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STRONGLY $\oplus - g-$RADICAL SUPPLEMENTED MODULES

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Abstract. In this work, strongly $\oplus - g-$radical supplemented modules are defined and some properties of these modules are investigated. Every ring has unity and every module is unital left module in this work. It is proved that every direct summand of a strongly $\oplus - g-$radical supplemented module is strongly $\oplus - g-$radical supplemented. Let $f : M \longrightarrow N$ be an $R-$module epimorphism and $Ker(f)$ be a direct summand of M. If M is strongly $\oplus - g-$radical supplemented, then N is also strongly $\oplus - g-$radical supplemented. Let M be a strongly $\oplus - g-$ radical supplemented $R-$module and $K \leq M$. If V is a g-radical supplement submodule in M for every g-radical supplement submodule V/K in M/K, then M/K is strongly $\oplus - g-$radical supplemented.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an $R-$module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any $R-$module M, we have $M = M \oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then it is defined $RadM = M$. M is said to be *semilocal* if $M/RadM$ is semisimple. A submodule N of an $R-$module M is called an *essential submodule* of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $K = 0$ for every $K \leq M$ with $N \cap K = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (or briefly, $g-small$) *submodule* of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$ (in [15], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let M be an R -module. M is called a *hollow module* if every proper submodule of M is small in M. M is called a *local module* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a $\oplus-$*supplemented module*. Let M be a supplemented $R-$module. If every supplement submodule of M is a direct summand of M, then M is called a *strongly* $\oplus-$*supplemented module*. Let M be an R $-$module and $U,V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be $\oplus - g-$*supplemented* if every submodule of M has a g-supplement that is a direct summand in M (see [8]). Let M be an R-module and $U,V \leq M$. If $M = U + V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. M is said to be *generalized* (*radical*) $\oplus-$*supplemented* (briefly, *Rad*$-\oplus-$*supplemented*) if every submodule of M has a Rad-supplement that is a direct summand in M. M said to be *strongly Rad*$-\oplus$-*supplemented* if M is Rad-supplemented and every Rad-supplement submodule in M is a direct summand of M (see [12]). The intersection of all essential maximal submodules of an $R-$module M is called the *generalized radical* of M and denoted by Rad_gM (in [15], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM = M$. An $R-$module M is said to be *g-semilocal* if M/Rad_gM is semisimple (see [9]). Let M be an $R-$module and $U,V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_gV$, then V is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of U in M. M is said to be *generalized radical supplemented* (briefly, *g-radical supplemented*) if every submodule of M has a g-radical supplement in M. M is said to be $\oplus - g - Rad-$*supplemented* if every submodule of M has a g-radical supplement that is a direct summand in M (see [10]).

More informations about supplemented modules are in [2, 14]. More results about $\oplus-$supplemented modules are in [4]. More details about strongly $\oplus-$supplemented modules are in [7]. More details about generalized (radical) supplemented modules are in [13]. More details about generalized (radical) $\oplus-$supplemented modules are in [1, 3]. More informations about g-supplemented modules are in [5]. More informations about g-radical supplemented modules are in [6]. Now we will give some important properties of the generalized radical of any module.

Lemma 1. *Let M be an $R-$module. The following conditions hold.*

(1) *$Rad_gM = \sum\limits_{L \ll_g M} L$.*
(2) *$Rm \ll_g M$ for every $m \in Rad_gM$.*
(3) *If $N \leq M$, then $Rad_gN \leq Rad_gM$.*
(4) *If $K, L \leq M$, then $Rad_gK + Rad_gL \leq Rad_g(K+L)$.*
(5) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. Then $f(Rad_gM) \leq Rad_gN$.*
(6) *If $K, L \leq M$, then $\frac{Rad_gK+L}{L} \leq Rad_g\frac{K+L}{L}$.*
(7) *If $M = \bigoplus\limits_{i \in I} M_i$, then $Rad_gM = \bigoplus\limits_{i \in I} Rad_gM_i$.*

Proof. See [6, Lemma 2, Lemma 3 and Lemma 4]. □

2. STRONGLY $\oplus - g-$RADICAL SUPPLEMENTED MODULES

Definition 1. Let M be a g-supplemented $R-$module. If every g-supplement submodule is a direct summand in M, then M is called a strongly $\oplus - g-$ supplemented module.

Definition 2. Let M be a g-radical supplemented $R-$module. If every g-radical supplement submodule is a direct summand in M, then M is called a strongly $\oplus-$ generalized radical supplemented (briefly, strongly $\oplus - g -$ radical supplemented) module. (See also [11])

Clearly we can see that every strongly $\oplus - g-$radical supplemented module is $\oplus - g-$radical supplemented. But the converse of this statement is not true in general (see Example 1, Example 2 and Example 3). Let M be a strongly $\oplus - g-$radical supplemented $R-$module. Then M is g-radical supplemented and by [6, Theorem 1], M is g-semilocal.

Proposition 1. *Let M be a strongly $\oplus - g-$radical supplemented $R-$module. If $Rad_gM \ll_g M$, then M is strongly $\oplus - g-$supplemented.*

Proof. Let $U \leq M$ and V be any g-radical supplement of U in M. Here $M = U + V$ and $U \cap V \leq Rad_gV$. Since M is strongly $\oplus - g-$radical supplemented, V is a direct summand of M. By Lemma 1, $Rad_gV \leq Rad_gM \ll_g M$. Since V is a direct summand of M, we can see that $Rad_gV \ll_g V$. Hence $U \cap V \ll_g V$ and V is a g-supplement of U in M. Therefore, M is stronlgy $\oplus - g-$ supplemented. □

Proposition 2. *Let M be a strongly $\oplus - g-$radical supplemented $R-$module. If $Rad_gM \ll M$, then M is strongly $\oplus-$supplemented.*

Proof. Let $U \leq M$ and V be any g-radical supplement of U in M. Here $M = U + V$ and $U \cap V \leq Rad_gV$. Since M is strongly $\oplus - g-$radical supplemented, V is a direct summand of M. By Lemma 1, $Rad_gV \leq Rad_gM \ll M$. Since V is a direct summand of M, we can see that $Rad_gV \ll V$. Hence $U \cap V \ll V$ and V is a supplement of U in M. Therefore, M is stronlgy $\oplus-$supplemented. □

Lemma 2. *Let $M = M_1 \oplus M_2$ and $X, Y \leq M_2$. Then Y is a g-radical supplement of X in M_2 if and only if Y is a g-radical supplement of $M_1 + X$ in M.*

Proof. ($\Longrightarrow$)Since Y is a g-radical supplement of X in M_2, $M_2 = X + Y$ and $X \cap Y \leq Rad_g Y$. Then $M = M_1 + M_2 = M_1 + X + Y$ and by Modular Law, $(M_1 + X) \cap Y = (M_1 + X) \cap M_2 \cap Y = (M_1 \cap M_2 + X) \cap Y = (0 + X) \cap Y = X \cap Y \leq Rad_g Y$. Hence Y is a g-radical supplement of $M_1 + X$ in M.

($\Longleftarrow$) Since Y is a g-radical supplement of $M_1 + X$ in M, $M = M_1 + X + Y$ and $(M_1 + X) \cap Y \leq Rad_g Y$. Then by Modular Law, $M_2 = M_2 \cap M = M_2 \cap (M_1 + X + Y) = M_1 \cap M_2 + X + Y = 0 + X + Y = X + Y$ and $X \cap Y \leq (M_1 + X) \cap Y \leq Rad_g Y$. Hence Y is a g-radical supplement of X in M_2. □

Lemma 3. *Let M be a strongly $\oplus - g -$radical supplemented module. Then every direct summand of M is strongly $\oplus - g -$radical supplemented.*

Proof. Let N be a direct summand of M and $M = N \oplus K$ with $K \leq M$. Since M is g-radical supplemented, by [6, Lemma 9], M/K is g-radical supplemented. By $\frac{M}{K} = \frac{N \oplus K}{K} \cong \frac{N}{N \cap K} = \frac{N}{0} \cong N$, N is also g-radical supplemented. Let $X \leq N$ and Y be any g-radical supplement of X in N. Since $M = N \oplus K$, by Lemma 2, Y is a g-radical supplement of $K + X$ in M. Since M is strongly $\oplus - g -$radical supplemented, every g-radical supplement submodule in M is a direct summand of M and hence Y is a direct summand of M. By $Y \leq N$, Y is also a direct summand of N. Hence N is strongly $\oplus - g -$radical supplemented. □

Corollary 1. *Let M be a strongly $\oplus - g -$radical supplemented module. Then M/K is strongly $\oplus - g -$radical supplemented for every direct summand K of M.*

Proof. Let K be any direct summand of M and $M = K \oplus X$ with $X \leq M$. By Lemma 3, X is strongly $\oplus - g -$radical supplemented. Then by $M/K = (K + X)/K \cong X/(K \cap X) = X/0 \cong X$, M/K is strongly $\oplus - g -$radical supplemented. □

Corollary 2. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism and $Ker(f)$ be a direct summand of M. If M is strongly $\oplus - g -$radical supplemented, then N is also strongly $\oplus - g -$radical supplemented.*

Proof. By Corollary 1, $M/Ker(f)$ is strongly $\oplus - g -$radical supplemented. Since $M/Ker(f) \cong f(M) = N$, N is also strongly $\oplus - g -$radical supplemented. □

Proposition 3. *Let M be a strongly $\oplus - g -$radical supplemented $R-$module and $K \leq M$. If V is a g-radical supplement submodule in M for every g-radical supplement submodule V/K in M/K, then M/K is strongly $\oplus - g -$ radical supplemented.*

Proof. By [6, Lemma 9], M/K is g-radical supplemented. Let V/K be any g-radical supplement submodule in M/K. By hypothesis, V is a g-radical supplement submodule in M. Since M is strongly $\oplus - g -$radical supplemented, V is a direct summand of M. Then there exists $X \leq M$ with $M = V \oplus X$. Since $M = V \oplus X$,

$M/K = V/K + (X+K)/K$ and $V/K \cap (X+K)/K = (X \cap V + K)/K = (0+K)/K = 0$. Hence $M/K = V/K \oplus (X+K)/K$. Thus M/K is strongly $\oplus - g-$radical supplemented. $\square$

Lemma 4. *Let M be a strongly $\oplus - g-$radical supplemented $R-$module, $K \leq M$ and $K = (K \cap M_1) \oplus (K \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then M/K is $\oplus - g-$radical supplemented.*

Proof. Let $U/K \leq M/K$. Since M is strongly $\oplus - g-$radical supplemented, U has a g-radical supplement V that is a direct summand in M. Here there exists $X \leq M$ such that $M = V \oplus X$. By hypothesis, $K = (K \cap V) \oplus (K \cap X)$. Since V is a g-radical supplement of U in M and $K \leq U$, by [6, Lemma 8], $\frac{V+K}{K}$ is a g-radical supplement of U/K in M/K. Since $M = V \oplus X$, $\frac{M}{K} = \frac{V+K}{K} + \frac{X+K}{K}$. Here $\frac{V+K}{K} \cap \frac{X+K}{K} = \frac{(V+K)\cap(X+K)}{K} = \frac{(V+K)\cap X+K}{K} = \frac{(V+K\cap V+K\cap X)\cap X+K}{K} = \frac{(V+K\cap X)\cap X+K}{K} = \frac{V\cap X+K\cap X+K}{K} = \frac{0+K}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{X+K}{K}$. Thus M/K is $\oplus - g-$radical supplemented. $\square$

Corollary 3. *Let M be a strongly $\oplus - g-$radical supplemented $R-$module, $f : M \longrightarrow N$ be an $R-$module epimorphism with N be an $R-$module and $Ker(f) = (Ker(f) \cap M_1) \oplus (Ker(f) \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then N is $\oplus - g-$radical supplemented.*

Proof. Clear from Lemma 4, since $M/Ker(f) \cong Im(f) = N$. $\square$

Proposition 4. *Let M be a strongly $\oplus - g-$radical supplemented $R-$module and $K \leq M$. If $\frac{X+K}{K}$ is a direct summand of $\frac{M}{K}$ for every direct summand X of M, then $\frac{M}{K}$ is $\oplus - g-$radical supplemented.*

Proof. Let U/K be any submodule of M/K. Since M is strongly $\oplus - g-$ radical supplemented, U has a g-radical supplement X in M and X is a direct summand of M. Since X is a g-radical supplement of U in M and $K \leq U$, by [6, Lemma 8], $\frac{X+K}{K}$ is a g-radical supplement of U/K in M/K. Since X is a direct summand of M, by hypothesis, $\frac{X+K}{K}$ is a direct summand of M/K. Hence M/K is $\oplus - g-$radical supplemented. $\square$

Remark 1. Let M be a Rad-supplemented module. If M is strongly $\oplus - g-$radical supplemented, then M is strongly $Rad - \oplus-$supplemented. But the converse of this statement is not true in general (see Example 2 and Example 3).

Remark 2. Let M be a supplemented module. If M is strongly $\oplus - g-$radical supplemented, then M is strongly $\oplus-$supplemented. But the converse of this statement is not true in general (see Example 1, Example 2 and Example 3).

Example 1. Let M be a local module with $Rad_g(Rad_g M) = Rad_g M = Rad M \neq 0$. Then M is supplemented and since the supplement submodules in M are only 0 and M, M is strongly $\oplus-$ supplemented. Also M is $\oplus - g-$radical supplemented. Here $Rad_g M$ is the largest submodule of M. Since $Rad_g M$ is a g-radical supplement of

M in M and $Rad_g M$ is not a direct summand of M, M is not strongly $\oplus - g-$radical supplemented.

Example 2. Consider the $\mathbb{Z}-$module $\mathbb{Z}_4$. Here $\mathbb{Z}_4$ is local and $Rad_g\left(Rad_g\mathbb{Z}_4\right) = Rad_g\mathbb{Z}_4 = Rad\mathbb{Z}_4 \neq 0$. By Example 1, $\mathbb{Z}_4$ is strongly $\oplus-$supplemented and $\oplus - g-$radical supplemented. But $\mathbb{Z}_4$ is not strongly $\oplus - g-$radical supplemented. Since the Rad-supplement submodules in $\mathbb{Z}_4$ are only 0 and $\mathbb{Z}_4$, $\mathbb{Z}_4$ is strongly $Rad - \oplus-$supplemented.

Example 3. Consider the $\mathbb{Z}-$module $\mathbb{Z}_8$. Here $\mathbb{Z}_8$ is local, $Rad_g\mathbb{Z}_8 = Rad\mathbb{Z}_8 = \left\{\overline{0},\overline{2},\overline{4},\overline{6}\right\}$ and $Rad_g\left(Rad_g\mathbb{Z}_8\right) = \left\{\overline{0},\overline{4}\right\}$. Since the supplement submodules in $\mathbb{Z}_8$ are only 0 and $\mathbb{Z}_8$, $\mathbb{Z}_8$ is strongly $\oplus-$supplemented. Also $\mathbb{Z}_8$ is $\oplus - g-$radical supplemented. Since $\left\{\overline{0},\overline{4}\right\}$ is a g-radical supplement of $\mathbb{Z}_8$ in $\mathbb{Z}_8$ and $\left\{\overline{0},\overline{4}\right\}$ is not a direct summand of $\mathbb{Z}_8$, $\mathbb{Z}_8$ is not strongly $\oplus - g-$radical supplemented. Since the Rad-supplement submodules in $\mathbb{Z}_8$ are only 0 and $\mathbb{Z}_8$, $\mathbb{Z}_8$ is strongly $Rad - \oplus-$supplemented.

REFERENCES

[1] H. Çalışıcı and E. Türkmen, "Generalized $\oplus$-supplemented modules." *Algebra Discrete Math.*, vol. 10, no. 2, pp. 10–18, 2010.

[2] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules. Supplements and projectivity in module theory.* Basel: Birkhäuser, 2006.

[3] Ş. Ecevit, M. T. Koşan, and R. Tribak, "Rad-$\oplus$-supplemented modules and cofinitely Rad-$\oplus$-supplemented modules." *Algebra Colloq.*, vol. 19, no. 4, pp. 637–648, 2012, doi: 10.1142/S1005386712000508.

[4] A. Harmanci, D. Keskin, and P. F. Smith, "On $\oplus$-supplemented modules," *Acta Math. Hung.*, vol. 83, no. 1-2, pp. 161–169, 1999, doi: 10.1023/A:1006627906283.

[5] B. Koşar, C. Nebiyev, and N. Sökmez, "g-supplemented modules," *Ukr. Math. J.*, vol. 67, no. 6, pp. 975–980, 2015, doi: 10.1007/s11253-015-1127-8.

[6] B. Koşar, C. Nebiyev, and A. Pekin, "A generalization of g-supplemented modules," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 345–352, 2019, doi: 10.18514/MMN.2019.2586.

[7] C. Nebiyev and A. Pancar, "On strongly $\oplus$-supplemented modules." *Ukr. Math. J.*, vol. 63, no. 5, pp. 768–775, 2011, doi: 10.1007/s11253-011-0541-9.

[8] C. Nebiyev and H. H. Ökten, "Some Properties of $\oplus - g-$Supplemented Modules," *Miskolc Math. Notes*, (Accepted).

[9] C. Nebiyev and H. H. Ökten, "Weakly g-supplemented modules," *Eur. J. Pure Appl. Math.*, vol. 10, no. 3, pp. 521–528, 2017.

[10] C. Nebiyev and H. B. Özdemir, "$\oplus - g - rad-$supplemented modules," 2020, 9th International Eurasian Conference on Mathematical Science and Applications (IECMSA).

[11] C. Nebiyev and H. B. Özdemir, "Strongly $\oplus - g - rad-$supplemented modules," 2021, 4th International E-Conference on Mathematical Advances and Applications (ICOMAA).

[12] Y. Talebi, A. R. Moniri Hamzekolaee, and A. Tercan, "Goldie-Rad-supplemented modules." *An. Științ. Univ. "Ovidius" Constanța, Ser. Mat.*, vol. 22, no. 3, pp. 205–218, 2014.

[13] Y. Wang and N. Ding, "Generalized supplemented modules." *Taiwanese J. Math.*, vol. 10, no. 6, pp. 1589–1601, 2006, doi: 10.11650/twjm/1500404577.

[14] R. Wisbauer, *Foundations Of Module And Ring Theory. A Handbook For Study And Research.* Philadelphia etc.: Gordon and Breach Science Publishers, 1991, vol. 3.

[15] D. X. Zhou and X. R. Zhang, "Small-essential submodules and Morita duality." *Southeast Asian Bull. Math.*, vol. 35, no. 6, pp. 1051–1062, 2011.

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