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AMPLY ESSENTIAL G-SUPPLEMENTED MODULES

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Abstract. Let M be an $R-$module. If every essential submodule of M has a g-supplement in M, then M is called an essential g-supplemented (or briefly, eg-supplemented) module. If every essential submodule has ample g-supplements in M, then M is called an amply essential g-supplemented (or briefly, amply eg-supplemented) module. In this work, some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of an amply eg-supplemented module are amply eg-supplemented. Let M be a projective and eg-supplemented module. Then every finitely $M-$ generated $R-$module is amply eg-supplemented.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*)submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$. It is clear that every small submodule is generalized small but the converse is not true generally. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M.

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Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement $V^{'}$with $V^{'} \leq V$, we say U has *ample supplements* in M. M is said to be *amply supplemented* if every submodule of M has ample supplements in M. M is said to be *amply essential supplemented* (briefly, *amply e-supplemented*) if every essential submodule of M has ample supplements in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a g-supplement $V^{'}$with $V^{'} \leq V$, we say U has *ample g-supplements* in M. M is said to be *amply g-supplemented* if every submodule of M has ample g-supplements in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The intersection of essential maximal submodules of an R-module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM. If M have no essential maximal submodules, then we denote $Rad_gM = M$. An $R-$module M is said to be *g-semilocal* if M/Rad_gM is semisimple (see [13]). The sum of all simple submodules of an $R-$module M is called the *socle* of M and denoted by $SocM$. Let M be an $R-$module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More details about (amply) supplemented modules are in [2, 9, 15]. More details about (amply) essential supplemented modules are in [8, 12]. More informations about g-small submodules and g-supplemented modules are in [3, 4, 10].

Lemma 1. *Let M be an R -module and $K \leq N \leq M$. If K is a generalized small submodule of N, then K is a generalized small submodule in every submodule of M which contains N.*

Proof. See [3, Lemma 1 (2)]. □

Lemma 2. *Let M be an R -module. Then $Rad_gM = \sum_{L<<_gM} L$.*

Proof. See [3, Lemma 5 and Corollary 5]. □

Lemma 3. *The following assertions are hold.*

(1) *If M is an $R-$module, then $Rm \ll_g M$ for every $m \in Rad_gM$.*
(2) *If $N \leq M$, then $Rad_gN \leq Rad_gM$.*

Proof. See [4, Lemma 3]. □

2. Essential g-Supplemented Modules

Definition 1. Let M be an $R-$module. If every essential submodule of M has a g-supplement in M, then M is called an essential g-supplemented (or briefly, eg-supplemented) module. (See also [6, 7, 11]).

Definition 2 ([11])**.** Let M be an $R-$module and $X \leq M$. If X is a g-supplement of an essential submodule of M, then X is called an eg-supplement submodule in M.

Proposition 1. *Let M be an eg-supplemented module. Then M is g-semilocal.*

Proof. Since M is eg-supplemented, by [11, Proposition 2.5], M/Rad_gM have no proper essential submodules. Then by [15, 21.1], $Soc(M/Rad_gM) = M/Rad_gM$. Hence M/Rad_gM is semisimple and M is g-semilocal. □

Lemma 4. *Let M be an $R-$module, V be an eg-supplement in M and $K \ll_g M$. Then $K \cap V \ll_g V$.*

Proof. Since V is an eg-supplement in M, there exists $U \trianglelefteq M$ such that V is a g-supplement of U in M. Let $K \cap V + T = V$ with $T \trianglelefteq V$. Then $M = U + V = U + T + K \cap V$, and since $K \cap V \leq K \ll_g M$ and $(U + T) \trianglelefteq M$, $U + T = M$. By V being a g-supplement of U in M and $T \trianglelefteq V$, $T = V$. Hence $K \cap V \ll_g V$. □

Corollary 1 ([10, Lemma 2.3])**.** *Let M be an $R-$module, V be an eg-supplement in M and $K \leq V$. Then $K \ll_g M$ if and only if $K \ll_g V$.*

Proof. Clear from Lemma 1 and Lemma 4. □

Corollary 2 ([10, Theorem 2.4])**.** *Let M be an $R-$module and V be an eg-supplement in M. Then $Rad_gV = V \cap Rad_gM$.*

Proof. By Lemma 3, $Rad_gV \leq V \cap Rad_gM$. Let $y \in V \cap Rad_gM$. Then $y \in V$ and $y \in Rad_gM$. Since $y \in Rad_gM$, by Lemma 3, $Ry \ll_g M$ and by Lemma 4, $Ry \ll_g V$. By Lemma 2, $Ry \leq Rad_gV$ and $y \in Rad_gV$. Hence $V \cap Rad_gM \leq Rad_gV$. Therefore, $Rad_gV = V \cap Rad_gM$. □

We can also prove this corollary as follows:

Proof. Since V is an eg-supplement in M, there exists $U \trianglelefteq M$ such that V is a g-supplement of U in M. Here $M = U + V$ and $U \cap V \ll_g V$. Let K be a generalized maximal submodule of V. Since $U \cap V \ll_g V$, by Lemma 2, $U \cap V \leq Rad_gV \leq K$. By $\frac{M}{U+K} = \frac{U+K+V}{U+K} \cong \frac{V}{V\cap(U+K)} = \frac{V}{U\cap V+K} = \frac{V}{K}$ and $U + K \trianglelefteq M$, $U + K$ is a generalized maximal submodule of M and $Rad_gM \leq U + K$. This case $V \cap Rad_gM \leq V \cap (U + K) = U \cap V + K = K$ and $V \cap Rad_gM \leq Rad_gV$. By Lemma 3, $Rad_gV \leq V \cap Rad_gM$. Hence $Rad_gV = V \cap Rad_gM$. □

Lemma 5 ([10, Lemma 2.5])**.** *Let M be an $R-$module, V be a g-supplement of U in M and $K \trianglelefteq V$. Then for $T \leq V$, T is a g-supplement of K in V if and only if T is a g-supplement of $U + K$ in M.*

Lemma 6 ([11, Lemma 2.11])**.** *Let M be an eg-supplemented module. Then every finitely $M-$generated $R-$module is eg-supplemented.*

3. Amply Essential g-Supplemented Modules

Definition 3. Let M be an $R-$module. If every essential submodule of M has ample g-supplements in M, then M is called an amply essential g-supplemented (or briefly, amply eg-supplemented) module. (See also [5]).

Clearly, every amply essential g-supplemented module is essential g-supplemented.

Lemma 7. *Let M be an amply eg-supplemented module. Then M/Rad_gM have no proper essential submodules.*

Proof. Since M is amply eg-supplemented, then M is eg-supplemented. Then by [11, Proposition 2.5], M/Rad_gM have no proper essential submodules. □

Corollary 3. *If M is amply eg-supplemented, then M is g-semilocal.*

Proof. By Lemma 7, M/Rad_gM have no proper essential submodules. Then by [15, 21.1], $Soc(M/Rad_gM) = M/Rad_gM$. Hence M/Rad_gM is semisimple and M is g-semilocal. □

Proposition 2. *Let M be an amply eg-supplemented module. Then every eg-supplement submodule in M is amply eg-supplemented.*

Proof. Let V be an eg supplement submodule in M. Then there exists $U \trianglelefteq M$ such that V is a g-supplement of U in M. Let $V = K + X$ with $K \trianglelefteq V$ and $X \leq V$. Here $M = U + K + X$. Since M is amply eg-supplemented and $U + K \trianglelefteq M$, $U + K$ has a g-supplement T in M such that $T \leq X$. By Lemma 5, T is a g-supplement of K in V. Moreover, $T \leq X$. Hence V is amply eg-supplemented. □

Lemma 8. *Let M be an amply eg-supplemented module. Then every factor module of M is amply eg-supplemented.*

Proof. Let $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K} \trianglelefteq \frac{M}{K}$ and $\frac{M}{K} = \frac{U}{K} + \frac{V}{K}$. Then $U \trianglelefteq M$ and $M = U + V$. Since M is amply eg-supplemented, U has a g-supplement X in M with $X \leq V$. Since $K \leq U$, by [3, Lemma 4], $\frac{X+K}{K}$ is a g-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Moreover, $\frac{X+K}{K} \leq \frac{V}{K}$. Hence $\frac{M}{K}$ is amply eg-supplemented. □

Corollary 4. *Every homomorphic image of an amply eg-supplemented module is amply eg-supplemented.*

Proof. Clear from Lemma 8. □

Lemma 9. *If M is a π-projective and eg-supplemented module, then M is an amply eg-supplemented module.*

Proof. Let $U \trianglelefteq M$, $M = U + V$ and X be a g-supplement of U in M. Here $M = U + X$ and $U \cap X \ll_g X$. Since M is π -projective and $M = U + V$, there exists an R -module homomorphism $f : M \to M$ such that $Imf \subset V$ and $Im(1-f) \subset U$. So, we have $M = f(M) + (1-f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1-f)(x)$ and $(1-f)(x) \in U$, we have $x = a + (1-f)(x) \in U$. Thus $x \in U \cap X$ and so, $a = f(x) \in f(U \cap X)$. Therefore, we have $U \cap f(X) \leq f(U \cap X)$. Since $U \cap X \ll_g X$, $f(U \cap X) \ll_g f(X)$. Hence $U \cap f(X) \ll_g f(X)$ and since $M = U + f(X)$, $f(X)$ is a g-supplement of U in M. Moreover, $f(X) \subset V$. Therefore, M is amply eg-supplemented. □

Corollary 5. *If M is a projective and eg-supplemented module, then M is amply eg-supplemented.*

Proof. Clear from Lemma 9. □

Lemma 10. *Let M be a π-projective $R-$module. If every essential submodule of M is β_g^* equivalent to an eg-supplement submodule in M, then M is amply eg-supplemented. (The definition of β_g^* relation and some properties of this relation are in* [14].*)*

Proof. By [11, Lemma 2.13], M is eg-supplemented. Then by Lemma 9, M is amply eg-supplemented. □

Corollary 6. *Let M be a projective $R-$module. If every essential submodule of M is β_g^* equivalent to an eg-supplement submodule in M, then M is amply eg-supplemented.*

Proof. Clear from Lemma 10. □

Corollary 7. *Let M be a π-projective $R-$module. If every essential submodule of M is β^* equivalent to an eg-supplement submodule in M, then M is amply eg-supplemented. (The definition of β^* relation and some properties of this relation are in* [1].*)*

Proof. Clear from Lemma 10. □

Corollary 8. *Let M be a π-projective $R-$module. If every essential submodule of M lies above an eg-supplement submodule in M, then M is amply eg-supplemented.*

Proof. Clear from Corollary 7. □

Corollary 9. *Let M be a projective $R-$module. If every essential submodule of M is β^* equivalent to an eg-supplement submodule in M, then M is amply eg-supplemented.*

Proof. Clear from Corollary 7. □

Corollary 10. *Let M be a projective $R-$module. If every essential submodule of M lies above an eg-supplement submodule in M, then M is amply eg-supplemented.*

Proof. Clear from Corollary 9. □

Lemma 11. *Let Λ be a finite index set and $\{M_\lambda\}_\Lambda$ be a family of projective $R-$modules. If M_λ is eg-supplemented for every $\lambda \in \Lambda$, then $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is amply eg-supplemented.*

Proof. Since M_λ is eg-supplemented for every $\lambda \in \Lambda$, by [11, Corollary 2.8], $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is eg-supplemented. Since M_λ is projective for every $\lambda \in \Lambda$, by [15, 18.1], $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is projective. Since $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is projective and eg-supplemented, by Corollary 5, $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is amply eg-supplemented. □

Corollary 11. *Let M be a projective $R-$module. If M is eg-supplemented, then $M^{(\Lambda)}$ is amply eg-supplemented for every finite index set Λ.*

Proof. Clear from Lemma 11. □

Corollary 12. *Let M be a projective $R-$module. If M is eg-supplemented, then every finitely $M-$generated $R-$module is amply eg-supplemented.*

Proof. Let N be a finitely $M-$generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is projective and eg-supplemented, by Corollary 11, $M^{(\Lambda)}$ is amply eg-supplemented. Then by Corollary 4, N is amply eg-supplemented. □

Lemma 12. *Let M be an $R-$module. If every submodule of M is eg-supplemented, then M is amply eg-supplemented.*

Proof. Let $U \trianglelefteq M$ and $M = U + V$ with $V \leq M$. Since $U \trianglelefteq M$, $U \cap V \trianglelefteq V$. By hypothesis, V is eg-supplemented. Then $U \cap V$ has a g-supplement X in V. By this, $V = U \cap V + X$ and $U \cap X = U \cap V \cap X \ll_g X$. Then $M = U + V = U + U \cap V + X = U + X$ and $U \cap X \ll_g X$. Moreover, $X \leq V$. Hence M is amply eg-supplemented. □

Proposition 3. *Let R be any ring. Then every $R-$module is eg-supplemented if and only if every $R-$module is amply eg-supplemented.*

Proof. $(\Longrightarrow)$ Let M be any $R-$module. Since every $R-$module is eg-supplemented, every submodule of M is eg-supplemented. Then by Lemma 12, M is amply eg-supplemented.

$(\Longleftarrow)$ Clear. □

Proposition 4. *Let R be a ring. The following assertions are equivalent.*

(i) *${}_RR$ is eg-supplemented.*
(ii) *${}_RR$ is amply eg-supplemented.*
(iii) *Every finitely generated $R-$module is eg-supplemented.*
(iv) *Every finitely generated $R-$module is amply eg-supplemented.*

Proof. $(i) \Longleftrightarrow (ii)$ Clear from Corollary 5, since ${}_RR$ is projective.

$(i) \Longrightarrow (iii)$ Clear from Lemma 6.

$(iii) \Longrightarrow (iv)$ Let M be a finitely generated $R-$module. Since every finitely generated $R-$module is eg-supplemented, ${}_RR$ is eg-supplemented. Then by Corollary 12, M is amply eg-supplemented.

$(iv) \Longrightarrow (i)$ Clear. □

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