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STATE ESTIMATION OF MEMRISTOR-BASED STOCHASTIC NEURAL NETWORKS WITH MIXED VARIABLE DELAYS

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Abstract. This paper studies the state estimation problem for memristor-based stochastic neural networks (MSNNs) with mixed variable delays. A new Lyapunov-Krasovskii functional (LKF) with quadruple integral terms is incorporated. Then, asymptotic stability conditions are established for the error system using a linear matrix inequality technique. The estimator gain can be obtained by solving the linear matrix inequalities. Numerical simulations are given to demonstrate the effectiveness and superiority of the new scheme.

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Keywords: distributed variable delay, memristor-based stochastic neural networks, quadruple integral, state estimation

1. Introduction

The memristor-based neural networks (MNNs) made of hybrid complementary metal-oxide semiconductors have a vast range of applications in bioinspired engineering [5, 7, 8]. One can get a good range of useful applications in new classes of artificial neural systems [8], neural learning circuits [15], associative memories [5] by investigating MNNs. Recently, many research works have been published on MNNs [4, 12]. In the electronic circuit implementation of the neural networks (NNs), time delay plays a vital role due to the finite switching speed of amplifiers [3, 16]. The existence of time delays may cause divergence and instability in system performance [16]. A particular class of time delay exists in engineering systems, called time-varying interval delays, which generally exists in networked control systems [23]. In the previous decade, distributed delays were found in the realistic NN model, and significant research attention was brought to the stability of NNs with discrete and distributed time-varying delays [1, 24]. Thus, analyzing the dynamics of NNs with mixed variable delays has attracted more research attention [4, 12].

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In practice, synaptic transmission is a silent process brought on by the cause of random probabilities in the actual neuron system. The methods in [1, 20] show that some stochastic inputs can destabilize or degrade NNs. So, it is essential to study stochastic perturbations, and their effects on the NNs [18, 22]. In addition, the state estimation problem for stochastic neural networks (SNNs) is a more challenging task than for the deterministic NNs. Thus, it is of great importance to study state estimation problems for NNs with stochasticities [6, 19].

The state estimation problem for NNs has brought up a lot of research interest in the current decade (see [9, 10]). There are some difficulties in estimating the neurons in NNs due to the complicated composition of NNs. Therefore, it is challenging and essential to study the state estimation problem of NNs. In recent years, the state estimation of NNs was proposed in [2, 11, 14, 17]. In [18], synchronization of MSNNs with mixed delays has been proposed. Stability criteria for MSNNs with mixed delays were considered in [14]. State estimation problem for delayed generalized NNs was studied in [16]. Exponential stability of uncertain SNNs with mixed delays was established in [6]. Stability criteria for Markovian jump static SNNs with time delays are considered in [19]. To the author's best knowledge, state estimation of MSNNs with mixed variable delays has not been completely studied yet. Hence, our primary intention is to study the state estimation for delayed MSNNs.

The main contribution of this paper are as follows:

(1) Most of the existing results in the literature consider the state estimation problems for only deterministic cases. Therefore, the state estimation problem for MSNNs with mixed variable delays is considered in this paper. A novel approach is proposed to solve the considered problem.
(2) In this article, a stochastic quadruple integral LKF is proposed to deal with the state estimation problem for MSNNs with mixed time-varying delays.
(3) The quadruple integral term of activation function such as

$$\int_{-\vartheta_1}^{0}\int_{\rho}^{0}\int_{\chi}^{0}\int_{t+\kappa}^{t} e(s)\,ds\,d\kappa\,d\chi\,d\rho \quad \text{and} \quad \int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{\chi}^{0}\int_{t+\kappa}^{t} e(s)\,ds\,d\kappa\,d\chi\,d\rho$$

are utilized in estimating the time derivative of the LKF.

From the above comments, the main focus is on the state estimation problem for MSNNs with mixed variable delays. By establishing a new quadruple stochastic integral LKF and using the linear matrix inequalities (LMIs) technique, a sufficient condition for the feasibility of the state estimation problem has been presented, which assures the asymptotic stability of the estimation error system. At last, an example is provided to reveal the usefulness and superiority of our approach.

Notation 1. $\mathfrak{R}^n$ denotes the n-dimensional Euclidean space, and $\mathfrak{R}^{n\times n}$ denotes the set of all $n\times n$ real matrices. Y^T denotes the matrix transpose of Y. $X > 0$ denotes that X is a real symmetric positive definite matrix. I denotes the identity matrix. $*$

denotes the term that is induced by symmetry. $\mathrm{diag}\{a, b, \ldots, z\}$ denotes the block-diagonal matrix with $a, b, \ldots, z$ in the diagonal entries.

2. Problem statement and Preliminaries

As in [5], the memductance of a memristor is given as $\varpi(\phi) = \frac{d\hat{q}(\phi)}{d\phi}$, where $d\hat{q}(\phi)$ denotes the charge passing through the memristor and ϕ denotes the flux. Since q and ϕ, respectively, represent the time integral of the current $i(t)$ and voltage $\nu(t)$ applied to the memristor; namely $q(t) = \int_{-\infty}^{t} i(\tau)d\tau$, and $\phi(t) = \int_{-\infty}^{t} \nu(\tau)d\tau$ the memductance ϖ is the function of voltage v and $\varpi(v) = \frac{d\hat{q}(\phi)}{d\phi} = \frac{dq/dt}{d\phi/dt} = \frac{i(t)}{\nu(t)}$.

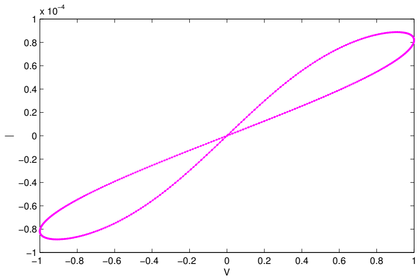

FIGURE 1. Typical current voltage characteristic of memristor with a sinusoidal current source [8].

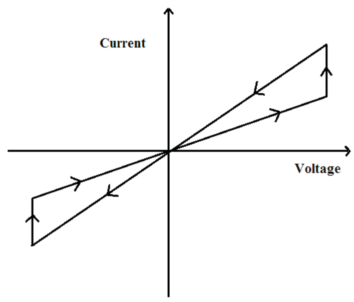

FIGURE 2. Characteristic of the Piece-wise linear model of memristor [8].

The mathematical model of the memductance is presented to accurately describe the memory function and the hysteresis properties of the memory, according to the current-voltage characteristics of the memory in Figure 1:

$$\varpi(\nu(t)) = \begin{cases} \varpi'(\nu(t)), & \nu(s) \downarrow, s \in (t-\rho, t]; \\ \varpi''(\nu(t)), & \nu(s) \uparrow, s \in (t-\rho, t]; \\ \lim_{s \to t^-} \varpi(\nu(s)), & \nu(s) \text{ unchage}, s \in (t-\rho, t], \end{cases} \tag{2.1}$$

where $\uparrow$ means increase, $\downarrow$ means decrease, ρ is a sufficiently small positive constant, $\lim_{s \to t^-}$ is either equal to $\varpi'(\nu(t))$ or $\varpi''(\nu(t))$, and it means that the memductance keeps the voltage value. Hence, the memductance function may be discontinuous. As mentioned in [5], the memristor must display two adequate equilibrium levels, R_0 and R_1, where $R_0 \gg R1$, which makes it easy to switch between high resistance levels. Fast resistance as low as possible, or vice versa when consuming as little energy as possible. Therefore, with this property a memductance of the memristor can be defined as (2.1), where the constants defined by $\varpi'(\nu(t)) = \varpi'$ and $\varpi''(\nu(t)) =$

ϖ''. Figure 2 shows the current-voltage characteristics of this memristor. The use of memristors to change the resistance of the circuit sensor in the connection of the NN called the MNN. Now, consider the MSNN model based on the methods in [8]:

$$\begin{aligned} dx(t) = [&-Ax(t) + W_0(x)f(x(t)) + W_1(x)f(x(t-\vartheta(t))) \\ &+ W_2(x)\int_{t-d(t)}^{t} f(x(s))ds]dt + [Ex(t) + E_1x(t-\vartheta(t)) \\ &+ H_0(x)f(x(t)) + H_1(x)f(x(t-\vartheta(t))) \\ &+ H_2(x)\int_{t-d(t)}^{t} f(x(s))ds]dv(t), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} x(t) &= [x_1(t), x_2(t), \dots, x_n(t)]^T \text{ is the state vector;} \\ A &= \operatorname{diag}\{a_1(t), a_2(t), \dots, a_n(t)\}, \\ W_0(x) &= [w_{0ij}(f_j(x_j(t)) - x_i(t))]_{n\times n}, \\ W_1(x) &= [w_{1ij}(f_j(x_j(t-\vartheta(t))) - x_i(t))]_{n\times n}, \\ W_2(x) &= \left[w_{2ij}\left(\int_{t-d(t)}^{t} f_j(x_j(s))ds - x_i(t)\right)\right]_{n\times n}, \\ H_0(x) &= [h_{0ij}(f_j(x_j(t)) - x_i(t))]_{n\times n}, \\ H_1(x) &= [h_{1ij}(f_j(x_j(t-\vartheta(t))) - x_i(t))]_{n\times n}, \\ H_2(x) &= \left[h_{2ij}(\int_{t-d(t)}^{t} f_j(x_j(s))ds) - x_i(t))\right]_{n\times n} \end{aligned}$$

are the feedback connection weight matrices. $\vartheta(t)$, and $d(t)$ are respectively representing the time-varying and time-varying distributed delays.

Assumption 1. *There exist scalars $\vartheta_1 > 0$, $\vartheta_2 > 0$, $\mu \geq 0$ and $d > 0$ such that $0 \leq \vartheta_1 \leq \vartheta(t) \leq \vartheta_2$, $\dot{\vartheta}(t) \leq \mu$ and $0 \leq d(t) \leq d$, respectively.*

Assumption 2. *Each neuron activation function $f_i(t)$ $(i = 1, 2, \dots, n)$ is continuous and bounded, and satisfies the Lipschitz condition: $|f(x_1) - f(x_2)| \leq F|(x_1 - x_2)|$, with $F \in \mathfrak{R}^{n\times n}$ and $x_1, x_2 \in \mathfrak{R}$, $x_1 \neq x_2$ are constants. The network measurements are assumed to satisfy:*

$$y(t) = Cx(t) + l(t, x(t)),$$

where $y(t) \in \mathfrak{R}^n$ is the network measurement, C is a known constant matrix with appropriate dimension, $l\colon \mathfrak{R} \times \mathfrak{R}^n \to \mathfrak{R}^m$ is the neuron-dependent nonlinear disturbances on the network outputs, and satisfies: $|l(t, x_1) - l(t, x_2)| \leq |L(x_1 - x_2)|$, with $L \in \mathfrak{R}^{n\times n}$. Consider the full-order state estimator ([21]):

$$d\hat{x}(t) = [-A\hat{x}(t) + W_0(x)f(\hat{x}(t)) + W_1(x)f(\hat{x}(t-\vartheta(t)))$$

$$+W_2(x)\int_{t-d(t)}^{t} f(\hat{x}(s))ds + K[y(t) - C\hat{x}(t) - l(t,\hat{x}(t))]]dt + [E\hat{x}(t) + E_1\hat{x}(t-\vartheta(t)) + H_0(x)f(\hat{x}(t)) + H_1(x)f(\hat{x}(t-\vartheta(t))) + H_2(x)\int_{t-d(t)}^{t} f(\hat{x}(s))ds]dv(t), \tag{2.3}$$

where $\hat{x}(t) \in \mathfrak{R}^n$ denotes the estimated state, and $K \in \mathfrak{R}^{n\times m}$ is the estimator gain matrix to be calculated. Define the error state $e(t) = x(t) - \hat{x}(t)$. From (2.2) and (2.3), one can obtain the following error system:

$$\begin{aligned} de(t) = [&-Ae(t) + W_0(x)\varphi(t) + W_1(x)\varphi(t-\vartheta(t)) + W_2(x)\Upsilon_1 \\ &- KC\psi(t)]dt + [Ee(t) + E_1e(t-\vartheta(t)) + H_0(x)\varphi(t) \\ &+ H_1(x)\varphi(t-\vartheta(t)) + H_2(x)\Upsilon_1]dv(t), \end{aligned} \tag{2.4}$$

where $\Upsilon_1 = \int_{t-d(t)}^{t}\varphi(s)ds$, $e(t) = [e_1(t), e_2(t), \ldots, e_n(t)]^T \in \mathfrak{R}^n$ is the state vector of the transformed system and $\varphi(t) = f(x(t)) - f(\widehat{x}(t))$ with $\varphi(t) = [\varphi_1(t), \ldots, \varphi_n(t)]^T$.

Definition 1. For the error-state system (2.4) and every $\varsigma \in L^2_{\mathcal{F}_0}([-\vartheta_2, 0]; \mathcal{R}^n)$ where ϑ_2 is the upper bound of the time delay, the trivial solution is globally asymptotically stable if it is locally stable in the sense of Lyapunov and globally attractive.

3. Main Results

Theorem 1. *For given positive scalars $\vartheta_1, \vartheta_2, \mu, d$ and the matrix K, the error-state system (2.4) of MSNN (2.2) and (2.3) is globally asymptotically mean-square stable if there exist matrices $\mathbb{P}_{5\times 5} > 0$, $R_j > 0$, $(j = 1,2,3)$, $S_p > 0$, $(p = 1,2,3,4,5)$, $T_q > 0$, $Z_q > 0$, $q = (1,2,3,4,5,6)$ and any appropriate dimensional matrices $\mathcal{P}_s$, $\mathcal{Q}_s$, $\mathcal{R}_s$, $\mathcal{S}_s$, $\mathcal{T}_s$, $\mathcal{U}_s$, $\mathcal{V}_s$, $(s = 1,2)$, such that the following LMI hold for $i = 1,2,3$:*

$$\widehat{\Pi}_i = \begin{bmatrix} \Pi_1^i & \Pi_2^i \\ * & \Pi_3^i \end{bmatrix} < 0, \tag{3.1}$$

where

$$\Pi_1^i = \Pi_{r\times s}, \quad (i = 1,2,3,\ r,s = 1,2,\ldots,12),$$

$$\mathbb{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\ * & P_{22} & P_{23} & P_{24} & P_{25} \\ * & * & P_{33} & P_{34} & P_{35} \\ * & * & * & P_{44} & P_{45} \\ * & * & * & * & P_{55} \end{bmatrix}, \quad R_j = \begin{bmatrix} R_{11}^j & R_{12}^j \\ * & R_{22}^j \end{bmatrix}, \quad (j = 1,2,3)$$

$$\begin{aligned} \Pi_{1,1} = &-P_{11}A - A^TP_{11} + P_{12} + P_{12}^T + R_{11}^1 + \vartheta_1S_1 + \vartheta_{21}S_2 + \mathcal{P}_1 + \mathcal{P}_1^T + \vartheta_1\mathcal{S}_1 \\ &+ \vartheta_1\mathcal{S}_1^T + \vartheta_{21}\mathcal{T}_1 + \vartheta_{21}\mathcal{T}_1^T + \frac{\vartheta_1^2}{2}\mathcal{U}_1 + \frac{\vartheta_1^2}{2}\mathcal{U}_1^T + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_1 + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_1^T \\ &+ \varepsilon_1F^TF + \varepsilon_3L^TL, \end{aligned}$$

$$\Pi_{1,2} = \mathcal{P}_2^T - \mathcal{Q}_1 + \mathcal{R}_1 + \vartheta_1\mathcal{S}_2^T + \vartheta_{21}\mathcal{T}_2^T + \frac{\vartheta_1^2}{2}\mathcal{U}_2^T + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_2^T,$$
$$\Pi_{1,3} = -P_{12} + P_{13} - \mathcal{P}_1 + \mathcal{Q}_1, \quad \Pi_{1,4} = -\mathcal{R}_1 - P_{13}, \quad \Pi_{1,5} = P_{11}W_0 + P_{14} + R_{12}^1,$$
$$\Pi_{1,6} = P_{11}W_1, \quad \Pi_{1,7} = P_{15} - P_{14}, \quad \Pi_{1,8} = -P_{15}, \quad \Pi_{1,9} = -P_{11}KC,$$
$$\Pi_{1,10} = P_{11}W_2, \quad \Pi_{1,11} = -A^TP_{14} + P_{24}, \quad \Pi_{1,12} = -A^TP_{15} + P_{25},$$
$$\Pi_{2,2} = -(1-\mu)R_{11}^2 - \mathcal{Q}_2 - \mathcal{Q}_2^T + \mathcal{R}_2 + \mathcal{R}_2^T + \varepsilon_2F^TF, \quad \Pi_{2,3} = -\mathcal{P}_2 + \mathcal{Q}_2,$$
$$\Pi_{2,4} = -\mathcal{R}_2, \quad \Pi_{2,6} = -(1-\mu)R_{12}^2, \quad \Pi_{3,3} = -R_{11}^1 + R_{11}^2 + R_{11}^3,$$
$$\Pi_{3,7} = -R_{12}^1 + R_{12}^2 + R_{12}^3, \quad \Pi_{3,11} = -P_{24} + P_{34}, \quad \Pi_{3,12} = -P_{25} + P_{35},$$
$$\Pi_{4,4} = -R_{11}^3, \quad \Pi_{4,8} = -R_{12}^3, \quad \Pi_{4,11} = -P_{34}, \quad \Pi_{4,12} = -P_{35},$$
$$\Pi_{5,5} = R_{22}^1 + d^2S_3 + \vartheta_1^2S_4 + \vartheta_{21}^2S_5 - \varepsilon_1, \quad \Pi_{5,11} = W_0^TP_{14} + P_{44},$$
$$\Pi_{5,12} = W_0^TP_{15} + P_{45}, \quad \Pi_{6,6} = -(1-\mu)R_{22}^2 - \varepsilon_2, \quad \Pi_{6,11} = W_1^TP_{14},$$
$$\Pi_{6,12} = W_1^TP_{15}, \quad \Pi_{7,7} = -R_{22}^1 + R_{22}^2 + R_{22}^3, \quad \Pi_{7,11} = -P_{44} + P_{45}^T,$$
$$\Pi_{7,12} = -P_{45} + P_{55}, \quad \Pi_{8,8} = -R_{22}^3, \quad \Pi_{8,11} = -P_{45}^T,$$
$$\Pi_{8,12} = -P_{55}, \quad \Pi_{9,9} = -\varepsilon_3, \quad \Pi_{9,11} = -C^TK^TP_{14},$$
$$\Pi_{9,12} = -C^TK^TP_{15}, \quad \Pi_{10,10} = -S_3, \quad \Pi_{10,11} = W_2^TP_{14},$$
$$\Pi_{10,12} = W_2^TP_{15}, \quad \Pi_{11,11} = -S_4, \quad \Pi_{12,12} = -S_5, \quad \vartheta_{21} = \vartheta_2 - \vartheta_1,$$
$$\Pi_2^1 = \Big[\Omega_1 \;\; \Big(-\vartheta_1\widetilde{\mathcal{P}} - \frac{\vartheta_1^2}{2}\widetilde{\mathcal{S}} - \frac{\vartheta_1^3}{6}\widetilde{\mathcal{U}}\Big) \;\; \widehat{E}^TP_{11} \;\; \vartheta_1\widehat{A}^TT_1 \;\; \frac{\vartheta_1^2}{2}\widehat{A}^TT_3 \;\; \frac{\vartheta_1^3}{6}\widehat{A}^TT_5$$
$$\vartheta_1\widehat{E}^TZ_1 \;\; \frac{\vartheta_1^2}{2}\widehat{E}^TZ_3 \;\; \frac{\vartheta_1^3}{6}\widehat{E}^TZ_5 \;\; \widetilde{\mathcal{P}} \;\; \vartheta_1\widetilde{\mathcal{S}} \;\; \frac{\vartheta_1^2}{2}\widetilde{\mathcal{U}}\Big],$$
$$\Pi_2^2 = \Big[\Omega_2 \;\; \Big(-\vartheta_{21}\widetilde{\mathcal{Q}} - \frac{\vartheta_2^2 - \vartheta_1^2}{2}\widetilde{\mathcal{T}} - \frac{\vartheta_2^3 - \vartheta_1^3}{6}\widetilde{\mathcal{V}}\Big) \;\; \widehat{E}^TP_{11} \;\; \vartheta_{21}\widehat{A}^TT_2$$
$$\frac{\vartheta_2^2 - \vartheta_1^2}{2}\widehat{A}^TT_4 \;\; \frac{\vartheta_2^3 - \vartheta_1^3}{6}\widehat{A}^TT_6 \;\; \vartheta_{21}\widehat{E}^TZ_2 \;\; \frac{\vartheta_2^2 - \vartheta_1^2}{2}\widehat{E}^TZ_4 \;\; \frac{\vartheta_2^3 - \vartheta_1^3}{6}\widehat{E}^TZ_6$$
$$\widetilde{\mathcal{Q}} \;\; \vartheta_{21}\widetilde{\mathcal{T}} \;\; \frac{\vartheta_2^2 - \vartheta_1^2}{2}\widetilde{\mathcal{V}}\Big],$$
$$\Pi_2^3 = \Big[\Omega_2 \;\; \Big(-\vartheta_{21}\widetilde{\mathcal{R}} - \frac{\vartheta_2^2 - \vartheta_1^2}{2}\widetilde{\mathcal{T}} - \frac{\vartheta_2^3 - \vartheta_1^3}{6}\widetilde{\mathcal{V}}\Big) \;\; \widehat{E}^TP_{11} \;\; \vartheta_{21}\widehat{A}^TT_2$$
$$\frac{\vartheta_2^2 - \vartheta_1^2}{2}\widehat{A}^TT_4 \;\; \frac{\vartheta_2^3 - \vartheta_1^3}{6}\widehat{A}^TT_6 \;\; \vartheta_{21}\widehat{E}^TZ_2 \;\; \frac{\vartheta_2^2 - \vartheta_1^2}{2}\widehat{E}^TZ_4 \;\; \frac{\vartheta_2^3 - \vartheta_1^3}{6}\widehat{E}^TZ_6$$
$$\widetilde{\mathcal{R}}\vartheta_{21}\widetilde{\mathcal{T}} \quad \frac{\vartheta_2^2 - \vartheta_1^2}{2}\widetilde{\mathcal{V}}\Big],$$

$$\Pi_3^1 = \Big[-\vartheta_1 S_1 \quad \Big(-\vartheta_1 T_1 - \frac{\vartheta_1^2}{2} T_3 - \frac{\vartheta_1^3}{6} T_5\Big) \quad -P_{11} \quad -\vartheta_1 T_1 \quad -\frac{\vartheta_1^2}{2} T_3 \quad -\frac{\vartheta_1^3}{6} T_5$$
$$-\vartheta_1 Z_1 \quad -\frac{\vartheta_1^2}{2} Z_3 \quad -\frac{\vartheta_1^3}{6} Z_5 \quad -Z_1 \quad -\vartheta_1 Z_3 \quad -\frac{\vartheta_1^2}{2} Z_5\Big],$$
$$\Pi_3^2 = \Big[-\vartheta_{21} S_2 \quad \Big(-\vartheta_1 T_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2} T_4 - \frac{\vartheta_2^3-\vartheta_1^3}{6} T_6\Big) \quad -P_{11} \quad -\vartheta_{21} T_2$$
$$-\frac{\vartheta_2^2-\vartheta_1^2}{2} T_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6} T_6 \quad -\vartheta_{21} Z_2 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2} Z_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6} Z_6$$
$$-Z_2 \quad -\vartheta_{21} Z_4 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2} Z_6\Big],$$
$$\Pi_3^3 = \Big[-\vartheta_{21} S_2 \quad \Big(-\vartheta_{21} T_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2} T_4 - \frac{\vartheta_2^3-\vartheta_1^3}{6} T_6\Big) \quad -P_{11} \quad -\vartheta_{21} T_2$$
$$-\frac{\vartheta_2^2-\vartheta_1^2}{2} T_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6} T_6 \quad -\vartheta_{21} Z_2 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2} Z_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6} Z_6$$
$$-Z_2 \quad -\vartheta_{21} Z_4 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2} Z_6\Big],$$
$$\Omega_1 = \Big[\Big(\vartheta_1 P_{22} - \vartheta_1 A^T P_{12} - \vartheta_1 \mathcal{S}_1 - \frac{\vartheta_1^2}{2}\mathcal{U}_1\Big) \quad \Big(-\vartheta_1 \mathcal{S}_2 - \frac{\vartheta_1^2}{2}\mathcal{U}_2\Big)$$
$$\big(-\vartheta_1 P_{22} + \vartheta_1 P_{23}^T\big) \quad -\vartheta_1 P_{23}^T \quad \big(\vartheta_1 W_0^T P_{12} + \vartheta_1 P_{24}^T\big) \quad \vartheta_1 W_1^T P_{12}$$
$$\big(-\vartheta_1 P_{24}^T + \vartheta_1 P_{25}^T\big) \quad -\vartheta_1 P_{25}^T \quad -\vartheta_1 C^T K^T P_{12} \quad \vartheta_1 W_2^T P_{12} \quad 0 \quad 0\big],$$
$$\Omega_2 = \Big[\Big(\vartheta_{21} P_{23} - \vartheta_{21} A^T P_{13} - \vartheta_{21} \mathcal{T}_1 - \frac{\vartheta_2^2-\vartheta_1^2}{2}\mathcal{V}_1\Big) \quad \Big(-\vartheta_{21} \mathcal{T}_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2}\mathcal{V}_2\Big)$$
$$\big(-\vartheta_{21} P_{23} + \vartheta_{21} P_{33}\big) \quad -\vartheta_{21} P_{33} \quad \big(\vartheta_{21} W_0^T P_{13} + \vartheta_{21} P_{34}^T\big)$$
$$\vartheta_{21} W_1^T P_{13} \quad \big(-\vartheta_{21} P_{34}^T + \vartheta_{21} P_{35}^T\big) \quad -\vartheta_{21} P_{35}^T \quad -\vartheta_{21} C^T K^T P_{13}$$
$$\vartheta_{21} W_2^T P_{13} \quad 0 \quad 0\big],$$
$$\widehat{E} = [E \quad E_1 \quad 0 \quad 0 \quad H_0 \quad H_1 \quad 0 \quad 0 \quad 0 \quad H_2 \quad 0 \quad 0],$$
$$\widehat{A} = [A \quad 0 \quad 0 \quad 0 \quad W_0 \quad W_1 \quad 0 \quad 0 \quad -KC \quad W_2 \quad 0 \quad 0],$$
$$\widetilde{\mathcal{P}}^T = \big[\mathcal{P}_1^T \quad \mathcal{P}_2^T \quad 0_{1\times 10}\big], \quad \widetilde{\mathcal{Q}}^T = \big[\mathcal{Q}_1^T \quad \mathcal{Q}_2^T \quad 0_{1\times 10}\big],$$
$$\widetilde{\mathcal{R}}^T = \big[\mathcal{R}_1^T \quad \mathcal{R}_2^T \quad 0_{1\times 10}\big], \quad \widetilde{\mathcal{S}}^T = \big[\mathcal{S}_1^T \quad \mathcal{S}_2^T \quad 0_{1\times 10}\big], \quad \widetilde{\mathcal{T}}^T = \big[\mathcal{T}_1^T \quad \mathcal{T}_2^T \quad 0_{1\times 10}\big],$$
$$\widetilde{\mathcal{U}}^T = \big[\mathcal{U}_1^T \quad \mathcal{U}_2^T \quad 0_{1\times 10}\big], \quad \widetilde{\mathcal{V}}^T = \big[\mathcal{V}_1^T \quad \mathcal{V}_2^T \quad 0_{1\times 10}\big].$$

Proof. Choose the LKF:

$$V(e_t)=\sum_{p=1}^{5}V_p(e_t), \tag{3.2}$$

where

$$\begin{aligned}
V_1(e_t)&=\eta_1^T(t)\mathbb{P}\eta_1(t),\\
V_2(e_t)&=\int_{t-\vartheta_1}^{t}\eta_2^T(s)R_1\eta_2(s)ds+\int_{t-\vartheta(t)}^{t-\vartheta_1}\eta_2^T(s)R_2\eta_2(s)ds+\int_{t-\vartheta_2}^{t-\vartheta_1}\eta_2^T(s)R_3\eta_2(s)ds,\\
V_3(e_t)&=\Upsilon_2 e^T(s)S_1e(s)dsd\rho+\Upsilon_3e^T(s)S_2e(s)dsd\rho+d\int_{-d}^{0}\int_{t+\rho}^{t}\varphi^T(s)S_3\varphi(s)dsd\rho\\
&\quad+\vartheta_1\Upsilon_2\varphi^T(s)S_4\varphi(s)dsd\rho+\vartheta_{21}\Upsilon_3\varphi^T(s)S_5\varphi(s)dsd\rho,\\
V_4(e_t)&=\Upsilon_2\mathfrak{z}^T(s)T_1\mathfrak{z}(s)dsd\rho+\Upsilon_3\mathfrak{z}^T(s)T_2\mathfrak{z}(s)dsd\rho+\Upsilon_4\mathfrak{z}^T(s)T_3\mathfrak{z}(s)dsd\chi d\rho\\
&\quad+\Upsilon_5\mathfrak{z}^T(s)T_4\mathfrak{z}(s)dsd\chi d\rho+\Upsilon_6\mathfrak{z}^T(s)T_5\mathfrak{z}(s)dsd\kappa d\chi d\rho\\
&\quad+\Upsilon_7\mathfrak{z}^T(s)T_6\mathfrak{z}(s)dsd\kappa d\chi d\rho,\\
V_5(e_t)&=\Upsilon_2\mathfrak{y}^T(s)Z_1\mathfrak{y}(s)dsd\rho+\Upsilon_3\mathfrak{y}^T(s)Z_2\mathfrak{y}(s)dsd\rho+\Upsilon_4\mathfrak{y}^T(s)Z_3\mathfrak{y}(s)dsd\chi d\rho\\
&\quad+\Upsilon_5\mathfrak{y}^T(s)Z_4\mathfrak{y}(s)dsd\chi d\rho+\Upsilon_6\mathfrak{y}^T(s)Z_5\mathfrak{y}(s)dsd\kappa d\chi d\rho\\
&\quad+\Upsilon_7\mathfrak{y}^T(s)Z_6\mathfrak{y}(s)dsd\kappa d\chi d\rho,
\end{aligned}$$

and

$$\begin{aligned}
\eta_1(t)&=\left[e^T(t)\ \ \int_{t-\vartheta_1}^{t}e^T(s)ds\ \ \int_{t-\vartheta_2}^{t-\vartheta_1}e^T(s)ds\ \ \int_{t-\vartheta_1}^{t}\varphi^T(s)ds\ \ \int_{t-\vartheta_2}^{t-\vartheta_1}\varphi^T(s)ds\right]^T,\\
\eta_2(t)&=\left[e^T(s)\ \ \varphi^T(s)\right]^T,\\
\mathfrak{z}(s)&=-Ae(t)+W_0\varphi(t)+W_1\varphi(t-\vartheta(t))+W_2\Upsilon_1-KC\psi(t),\\
\mathfrak{y}(s)&=Ee(t)+E_1e(t-\vartheta(t))+H_0\varphi(t)+H_1\varphi(t-\vartheta(t))+H_2\Upsilon_1,
\end{aligned}$$

$$\begin{aligned}
\Upsilon_2&=\int_{-\vartheta_1}^{0}\int_{t+\rho}^{t}, &\quad \Upsilon_3&=\int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\rho}^{t}, &\quad \Upsilon_4&=\int_{-\vartheta_1}^{0}\int_{\rho}^{0}\int_{t+\chi}^{t},\\
\Upsilon_5&=\int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{t+\chi}^{t}, &\quad \Upsilon_6&=\int_{-\vartheta_1}^{0}\int_{\rho}^{0}\int_{\chi}^{0}\int_{t+\kappa}^{t}, &\quad \Upsilon_7&=\int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{\chi}^{0}\int_{t+\kappa}^{t}.
\end{aligned}$$

Using Ito's formula ([13]), the generator $\mathcal{L}V$ for the evolution of $V(x_t)$ in (3.2) is therefore given by

$$dV(e_t) = \mathcal{L}V(x_t)dt + 2[e^T(t)P_{11}\mathfrak{y}(t) + \int_{t-\vartheta_1}^{t} e^T(s)dsP_{12}^T\mathfrak{y}(t) + \int_{t-\vartheta_2}^{t-\vartheta_1} e^T(s)dsP_{13}^T\mathfrak{y}(t)$$
$$+ \int_{t-\vartheta_1}^{t} \varphi^T(s)dsP_{14}^T\mathfrak{y}(t) + \int_{t-\vartheta_2}^{t-\vartheta_1} \varphi^T(s)dsP_{15}^T\mathfrak{y}(t)]dv(t), \tag{3.3}$$

where

$$\mathcal{L}V(e_t) = \sum_{p=1}^{5} \mathcal{L}V_p(x_t),$$

$$\mathcal{L}V_1(e_t) = 2\left[e^T(t) \quad \int_{t-\vartheta_1}^{t} e^T(s)ds \quad \int_{t-\vartheta_2}^{t-\vartheta_1} e^T(s)ds \quad \int_{t-\vartheta_1}^{t} \varphi^T(s)ds \quad \int_{t-\vartheta_2}^{t-\vartheta_1} \varphi^T(s)ds\right]$$
$$\mathbb{P} \times \big[\mathfrak{z}(t)^T \quad e(t)^T - e(t-\vartheta_1)^T \quad e(t-\vartheta_1)^T - e(t-\vartheta_2)^T$$
$$\varphi(t)^T - \varphi(t-\vartheta_1)^T \quad \varphi(t-\vartheta_1)^T - \varphi(t-\vartheta_2)^T\big]^T + \mathfrak{y}^T(t)P_{11}\mathfrak{y}(t),$$

$$\mathcal{L}V_2(e_t) \leq \eta_2^T(t)R_1\eta_2(t) - \eta_2^T(t-\vartheta_1)[-R_1 + R_2 + R_3]\eta_2(t-\vartheta_1)$$
$$- (1-\mu)\eta_2^T(t-\vartheta(t))R_2\eta_2(t-\vartheta(t)) - \eta_2^T(t-\vartheta_2)R_3\eta_2(t-\vartheta_2),$$

$$\mathcal{L}V_3(e_t) = e^T(t)[\vartheta_1 S_1 + \vartheta_{21}S_2]e(t) - \int_{t-\vartheta_1}^{t} e^T(s)S_1e(s)ds - \int_{t-\vartheta_2}^{t-\vartheta_1} e^T(s)S_2e(s)ds$$
$$+ \varphi^T(t)[d^2S_3 + \vartheta_1^2S_4 + \vartheta_{21}^2S_5]\varphi(t) - d\int_{t-d}^{t} \varphi^T(s)S_3\varphi(s)ds$$
$$- \vartheta_1 \int_{t-\vartheta_1}^{t} \varphi^T(s)S_4\varphi(s)ds - \vartheta_{21}\int_{t-\vartheta_2}^{t-\vartheta_1} \varphi^T(s)S_5\varphi(s)ds,$$

$$\mathcal{L}V_4(e_t) = \mathfrak{z}^T(t)[\vartheta_1 T_1 + \vartheta_{21}T_2 + \left(\frac{\vartheta_1^2}{2}\right)T_3 + \left(\frac{\vartheta_2^2 - \vartheta_1^2}{2}\right)T_4 + \left(\frac{\vartheta_1^3}{6}\right)T_5$$
$$+ \left(\frac{\vartheta_2^3 - \vartheta_1^3}{6}\right)T_6]\mathfrak{z}(t) - \int_{t-\vartheta_1}^{t} \mathfrak{z}^T(s)T_1\mathfrak{z}(s)ds - \int_{t-\vartheta_2}^{t-\vartheta_1} \mathfrak{z}^T(s)T_2\mathfrak{z}(s)ds$$
$$- \Upsilon_2\mathfrak{z}^T(s)T_3\mathfrak{z}(s)dsd\rho - \Upsilon_3\mathfrak{z}^T(s)T_4\mathfrak{z}(s)dsd\rho$$
$$- \Upsilon_4\mathfrak{z}^T(s)T_5\mathfrak{z}(s)dsd\chi d\rho - \Upsilon_5\mathfrak{z}^T(s)T_6\mathfrak{z}(s)dsd\chi d\rho,$$

$$\mathcal{L}V_5(e_t) = \mathfrak{y}^T(t)[\vartheta_1 Z_1 + \vartheta_{21}Z_2 + \left(\frac{\vartheta_1^2}{2}\right)Z_3 + \left(\frac{\vartheta_2^2 - \vartheta_1^2}{2}\right)Z_4 + \left(\frac{\vartheta_1^3}{6}\right)Z_5$$
$$+ \left(\frac{\vartheta_2^3 - \vartheta_1^3}{6}\right)Z_6]\mathfrak{y}(t) - \int_{t-\vartheta_1}^{t} \mathfrak{y}^T(s)Z_1\mathfrak{y}(s)ds - \int_{t-\vartheta_2}^{t-\vartheta_1} \mathfrak{y}^T(s)Z_2\mathfrak{y}(s)ds$$
$$- \Upsilon_2\mathfrak{y}^T(s)Z_3\mathfrak{y}(s)dsd\rho - \Upsilon_3\mathfrak{y}^T(s)Z_4\mathfrak{y}(s)dsd\rho$$
$$- \Upsilon_4\mathfrak{y}^T(s)Z_5\mathfrak{y}(s)dsd\chi d\rho - \Upsilon_5\mathfrak{y}^T(s)Z_6\mathfrak{y}(s)dsd\chi d\rho.$$

By using Jensen's inequality implies

$$-d\int_{t-d}^{t}\varphi^T(s)S_3\varphi(s)ds \le -\Upsilon_1^T S_3\Upsilon_1,$$
$$-\vartheta_1\int_{t-\vartheta_1}^{t}\varphi^T(s)S_4\varphi(s)ds \le -\Upsilon_8^T S_4\Upsilon_8,$$
$$-\vartheta_{21}\int_{t-\vartheta_2}^{t-\vartheta_1}\varphi^T(s)S_5\varphi(s)ds \le -\Upsilon_9^T S_5\Upsilon_9,$$

where $\Upsilon_8 = \int_{t-\vartheta_1}^{t}\varphi(s)ds$, $\Upsilon_9 = \int_{t-\vartheta_2}^{t-\vartheta_1}\varphi(s)ds$. To derive the stability criteria, zero equations with any matrices $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$ and $\mathcal{V}$ of appropriate dimensions, to be chosen as follows,

$$0 = 2\eta_3^T(t)\mathcal{P}[e(t)-e(t-\vartheta_1)-\int_{t-\vartheta_1}^{t}\mathfrak{z}(s)ds-\int_{t-\vartheta_1}^{t}\mathfrak{y}(s)dv(s)],$$
$$0 = 2\eta_3^T(t)\mathcal{Q}[e(t-\vartheta_1)-e(t-\vartheta(t))-\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{z}(s)ds-\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{y}(s)dv(s)],$$
$$0 = 2\eta_3^T(t)\mathcal{R}[e(t-\vartheta(t))-e(t-\vartheta_2)-\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{z}(s)ds-\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{y}(s)dv(s)],$$
$$0 = 2\eta_3^T(t)\mathcal{S}[\vartheta_1 e(t)-\int_{t-\vartheta_1}^{t}e(s)ds-\Upsilon_2\mathfrak{z}(s)dsd\rho-\Upsilon_2\mathfrak{y}(s)dv(s)d\rho],$$
$$0 = 2\eta_3^T(t)\mathcal{T}[\vartheta_{21} e(t)-\int_{t-\vartheta_2}^{t-\vartheta_1}e(s)ds-\Upsilon_3\mathfrak{z}(s)dsd\rho-\Upsilon_3\mathfrak{y}(s)dv(s)d\rho],$$
$$0 = 2\eta_3^T(t)\mathcal{U}[\left(\frac{\vartheta_1^2}{2}\right)e(t)-\Upsilon_2 e(s)dsd\rho-\Upsilon_4\mathfrak{z}(s)dsd\chi d\rho-\Upsilon_4\mathfrak{y}(s)dv(s)d\chi d\rho],$$
$$0 = 2\eta_3^T(t)\mathcal{V}[\left(\frac{\vartheta_2^2-\vartheta_1^2}{2}\right)e(t)-\Upsilon_3 e(s)dsd\rho-\Upsilon_5\mathfrak{z}(s)dsd\chi d\rho-\Upsilon_5\mathfrak{y}(s)dv(s)d\chi d\rho],$$

where

$$\eta_3^T(t) = \left[e^T(t)\ \ e^T(t-\vartheta(t))\right], \qquad \mathcal{P} = \left[\mathcal{P}_1^T\ \ \mathcal{P}_2^T\right]^T, \qquad \mathcal{Q} = \left[\mathcal{Q}_1^T\ \ \mathcal{Q}_2^T\right]^T,$$
$$\mathcal{R} = \left[\mathcal{R}_1^T\ \ \mathcal{R}_2^T\right]^T, \qquad \mathcal{S} = \left[\mathcal{S}_1^T\ \ \mathcal{S}_2^T\right]^T, \qquad \mathcal{T} = \left[\mathcal{T}_1^T\ \ \mathcal{T}_2^T\right]^T,$$
$$\mathcal{U} = \left[\mathcal{U}_1^T\ \ \mathcal{U}_2^T\right]^T, \qquad \mathcal{V} = \left[\mathcal{V}_1^T\ \ \mathcal{V}_2^T\right]^T.$$

It is clear that

$$-2\eta_3^T(t)\mathcal{P}\int_{t-\vartheta_1}^{t}\mathfrak{y}(s)dv(s)$$
$$\le \eta_3^T(t)\mathcal{P}Z_1^{-1}\mathcal{P}^T\eta_3(t)+[\int_{t-\vartheta_1}^{t}\mathfrak{y}(s)dv(s)]^T Z_1[\int_{t-\vartheta_1}^{t}\mathfrak{y}(s)dv(s)],$$
$$-2\eta_3^T(t)\mathcal{Q}\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{y}(s)dv(s)$$

$$\leq \eta_3^T(t)\mathcal{Q}Z_2^{-1}\mathcal{Q}^T\eta_3(t)+[\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{y}(s)dv(s)]^TZ_2[\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{y}(s)dv(s)],$$

$$-2\eta_3^T(t)\mathcal{R}\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{y}(s)dv(s)$$

$$\leq \eta_3^T(t)\mathcal{R}Z_2^{-1}\mathcal{R}^T\eta_3(t)+[\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{y}(s)dv(s)]^TZ_2[\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{y}(s)dv(s)],$$

$$-2\eta_3^T(t)\mathcal{S}\Upsilon_2\mathfrak{y}(s)dv(s)d\rho \leq \vartheta_1\eta_3^T(t)\mathcal{S}Z_3^{-1}\mathcal{S}^T\eta_3(t)+\int_{-\vartheta_1}^{0}\Upsilon_{10}^TZ_3\Upsilon_{10}d\rho,$$

$$-2\eta_3^T(t)\mathcal{T}\Upsilon_3\mathfrak{y}(s)dv(s)d\rho$$

$$\leq \vartheta_{21}\eta_3^T(t)\mathcal{T}Z_4^{-1}\mathcal{T}^T\eta_3(t)+\int_{-\vartheta_1}^{0}[\int_{-\vartheta_2}^{-\vartheta_1}\mathfrak{y}(s)dv(s)]^TZ_4[\int_{-\vartheta_2}^{-\vartheta_1}\mathfrak{y}(s)dv(s)]d\rho,$$

$$-2\eta_3^T(t)\mathcal{U}\Upsilon_4\mathfrak{y}(s)dv(s)d\chi d\rho$$

$$\leq \left(\frac{\vartheta_1^2}{2}\right)\eta_3^T(t)\mathcal{U}Z_5^{-1}\mathcal{U}^T\eta_3(t)+\int_{-\vartheta_1}^{0}\int_{\rho}^{0}\Upsilon_{10}^TZ_5\Upsilon_{10}d\chi d\rho,$$

$$-2\eta_3^T(t)\mathcal{V}\Upsilon_5\mathfrak{y}(s)dv(s)d\chi d\rho$$

$$\leq \left(\frac{\vartheta_2^2-\vartheta_1^2}{2}\right)\eta_3^T(t)\mathcal{V}Z_6^{-1}\mathcal{V}^T\eta_3(t)+\int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\Upsilon_{10}^TZ_6\Upsilon_{10}d\chi d\rho.$$

where $\Upsilon_{10}=\int_{t+\rho}^{t}\mathfrak{y}(s)dv(s)$. From inequalities in Assumption (A2) and in (2.3), one can have

$$\begin{aligned}
&0\leq \varepsilon_1[e^T(t)F^TFe(t)-\varphi^T(t)\varphi(t)],\\
&0\leq \varepsilon_2[e^T(t-\vartheta(t))F^TFe(t-\vartheta(t))-\varphi^T(t-\vartheta(t))\varphi(t-\vartheta(t))],\\
&0\leq \varepsilon_3[e^T(t)L^TLe(t)-\psi^T(t)\psi(t)].
\end{aligned}\tag{3.4}$$

Define

$$\xi^T(t)=\big[e^T(t)\;\; e^T(t-\vartheta(t))\;\; e^T(t-\vartheta_1)\;\; e^T(t-\vartheta_2)\;\; \varphi^T(t)\;\; \varphi^T(t-\vartheta(t))\;\; \varphi^T(t-\vartheta_1)\;\; \varphi^T(t-\vartheta_2)\;\; \psi^T(t)\;\; \Upsilon_1^T\;\; \Upsilon_8^T\;\; \Upsilon_9^T\big].$$

Combining (3.3) to (3.4) and applying Schur complement and the $It\widehat{o}$ isometry ([13]), one can deduce that

$$\begin{aligned}
\mathcal{L}V(x_t)\leq{}& \frac{6}{\vartheta_1^4}\int_{-\vartheta_1}^{0}\int_{\rho}^{0}\int_{t+\chi}^{t}\int_{-\vartheta_1}^{t}\xi^T(s)\widehat{\Pi}_1\xi(s)dsd\kappa d\chi d\rho+\left(\frac{6}{\vartheta_{21}(\vartheta_2^3-\vartheta_1^3)}\right)\\
&\times\int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{t+\chi}^{t}\int_{t-\vartheta(t)}^{t-\vartheta_1}\xi^T(s)\widehat{\Pi}_2\xi(s)dsd\sigma d\chi d\rho+\left(\frac{6}{\vartheta_{21}(\vartheta_2^3-\vartheta_1^3)}\right)\\
&\times\int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{t+\chi}^{t}\int_{t-\vartheta_2}^{t-\vartheta(t)}\xi^T(s)\widehat{\Pi}_3\xi(s)dsd\sigma d\chi d\rho.
\end{aligned}\tag{3.5}$$

From (3.5) and Schur's complement Lemma, it can be seen that the error system (2.4) is asymptotically stable. □

Theorem 2. *For given positive scalars $\vartheta_1,\vartheta_2,\mu,d$, the error-state system (2.4) of MSNN (2.2) and (2.3) is asymptotically stable if there exist matrices $\mathbb{P}>0$, $R_j>0$, $(j=1,2,3)$, $S_p>0$, $(p=1,2,3,4,5)$, $T_q>0$, $Z_q>0$, $q=(1,2,3,4,5,6)$, $G>0$, and any appropriate dimensional matrices $\mathcal{P}_s$, $\mathcal{Q}_s$, $\mathcal{R}_s$, $\mathcal{S}_s$, $\mathcal{T}_s$, $\mathcal{U}_s$, $\mathcal{V}_s$, $(s=1,2)$, such that the following LMI hold for $i=1,2,3$:*

$$\widehat{\Xi}_i=\begin{bmatrix}\Xi_1 & \Xi_2^i\\ * & \Xi_3^i\end{bmatrix}<0, \tag{3.6}$$

where

$$\Xi_1=\Xi_{r\times s}\quad \text{where } i=1,2,3,\ r,s\ =1,2,\ldots,12,$$

$$\begin{aligned}\Xi_{1,1}={}&-P_{11}A-A_{11}^T+\alpha_2P_{11}+R_{11}^1+\vartheta_1S_1+\vartheta_{21}S_2+\mathcal{P}_1+\mathcal{P}_1^T+\vartheta_1\mathcal{S}_1+\vartheta_1\mathcal{S}_1^T\\&+\vartheta_{21}\mathcal{T}_1+\vartheta_{21}\mathcal{T}_1^T+\frac{\vartheta_1^2}{2}\mathcal{U}_1+\frac{\vartheta_1^2}{2}\mathcal{U}_1^T+\frac{\vartheta_2^2-\vartheta_1^2}{2}\mathcal{V}_1+\frac{\vartheta_2^2-\vartheta_1^2}{2}\mathcal{V}_1^T\\&+\varepsilon_1F^TF+\varepsilon_3L^TL,\end{aligned}$$

$$\Xi_{1,2}=\Pi_{1,2},\quad \Xi_{1,3}=\alpha_3P_{11}-\alpha_2P_{11}-\mathcal{P}_1+\mathcal{Q}_1,\quad \Xi_{1,4}=-\mathcal{R}_1-\alpha_3P_{11},$$

$$\Xi_{1,5}=P_{11}W_0+\alpha_4P_{11}+R_{12}^1,\quad \Xi_{1,6}=P_{11}W_1,\quad \Xi_{1,7}=\alpha_5P_{11}-\alpha_4P_{11},$$

$$\Xi_{1,8}=-\alpha_5P_{11},\quad \Xi_{1,9}=-GC,\quad \Xi_{1,10}=P_{11}W_2,\quad \Xi_{1,11}=-\alpha_4A^TP_{11}+P_{24},$$

$$\Xi_{1,12}=-\alpha_5A^TP_{11}+P_{25},\quad \Xi_{2,2}=\Pi_{2,2},\quad \Xi_{2,3}=\Pi_{2,3},\quad \Xi_{2,4}=\Pi_{2,4},$$

$$\Xi_{2,6}=\Pi_{2,6},\quad \Xi_{3,3}=\Pi_{3,3},\quad \Xi_{3,7}=\Pi_{3,7},\quad \Xi_{3,11}=\Pi_{3,11},\quad \Xi_{3,12}=\Pi_{3,12},$$

$$\Xi_{4,4}=-R_{11}^3,\quad \Xi_{4,8}=-R_{12}^3,\quad \Xi_{4,11}=-P_{34},\quad \Xi_{4,12}=-P_{35},$$

$$\Xi_{5,5}=R_{22}^1+d^2S_3+\vartheta_1^2S_4+\vartheta_{21}^2S_5-\varepsilon_1,\quad \Xi_{5,11}=\alpha_4W_0^TP_{11}+P_{44},$$

$$\Xi_{5,12}=\alpha_5W_0^TP_{11}+P_{45},\quad \Xi_{6,6}=-(1-\mu)R_{22}^2-\varepsilon_2,\quad \Xi_{6,11}=\alpha_4W_1^TP_{11},$$

$$\Xi_{6,12}=\alpha_5W_1^TP_{11},\quad \Xi_{7,7}=-R_{22}^1+R_{22}^2+R_{22}^3,\quad \Xi_{7,11}=-P_{44}+P_{45}^T,$$

$$\Xi_{7,12}=-P_{45}+P_{55},\quad \Xi_{8,8}=-R_{22}^3,\quad \Xi_{8,11}=-P_{45}^T,\quad \Xi_{8,12}=-P_{55},$$

$$\Xi_{9,9}=-\varepsilon_3,\quad \Xi_{9,11}=-\alpha_4C^TG^T,\quad \Xi_{9,12}=-\alpha_5C^TG^T,\quad \Xi_{10,10}=-S_3,$$

$$\Xi_{10,11}=\alpha_4W_2^TP_{11},\quad \Xi_{10,12}=\alpha_5W_2^TP_{11},\quad \Xi_{11,11}=-S_4,\quad \Xi_{12,12}=-S_5,$$

$$\begin{aligned}\Xi_2^1=\Big[&\Sigma_1\ \left(-\vartheta_1\widetilde{\mathcal{P}}-\frac{\vartheta_1^2}{2}\widetilde{\mathcal{S}}-\frac{\vartheta_1^3}{6}\widetilde{\mathcal{U}}\right)\ \widehat{E}^TP_{11}\ \vartheta_1\widehat{A}^TP_{11}\ \frac{\vartheta_1^2}{2}\widehat{A}^TP_{11}\ \vartheta_1\widehat{E}^TP_{11}\\&\frac{\vartheta_1^3}{6}\widehat{A}^TP_{11}\ \frac{\vartheta_1^2}{2}\widehat{E}^TP_{11}\ \frac{\vartheta_1^3}{6}\widehat{E}^TP_{11}\ \widetilde{\mathcal{P}}\ \vartheta_1\widetilde{\mathcal{S}}\ \frac{\vartheta_1^2}{2}\widetilde{\mathcal{U}}\Big],\end{aligned}$$

$$\Xi_2^2=\Big[\Sigma_2\ \left(-\vartheta_{21}\widetilde{\mathcal{Q}}-\frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{T}}-\frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{V}}\right)\ \widehat{E}^TP_{11}\ \vartheta_{21}\widehat{A}^TP_{11}$$

$$\frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{A}^T P_{11} \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{A}^T P_{11} \;\; \vartheta_{21}\widehat{E}^T P_{11} \;\; \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{E}^T P_{11}$$
$$\frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{E}^T P_{11} \quad \widetilde{\mathcal{Q}} \quad \vartheta_{21}\widetilde{\mathcal{T}} \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}}\Big],$$

$$\Xi_2^3 = \Big[\Sigma_2 \;\; \left(-\vartheta_{21}\widetilde{\mathcal{R}} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{T}} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{V}}\right) \;\; \widehat{E}^T P_{11} \;\; \vartheta_{21}\widehat{A}^T P_{11}$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{A}^T P_{11} \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{A}^T P_{11} \;\; \vartheta_{21}\widehat{E}^T P_{11} \;\; \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{E}^T P_{11} \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{E}^T P_{11}$$
$$\widetilde{\mathcal{R}} \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{A}^T P_{11} \;\; \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}}\Big],$$

$$\Xi_3^1 = \Big[-\vartheta_1 S_1 \;\; \left(-\vartheta_1 T_1 - \frac{\vartheta_1^2}{2}T_3 - \frac{\vartheta_1^3}{6}T_5\right) \;\; -P_{11} \;\; \vartheta_1(-2P_{11}+T_1)$$
$$\frac{\vartheta_1^2}{2}(-2P_{11}+T_3) \;\; \frac{\vartheta_1^3}{6}(-2P_{11}+T_5) \;\; \vartheta_1(-2P_{11}+Z_1) \;\; \frac{\vartheta_1^2}{2}(-2P_{11}+Z_3)$$
$$\frac{\vartheta_1^3}{6}(-2P_{11}+Z_5) \;\; -Z_1 \;\; -\vartheta_1 Z_3 \;\; -\frac{\vartheta_1^2}{2}Z_5\Big],$$

$$\Xi_3^2 = \Big[-\vartheta_{21} S_2 \;\; \left(-\vartheta_{21} T_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2}T_4 - \frac{\vartheta_2^3-\vartheta_1^3}{6}T_6\right) \;\; -P_{11} \;\; \vartheta_{21}(-2P_{11}+T_2)$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}(-2P_{11}+T_4) \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}(-2P_{11}+T_6) \;\; \vartheta_{21}(-2P_{11}+Z_2)$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}(-2P_{11}+Z_4) \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}(-2P_{11}+Z_6)$$
$$-Z_2 \;\; -\vartheta_{21} Z_4 \;\; -\frac{\vartheta_2^2-\vartheta_1^2}{2}Z_6\Big],$$

$$\Xi_3^3 = \Big[-\vartheta_{21} S_2 \;\; \left(-\vartheta_{21} T_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2}T_4 - \frac{\vartheta_2^3-\vartheta_1^3}{6}T_6\right) \;\; -P_{11} \;\; \vartheta_{21}(-2P_{11}+T_2)$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}(-2P_{11}+T_4) \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}(-2P_{11}+T_6) \;\; \vartheta_{21}(-2P_{11}+Z_2)$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}(-2P_{11}+Z_4) \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}(-2P_{11}+Z_6)$$
$$-Z_2 \;\; -\vartheta_{21} Z_4 \;\; -\frac{\vartheta_2^2-\vartheta_1^2}{2}Z_6\Big],$$

$$\Sigma_1 = \Big[\left(\vartheta_1 P_{22} - \alpha_2\vartheta_1 A^T P_{11} - \vartheta_1 \mathcal{S}_1 - \frac{\vartheta_1^2}{2}\mathcal{U}_1\right)$$
$$\left(-\vartheta_1 P_{22} + \vartheta_1 P_{23}^T\right) \;\; -\vartheta_1 P_{23}^T \;\; \left(\alpha_2\vartheta_1 W_0^T P_{11} + \vartheta_1 P_{24}^T\right) \;\; \alpha_2\vartheta_1 W_1^T P_{11}$$

$$\left(-\vartheta_1 P_{24}^T + \vartheta_1 P_{25}^T\right) \quad -\vartheta_1 P_{25}^T \quad -\alpha_2\vartheta_1 C^T G^T \quad \alpha_2\vartheta_1 W_2^T P_{11} \quad 0 \quad 0\big],$$

$$\Sigma_2 = \Bigg[\left(\vartheta_{21}P_{23} - \alpha_3\vartheta_{21}A^T P_{11} - \vartheta_{21}\mathcal{T}_1 - \frac{\vartheta_2^2-\vartheta_1^2}{2}\mathcal{V}_1\right)$$

$$\left(-\vartheta_{21}\mathcal{T}_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2}\mathcal{V}_2\right) \quad (\vartheta_{21}P_{33} - \vartheta_{21}P_{23}) \quad -\vartheta_{21}P_{33}$$

$$(\alpha_3\vartheta_{21}W_0^T P_{11} + \vartheta_{21}P_{34}^T) \quad \alpha_3\vartheta_{21}W_1^T P_{11} \quad (\vartheta_{21}P_{35}^T - \vartheta_{21}P_{34}^T) \quad -\vartheta_{21}P_{35}^T$$

$$-\alpha_3\vartheta_{21}C^T G^T \quad \alpha_3\vartheta_{21}W_2^T P_{11} \quad 0 \quad 0\big],$$

$$\widehat{E} = [E \;\; E_1 \;\; 0 \;\; 0 \;\; H_0 \;\; H_1 \;\; 0 \;\; 0 \;\; 0 \;\; H_2 \;\; 0 \;\; 0],$$

$$\widehat{A} = \left[A \;\; 0 \;\; 0 \;\; 0 \;\; W_0 \;\; W_1 \;\; 0 \;\; 0 \;\; -P_{11}^{-1}GC \;\; W_2 \;\; 0 \;\; 0\right].$$

Furthermore, the gain matrix K of the state estimator of (2.4) can be designed as $K = P_{11}^{-1}G$.

Proof. Defining $P_{11}K = G$, $P_{1y} = \alpha_y P_{11}$, $(y = 2,3,4,5)$. Pre and post multiplying Π_1 in (3.1) by $\mathrm{diag}\{\overbrace{I,\ldots,I}^{15\text{ times}}, P_{11}T_1^{-1}, P_{11}T_3^{-1}, P_{11}T_5^{-1}, P_{11}Z_1^{-1}, P_{11}Z_3^{-1}, P_{11}Z_5^{-1}, I, I, I\}$ and $\mathrm{diag}\{\overbrace{I,\ldots,I}^{15\text{ times}}, T_1^{-1}P_{11}, T_3^{-1}P_{11}, T_5^{-1}P_{11}, Z_1^{-1}P_{11}, Z_3^{-1}P_{11}, Z_5^{-1}P_{11}, I, I, I\}$ respectively. As well as pre and post multiplying Π_2, Π_3 in (3.1) by $\mathrm{diag}\{\overbrace{I,\ldots,I}^{15\text{ times}}, P_{11}T_2^{-1}, P_{11}T_4^{-1}, P_{11}T_6^{-1}, P_{11}Z_2^{-1}, P_{11}Z_4^{-1}, P_{11}Z_6^{-1}, I, I, I\}$ and $\mathrm{diag}\{\overbrace{I,\ldots,I}^{15\text{ times}}, T_2^{-1}P_{11}, T_4^{-1}P_{11}, T_6^{-1}P_{11}, Z_2^{-1}P_{11}, Z_4^{-1}P_{11}, Z_6^{-1}P_{11}, I, I, I\}$ respectively. Then, through the inequalities (see [1])

$$(-2P_{11} + T_q) \geq -P_{11}T_q^{-1}P_{11}, \; (-2P_{11} + Z_q) \quad \geq -P_{11}Z_q^{-1}P_{11}, \; (q = 1,2,\ldots,6),$$

Ξ_i, (i=1,2,3) in (3.6) can be obtained. From Theorem 1, the error system (2.4) is asymptotically stable. □

4. EXAMPLES

A numerical example is given in this section. For convenience, denote $f_{ij}(t) = f_j(x_j(t)) - x_i(t)$, $f_{ij}(t-\vartheta(t)) = f_j(x_j(t-\vartheta(t))) - x_i(t)$ and $f_{ij}(t-d(t)) = \int_{t-d(t)}^{t} f_j(x_j(s))ds) - x_i(t))$.

Example 1. Consider a two-dimensional MSNNs (2.2):

$$w_{11}^0(t) = \begin{cases} -0.20, & f_{11}(t) \downarrow, s \in (t-\rho, t]; \\ 0.17, & f_{11}(t) \uparrow, s \in (t-\rho, t]; \\ \lim_{s\to t^-} w_{11}^0(s), & f_{11}(s) \text{ unchage } s \in (t-\rho, t], \end{cases}$$

$$w_{12}^{0}(t)=\begin{cases}0.15, & f_{12}(t)\ \downarrow, s\in(t-\rho,t];\\ -0.19, & f_{12}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{12}^{0}(s), & f_{12}(s)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{21}^{0}(t)=\begin{cases}0.30, & f_{21}(t)\ \downarrow, s\in(t-\rho,t];\\ 0.12, & f_{21}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{21}^{0}(s), & f_{21}(s)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{22}^{0}(t)=\begin{cases}0.10, & f_{22}(t)\ \downarrow, s\in(t-\rho,t];\\ -0.45, & f_{22}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{22}^{0}(s), & f_{22}(s)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{11}^{1}(t)=\begin{cases}0.15, & f_{11}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ 0.56, & f_{11}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{11}^{1}(s), & f_{11}(t-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{12}^{1}(t)=\begin{cases}0.35, & f_{12}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ -0.26, & f_{12}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{12}^{1}(s), & f_{12}(s-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{21}^{1}(t)=\begin{cases}-0.14, & f_{21}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ 0.35, & f_{21}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{21}^{1}(s), & f_{21}(s-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{22}^{1}(t)=\begin{cases}0.25, & f_{22}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ 0.05, & f_{22}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{22}^{1}(s), & f_{22}(s-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{11}^{2}(t)=\begin{cases}0.20, & f_{11}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ -0.13, & f_{11}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{11}^{2}(s), & f_{11}(s-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{12}^{2}(t)=\begin{cases}-0.01, & f_{12}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ -0.52, & f_{12}(t-0.2)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{12}^{2}(s), & f_{12}(s-0.2)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{21}^{2}(t))=\begin{cases}0.30, & f_{21}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ 0.05, & f_{21}(t-0.2)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{21}^{2}(s), & f_{21}(s-0.2)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$w_{22}^{2}(t)=\begin{cases}0.40, & f_{22}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ 0.14, & f_{22}(t-0.2)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{22}^{2}(s), & f_{22}(t-0.2)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{11}^{0}(t)=\begin{cases}0.12, & f_{11}(t)\ \downarrow, s\in(t-\rho,t];\\ -0.20, & f_{11}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{11}^{0}(s), & f_{11}(s)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{12}^{0}(t)=\begin{cases}0.02, & f_{12}(t)\ \downarrow, s\in(t-\rho,t];\\ -0.21, & f_{12}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{12}^{0}(s), & f_{12}(s)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{21}^{0}(t)=\begin{cases}0.30, & f_{21}(t)\ \downarrow, s\in(t-\rho,t];\\ 0.25, & f_{21}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{21}^{0}(s), & f_{21}(s)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{22}^{0}(t)=\begin{cases}-0.09, & f_{22}(t)\ \downarrow, s\in(t-\rho,t];\\ 0.12, & f_{22}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{22}^{0}(s), & f_{22}(s)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{11}^{1}(t)=\begin{cases}-0.11, & f_{11}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ -0.03, & f_{11}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{11}^{1}(s), & f_{11}(s-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{12}^{1}(t)=\begin{cases}0.20, & f_{12}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ 0.10, & f_{12}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{12}^{1}(s), & f_{12}(s-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{21}^{1}(t)=\begin{cases}-0.15, & f_{21}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ -0.10, & f_{21}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{21}^{1}(s), & f_{21}(s-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{22}^{1}(t)=\begin{cases}0.20, & f_{22}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ -0.05, & f_{22}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{22}^{1}(s), & f_{22}(s-0.7)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{11}^{2}(t)=\begin{cases}0.22, & f_{11}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ -0.19, & f_{11}(t-0.2)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{11}^{2}(s), & f_{11}(s-0.2)\ \text{unchage}\ s\in(t-\rho,t],\end{cases}$$

$$h_{12}^2(t) = \begin{cases} -0.10, & f_{12}(t-0.2) \downarrow, s \in (t-\rho, t]; \\ 0.73, & f_{12}(t-0.2) \uparrow, s \in (t-\rho, t]; \\ \lim_{s \to t^-} h_{12}^2(s), & f_{12}(s-0.2) \text{ unchage } s \in (t-\rho, t], \end{cases}$$

$$h_{21}^2(t) = \begin{cases} 0.10, & f_{21}(t-0.2) \downarrow, s \in (t-\rho, t]; \\ -0.26, & f_{21}(t-0.2) \uparrow, s \in (t-\rho, t]; \\ \lim_{s \to t^-} h_{21}^2(s), & f_{21}(s-0.2) \text{ unchage } s \in (t-\rho, t], \end{cases}$$

$$h_{22}^2(t) = \begin{cases} 0.50, & f_{22}(t-0.2) \downarrow, s \in (t-\rho, t]; \\ -0.14, & f_{22}(t-0.2) \uparrow, s \in (t-\rho, t]; \\ \lim_{s \to t^-} h_{22}^2(s), & f_{22}(s-0.2) \text{ unchage } s \in (t-\rho, t], \end{cases}$$

with the activation function $f_j(x_j(\cdot)), j = 1,2.$ and $L = 0.5I$, $F = I$, $C = 0.5I$, $a_1 = 2, a_2 = 1$, $be_1^0 = 0.2, e_2^0 = 0.3$, $e_{11}^1 = 0.1, e_{12}^1 = -0.1, e_{21}^1 = -0.05, e_{22}^1 = 0.05$, then, by solving the LMI in Theorem 2 with $d = 0.2, \vartheta_1 = 0.2, \vartheta_2 = 0.7$, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0.1$, and $\mu = 0.5$ the estimator gain matrix is obtained as $K = [0.2283, \ -0.0035; 0.0577, \ 0.0808]$, which verifies the feasibility of Theorem 2.

5. Conclusion

In this article, the state estimation problem for MSNNs with mixed variable delays via a new quadruple stochastic integral LKF has been investigated. A novel state estimation analysis have been considered in terms of LMIs. A desired state estimator gain matrix is obtained by solving LMIs. An example is provided to explain the usefulness of the proposed approach. In the future, it is possible to extend this results to the delayed bidirectional associative memory stochastic NNs.

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