})$ are satisfied. A quick calculation gives us

$$\overline{T}_{\gamma-1}^{m}T_{\alpha}\left(mC\Gamma(\alpha+1)+(m+1)(AT_{\lambda}+BT^{*}T_{\delta})\right) < \Gamma(\alpha+1).$$

So, $(\mathbf{H_5})$ holds. By virtue of Theorem 3 or 4 the problem (5.1)-(5.3) has at least one solution. Moreover, from Theorem 5, we have for any constant function $\theta : (0,1] \to [0,\infty)$ and $\zeta > 0$, the problem (5.1)-(5.3) is stable in the sense of Ulam-Hyers-Rassias according to (θ,ζ).

6. Conclusion

In this paper, we study the existence of a solution and its Ulam-Hyears-Rassias stability for certain pantograph fractional integro-differential equations with impulsive conditions. The significance of our work is that these conditions are nonlocal. The future consideration will be to consider non-instantaneous impulsive conditions.

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