Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1581–1603

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4024

VISCOSITY S-ITERATION ALGORITHM FOR FINDING COMMON FIXED POINT OF NONEXPANSIVE MAPPINGS AND APPLICATION TO NONEXPANSIVE SEMIGROUPS

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Received 11 December, 2021

Abstract. We study the problem of finding the common element of the set of fixed points of two nonexpansive mappings in Hilbert spaces. Some previous attempts in this direction make some restrictive assumptions which may be difficult to check in practice on the generated sequence. In this paper, we introduce a viscosity S-iteration scheme for finding the common fixed point of two nonexpansive mappings. Under some very mild conditions, we obtain a strong convergence theorem for the sequence generated by our algorithm. We apply our main result to approximating common fixed points of nonexpansive semigroups. We also provide numerical examples to support our main results and illustrate the efficiency and effectiveness of our algorithm by comparing with some existing algorithms in literature. This work generalizes and improves some existing works in the literature in this direction.

2010 *Mathematics Subject Classification:* 47H10; 47H20; 47J25; 65J15.

Keywords: viscosity approximation algorithm, S-iteration, nonexpansive semgroup, Hilbert space

1. INTRODUCTION

Let H be a real Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and C a nonempty closed and convex subset of H. A mapping $T: C\to C$ is called nonexpansive if $\|Tx-Ty\|\leq\|x-y\|$ for every $x,y\in C$. A mapping $f: C\to C$ is called a contraction if there exists $\theta\in[0,1)$ such that $\|f(x)-f(y)\|\leq\theta\|x-y\|$ for every $x,y\in C$. A point $x\in C$ is called the fixed point of T if $Tx=x$. We shall denote

The first author was supported by International Mathematical Union (IMU) Breakout Graduate Fellowship.

The third author was supported by Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS).

The fourth author was supported in part by the National Research Foundation (NRF), South Africa, Grant No. 119903.

the fixed point set of T by $\mathrm{Fix}(T)$. The study of fixed point theory for nonexpansive mappings has flourished in recent years due to its vast applications in fields like compressed sensing, economics and other applied sciences (see for instance, [4] and some of the references therein). In particular, some problems such as convex feasibility problems, convex optimization problems, monotone inclusion problems and image restoration problems can be seen as finding the fixed points of nonexpansive mapping (see [?, 6, 8, 20]). In the past and recent years, researchers have put considerable efforts in the study and in the formulation of algorithms to approximate the fixed points of nonexpansive mappings and related optimization problems, see [?, ?, 3, 15, 16, 19, 24]. We mention few of these algorithms:

The Mann iterative scheme is defined as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\} \subset [0,1]$. It is known that the Mann iterative scheme (1.1) converges weakly to a fixed point of T provided that $\{\alpha_n\} \subset [0,1]$ satisfies

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = +\infty.$$

Halpern [13] proposed the following recursive formula:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \ n \geq 0, \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1]$ and $u \in C$. Halpern shown that under the following control conditions:

(H1) $\lim_{n\to\infty} \alpha_n = 0$;
(H2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Oftentimes, in many real world problems arising in infinite dimensional spaces, strong convergence is much more desirable than weak convergence (see [7] and the references therein). Both Mann's and Halpern's algorithms have received considerable research efforts recently. Lions [17] proved that the Halpern iterative scheme (1.2) converges strongly to a fixed point of T provided $\{\alpha_n\}$ satisfies the following control conditions:

(A1) $\lim_{n\to\infty} \alpha_n = 0$;
(A2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(A3) $\lim_{n\to\infty} \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}^2} = 0$.

One can see that in [17], the sequence $\{\alpha_n\}$ excluded canonical choice like $\alpha_n = \frac{1}{n+1}$. Wittmann [26] considered the iterative scheme (1.2) where $\{\alpha_n\}$ satisfies the following control conditions:

(B1) $\lim_{n\to\infty} \alpha_n = 0$;

(B2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(B3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$,

and proved that $\{x_n\}$ converges strongly to $x^* \in F(T)$ such that $x^* = P_{F(T)}u$.

Moudafi [19] proposed the viscosity approximation method for finding the fixed points of nonexpansive mapping T: Let $x_1 \in C$ be arbitrary and define

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{1.3}$$

where $f\colon C \to C$ is a contraction and $\{\alpha_n\} \subset [0,1]$ satisfying the following control conditions:

(M1) $\lim_{n\to\infty} \alpha_n = 0$;
(M2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(M3) $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

It is obvious that (1.3) extends (1.2). Moudafi [19] proved the following result in Hilbert spaces:

Theorem 1 ([19, Theorem 2.1]). *If $\{\alpha_n\}$ satisfies the conditions (M1) - (M3) as above, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to a fixed point x^* of T, which also solves the variational inequality problem:*

$$\text{find } x^* \in F(T) \text{ such that } \langle f(x^*) - x^*, x - x^* \rangle \leq 0, \quad x \in F(T).$$

For other recent works on viscosity method for approximating fixed point of nonexpansive mappings, please see [16].

Agarwal et al. [1] proposes the following iterative scheme called the S-iteration process:

Algorithm 1. *Let C be a convex subset of a linear space X and T a mapping of C onto itself. Let $x_1 \in C$ and generate the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty.$$

It is known that Algorithm 1 does not reduce to the Mann iterative scheme (1.1).

Let $T, U\colon C \to C$ be two nonexpansive mappings, our aim in this paper is to find the common fixed points of T and U. Problems of this kind have been studied by authors very recently (see [9, 23] and some of the references therein).

Suparatulatorn et al. [23] introduced a modified S-iteration process defined as follows:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n S_1 x_n, \\ x_{n+1} = (1-\alpha_n)S_1 x_n + \alpha_n S_2 y_n, \quad n \geq 0, \end{cases}$$

where C is a nonempty subset of a real Banach space, the two sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $S_1, S_2 : C \to C$ are G-nonexpansive mappings and under some conditions proved weak and strong convergence theorems for finding common fixed point of the two G-nonexpansive mappings in a uniformly convex Banach space.

Also very recently, Ahmad et al. [2] studied the problem of finding the common fixed point of two nonexpansive mappings $T, U : C \to C$ in Hilbert spaces and proved the following theorem:

Theorem 2 ([2, Theorem 2.1])**.** *Let $T, U : C \to C$ be two nonexpansive mappings with $\Gamma := \mathrm{Fix}(T) \cap \mathrm{Fix}(U) \neq \varnothing$. Also let $f : C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Assume that the sequence $\{x_n\}$ in C generated by (1.4)*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T(x_n) + \gamma_n U(x_n), \quad n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying

(1) $\alpha_n + \beta_n + \gamma_n = 1$,
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} = \infty$,
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
(4) $\lim_{n\to\infty} \|U(x_n) - T(x_n)\| = 0$.

Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$, which satisfies the variational inequality

$$\langle (1-f)x^*, x - x^* \rangle \geq 0 \qquad \forall\, x \in \Gamma.$$

Looking at iterative scheme (1.4), one will see that the condition (4) is restrictive. This is because one will have to first check that $\lim_{n\to\infty} \|U(x_n) - T(x_n)\| = 0$ before the implementation. This then brought about the following question:

Question 1. Is it possible to modify (1.4) so that the conditions (1) - (4) are relaxed and also obtain strong convergence?

We answer this question in the affirmative.

In this paper, motivated by the works of Ahmad et al. [2], the above works and the ongoing research interest in this direction, we propose a viscosity S-iteration for finding the common fixed points of two nonexpansive mappings in Hilbert space. Under some mild conditions, we prove a strong convergence theorem and give some consequence of our main result. We apply our main results to nonexpansive semigroups.

This paper is organised as follows. In Section 2, we give some useful definitions, notations and lemmas which are needed for our algorithm's analysis. In Section 3, the algorithm and its strong convergence theorem is presented. In Section 4, we apply our main result to nonexpansive semigroups. In Section 5, we give numerical example to illustrate our algorithms and compare it with some existing algorithms in literature. We conclude in Section 6.

2. PRELIMINARIES

Throughout this article, we let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. C is a nonempty closed convex subset of H and $I\colon H\to H$ is the identity mapping on H. We denote by '$x_n \rightharpoonup x$' and '$x_n \to x$', the weak and the strong convergence of $\{x_n\}$ to a point x respectively.

We recall the following definitions:

Definition 1 ([8, Definition 4.1, Definition 22.1])**.** An operator $T\colon C\to C$ is said to be:

(i) Lipschitzian if there exists a constant $L>0$ such that

$$\|Tx-Ty\|\leq L\|x-y\| \qquad \forall\, x,y\in C;$$

(ii) firmly nonexpansive if

$$\|Tx-Ty\|^2+\|(I-T)x-(I-T)y\|^2\leq\|x-y\|^2 \qquad \forall\, x,y\in C,$$

or equivalently,

$$\|Tx-Ty\|^2\leq\langle x-y,Tx-Ty\rangle \qquad \forall\, x,y\in C;$$

(iii) β-strongly monotone if there exists $\beta>0$ such that

$$\langle Tx-Ty,x-y\rangle\geq\beta\|x-y\|^2 \qquad \forall\, x,y\in C.$$

One can verify that if the mapping $h\colon C\to C$ is a contraction with contractive constant $\tau\in[0,1)$, then $I-h$ is $2(1+\tau^2)$-Lipschitzian and $(1-\tau)$-strongly monotone. It can be deduced from the definition above that every firmly nonexpansive mapping is nonexpansive. It is known that the set of fixed points of nonexpansive mapping in Hilbert space is closed and convex. The metric projection of H onto C (see [**?**]), denotes as P_C, is the mapping that assigns every point $x\in H$ to its unique nearest point in C i.e.,

$$\|x-P_Cx\|\leq\|x-y\| \qquad \forall\, y\in C.$$

The metric projection is characterized by $P_Cx\in C$ and

$$\langle x-P_Cx,y-P_Cx\rangle\leq 0 \qquad \forall y\in C.$$

Moreover, P_C is nonexpansive and $\mathrm{Fix}(P_C)=C$.

Let $h\colon C\to C$ be a nonlinear operator. The Variational Inequality Problem (VIP) is to

$$\text{find } x^*\in C \text{ such that } \langle h(x^*),x-x^*\rangle\geq 0 \qquad \forall\, x\in C.$$

Lemma 1 ([28, Proposition 2.7]). *Let H be a real Hilbert space. Suppose that $h\colon H \to H$ is κ- Lipschitzian and β- strongly monotone over a closed convex subset $C \subset H$. Then, the following VIP*

$$\langle h(u^*), v-u^*\rangle \geq 0 \qquad \forall\, v \in C$$

has its unique solution $u^* \in C$.

Lemma 2. *[12] Let C be a nonempty closed convex subset of a real Hilbert space H and let $T: C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \varnothing$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, then $x^* = Tx^*$.*

Lemma 3 ([11, Page 1]). *Let H be a real Hilbert space, then the following assertions hold:*

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$ *for all $x, y \in H$;*
(ii) *for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2.$$

Lemma 4 ([27, Lemma 2.1]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$s_{n+1} \leq (1-t_n)s_n + t_n\rho_n, \quad n \geq n_0,$$

where $\{t_n\} \subset (0,1)$ and $\{\rho_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty} t_n = 0$, $\sum_{n=1}^{\infty} t_n = \infty$, and $\limsup_{n\to\infty} \rho_n \leq 0$. Then $s_n \to 0$ as $n \to \infty$.

Lemma 5 ([18, Lemma 3.1]). *Let $\{a_n\}$ be sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}_{i\geq 0}$ of $\{a_n\}$ with $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1,2,\cdots k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

3. Main results

In this section, we present our algorithm and study the convergence analysis.

Algorithm 2. *Let $T, U: C \to C$ be two nonexpansive mappings with $\Omega := \mathrm{Fix}(T) \cap \mathrm{Fix}(U) \neq \varnothing$. Also let $f\colon C \to C$ be a contraction with coefficient $\nu \in [0,1)$. Choose $x_0 \in C$ and let $\{x_n\}$ be the sequence generated by* (3.1)

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = \beta_n f(x_n) + \gamma_n T x_n + \xi_n U u_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\xi_n\}$are sequences in $(0,1)$ satisfying

(i) $\beta_n + \gamma_n + \xi_n = 1$;

(ii) $\liminf_{n\to\infty} \gamma_n \xi_n > 0$, $\liminf_{n\to\infty} \xi_n \alpha_n (1-\alpha_n) > 0$;
(iii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$.

Lemma 6. *The sequence $\{x_n\}$ generated by Algorithm 2 is bounded.*

Proof. Let $p \in \Omega$. Then

$$\begin{aligned} \|u_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n T x_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n \|T x_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n[f(x_n) - p] + \gamma_n[T x_n - p] + \xi_n[U u_n - p]\| \\ &\le \beta_n \|f(x_n) - p\| + \gamma_n \|x_n - p\| + \xi_n \|u_n - p\| \\ &\le \beta_n \nu \|x_n - p\| + \beta_n \|f(p) - p\| + \gamma_n \|x_n - p\| + \xi_n \|u_n - p\|. \end{aligned} \tag{3.3}$$

Substituting (3.2) into (3.3) gives

$$\begin{aligned} \|x_{n+1} - p\| &\le (1-\beta_n + \beta_n \nu)\|x_n - p\| + \beta_n \|f(p) - p\| \\ &= [1 - \beta_n(1-\nu)]\|x_n - p\| + \frac{\beta_n(1-\nu)\|f(p)-p\|}{1-\nu} \\ &\le \max\left\{\|x_n - p\|, \frac{\|f(p)-p\|}{1-\nu}\right\} \\ &\vdots \\ &\le \max\left\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-\nu}\right\}. \end{aligned}$$

Showing that $\{\|x_{n+1} - p\|\}$ is bounded. Hence $\{T x_n\}$, $\{U u_n\}$ and $\{x_n\}$ are bounded. □

Theorem 3. *The sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $p \in \Omega$, where p is the unique solution of the following VIP (3.4): Find $p \in \Omega$ such that*

$$\langle (I - f)p, x - p \rangle \ge 0 \qquad \forall\, x \in \Omega. \tag{3.4}$$

Proof. Since f is a ν-contraction mapping, it follows that $I - f$ is $2(1+\nu^2)$-Lipschitzian and $(1-\nu)$-strongly monotone ([**?**]). Therefore by Lemma 1, it follows that the VIP (3.4) has a unique solution $p \in \Omega$.

Set $K(x_n, u_n) = \theta_n T x_n + (1-\theta_n) U u_n$, where $\theta_n = \frac{\gamma_n}{1-\beta_n}$, $1 - \theta_n = \frac{\xi_n}{1-\beta_n}$. Note that by Lemma 3 (ii)

$$\|K(x_n, u_n) - p\|^2 = \|\theta_n T x_n + (1-\theta_n) U u_n - p\|^2$$

$$\begin{aligned}&\leq \theta_n\|Tx_n-p\|^2+(1-\theta_n)\|Uu_n-p\|^2\\&\leq \theta_n\|x_n-p\|^2+(1-\theta)\|u_n-p\|^2\\&\leq \|x_n-p\|^2.\end{aligned} \tag{3.5}$$

Also

$$\begin{aligned}\langle f(x_n)-p,K(x_n,u_n)-p\rangle &= \langle f(x_n)-f(p),K(x_n,u_n)-p\rangle\\&\quad+\langle f(p)-p,K(x_n,u_n)-p\rangle\\&\leq \|f(x_n)-f(p)\|\|K(x_n,u_n)-p\|\\&\quad+\langle f(p)-p,K(x_n,u_n)-p\rangle\\&\leq \frac{1}{2}[\|f(x_n)-f(p)\|^2+\|K(x_n,u_n)-p\|^2]\\&\quad+\langle f(p)-p,K(x_n,u_n)-p\rangle\\&\leq \frac{1}{2}[\nu^2\|x_n-p\|^2+\|x_n-p\|^2]\\&\quad+\langle f(p)-p,K(x_n,u_n)-p\rangle.\end{aligned} \tag{3.6}$$

Then we have by (3.5) and (3.6) that

$$\begin{aligned}\|x_{n+1}-p\|^2 =&\|\beta_nf(x_n)+(1-\beta_n)K(x_n,u_n)-p\|^2\\=&\beta_n^2\|f(x_n)-p\|^2+2\beta_n(1-\beta_n)\langle f(x_n)-p,K(x_n,u_n)-p\rangle\\&+(1-\beta_n)^2\|K(x_n,u_n)-p\|^2\\\leq&\beta_n^2\|f(x_n)-p\|^2+\beta_n(1-\beta_n)[(\nu^2+1)\|x_n-p\|^2]\\&+(1-\beta_n)^2\|x_n-p\|^2\\&+2\beta_n(1-\beta_n)\langle f(p)-p,K(x_n,u_n)-p\rangle\\=&(1-\beta_n(1-(1-\beta_n)\nu^2))\|x_n-p\|^2\\&+\beta_n[\beta_n\|f(x_n)-p\|^2+2(1-\beta_n)\langle f(p)-p,K(x_n,u_n)-p\rangle]\\=&(1-\lambda_n)\Gamma_n(p)+\lambda_n\delta_n,\end{aligned} \tag{3.7}$$

where $\lambda_n=\beta_n(1-(1-\beta_n)\nu^2)$, $\Gamma_n(p)=\|x_n-p\|^2$ and
$\delta_n=\frac{[\beta_n\|f(x_n)-p\|^2+2(1-\beta_n)\langle f(p)-p,K(x_n,u_n)-p\rangle]}{1-(1-\beta_n)\nu^2}$. We now consider the following cases:

Case 1: Suppose $\{\Gamma_n(p)\}$ is monotonically non-increasing for all $n\geq n_0$ for some $n_0\in\mathbb{N}$. Since $\{\Gamma_n(p)\}$ is bounded and assumed to be monotonically non-increasing, it then implies that it converges. So $\Gamma_n(p)-\Gamma_{n+1}(p)\to 0$ as $n\to\infty$. From (3.1) we derive that

$$\begin{aligned}\|x_{n+1}-p\|^2 =&\|\beta_nf(x_n)+\gamma_nTx_n+\xi_nUu_n-p\|^2\\\leq&\beta_n\|f(x_n)-p\|^2+\gamma_n\|Tx_n-p\|^2\end{aligned}$$

$$
\begin{aligned}
&+\xi_n\|Uu_n-p\|^2-\gamma_n\xi_n\|Tx_n-Uu_n\|^2\\
\leq&\beta_n\|f(x_n)-p\|^2+\gamma_n\|x_n-p\|^2+\xi_n\|x_n-p\|^2\\
&-\gamma_n\xi_n\|Tx_n-Uu_n\|^2-\xi_n\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2\\
=&\beta_n\|f(x_n)-p\|^2+(1-\beta_n)\|x_n-p\|^2\\
&-\gamma_n\xi_n\|Tx_n-Uu_n\|^2-\xi_n\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2.
\end{aligned}
$$

Then, we get that

$$
\begin{aligned}
\gamma_n\xi_n\|Tx_n-Uu_n\|^2+\xi_n\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2 \leq&\\
\beta_n\|f(x_n)-p\|^2+(1-\beta_n)\Gamma_n(p)-\Gamma_{n+1}(p).&\quad (3.8)
\end{aligned}
$$

Taking the limit in (3.8) as $n\to\infty$ and using the control conditions, we get

$$\|Tx_n-Uu_n\|\to 0 \tag{3.9}$$

and

$$\|Tx_n-x_n\|\to 0. \tag{3.10}$$

Therefore from (3.1) and (3.10), we obtain that

$$
\begin{aligned}
\|Ux_n-Uu_n\| &= \|Ux_n-U((1-\alpha_n)x_n+\alpha_nTx_n)\|\\
&\leq \|x_n-[(1-\alpha_n)x_n+\alpha_nTx_n]\|\\
&= \alpha_n\|x_n-Tx_n\|\to 0 \text{ as } n\to\infty. \qquad (3.11)
\end{aligned}
$$

Thus from (3.9) and (3.11), we obtain that

$$\|Tx_n-Ux_n\|\leq\|Tx_n-Uu_n\|+\|Uu_n-Ux_n\|\to 0 \text{ as } n\to\infty. \tag{3.12}$$

And from (3.10) and (3.12), we get

$$\|Ux_n-x_n\|\leq\|Ux_n-Tx_n\|+\|Tx_n-x_n\|\to 0 \text{ as } n\to\infty. \tag{3.13}$$

We next prove that $\|u_n-x_n\|\to 0$ and $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$. Indeed, from (3.1) and (3.10) we see that

$$\|u_n-x_n\|=\alpha_n\|x_n-Tx_n\|\to 0 \text{ as } n\to\infty.$$

Also from (3.10) and (3.13), we obtain that

$$
\begin{aligned}
&\|x_{n+1}-x_n\|=\|\beta_nf(x_n)+\gamma_nTx_n+\xi_nUu_n-x_n\|\\
&\leq\beta_n\|f(x_n)-x_n\|+\gamma_n\|Tx_n-x_n\|+\xi_n\|Uu_n-x_n\|\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$. By the demiclosedness of T and U and (3.10) (see Lemma 2) and (3.13), it then implies that $x^*\in \mathrm{Fix}(T)\cap\mathrm{Fix}(U)$. We will next show that $\limsup_{n\to\infty}\delta_n\leq 0$ and establish the strong convergence of the sequence $\{x_n\}$. First observe that from (3.10), (3.11) and (3.12), we have

$$K(x_{n_k},u_{n_k})=\theta_{n_k}Tx_{n_k}+(1-\theta_{n_k})Uu_{n_k}$$

$$\begin{aligned}&\to \theta_{n_k} x_{n_k} + (1-\theta_{n_k}) x_{n_k}\\ &\rightharpoonup x^* \text{ as } k \to \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(p)-p, K(x_n,u_n)-p\rangle\\ &\le \limsup_{k\to\infty}\langle f(p)-p, K(x_{n_k},u_{n_k})-p\rangle\\ &\le \langle f(p)-p, x^*-p\rangle \le 0, \qquad (3.14)\end{aligned}$$

where (3.14) follows from (3.4). Furthermore, it is easy to see that $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$. Hence, we conclude by Lemma 4 that $\lim_{n\to\infty}\Gamma_n(p) = 0$. Therefore $x_n \to p$ as $n \to \infty$.

Case 2: Suppose $\{\Gamma_n(p)\}$ is not monotonically decreasing. Then there exists a subsequence $\{n_r\}$ of $\{n\}$ such that $\Gamma_{n_r}(p) < \Gamma_{n_r+1}(p)$ for all $r \in \mathbb{N}$. By Lemma 5, there exists an increasing sequence $\{m_r\} \subset \mathbb{N}$ such that $m_r \to \infty$ and

$$0 \le \Gamma_{m_r}(p) \le \Gamma_{m_r+1}(p) \text{ for all } r \in \mathbb{N}. \qquad (3.15)$$

Then using similar argument as in Case 1, we see that as $m_r \to \infty$,

$$\|x_{m_r} - Tx_{m_r}\| \to 0\,, \|Ux_{m_r} - x_{m_r}\| \to 0$$

and

$$\limsup_{n\to\infty}\langle f(p)-p, K(x_{m_r},u_{m_r})-p\rangle \le 0.$$

It then follows from (3.7) and (3.15) that

$$\begin{aligned}0 &\le (1-\lambda_{m_r})\Gamma_{m_r}(p) - \Gamma_{m_r+1}(p) + \lambda_{m_r}\delta_{m_r}\\ &\le (1-\lambda_{m_r})\Gamma_{m_r+1}(p) - \Gamma_{m_r}(p) + \lambda_{m_r}\delta_{m_r}\\ &= -\lambda_{m_r}\Gamma_{m_r+1}(p) + \lambda_{m_r}\delta_{m_r}.\end{aligned}$$

Therefore,

$$\Gamma_{m_r+1}(p) \le \delta_{m_r}. \qquad (3.16)$$

Note that from the definition of δ_{m_r} we have $\limsup_{r\to\infty}\delta_{m_r} \le 0$. Hence taking the limit of (3.16) as $r \to \infty$, we get $\Gamma_{m_r+1}(p) \to 0$. Consequently, $\Gamma_{m_r}(p) \to 0$. By applying Lemma 5, we then get

$$0 \le \Gamma_r(p) \le \max\{\Gamma_r(p), \Gamma_{m_r}(p)\} \le \Gamma_{m_r+1}(p).$$

Hence $\lim_{r\to\infty}\Gamma_r(p) = 0$. Therefore, $\{x_n\}$ converges strongly to $p \in \Omega$.

So in both cases, we obtain that $x_n \to p \in \Omega$.

□

Taking $f(x) = u$ for all $x \in C$, we obtain the following Halpern S-iteration as a consequence of Theorem 3.

Corollary 1. *Let $T, U : C \to C$ be two nonexpansive mappings with $\Omega := \mathrm{Fix}(T) \cap \mathrm{Fix}(U) \neq \varnothing$ and $u \in C$ be arbitrary. Choose $x_0 \in C$ and let $\{x_n\}$ be the sequence generated by (3.17)*

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = \beta_n u + \gamma_n T x_n + \xi_n U u_n, \quad n \geq 1, \end{cases} \tag{3.17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\xi_n\}$are sequences in $(0,1)$ satisfying

(i) $\beta_n + \gamma_n + \xi_n = 1$;
(ii) $\liminf_{n\to\infty} \gamma_n \xi_n > 0$, $\liminf_{n\to\infty} \xi_n \alpha_n (1-\alpha_n) > 0$;
(iii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$.

Then, the sequence $\{x_n\}$ generated by Algorithm (3.17) converges strongly to $p \in \Omega$, where $p = P_\Omega u$.

4. Application to common fixed point of nonexpansive semigroups

Let H be a Hilbert space and C a nonempty closed convex subset of H. The one parameter family $\mathcal{T} := \{T(t) : 0 \leq t < \infty\}$ of mappings from C to C is said to be nonexpansive semigroup, if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) for each $x \in C$, the mapping $t \rightarrowtail T(t)x$ is continuous;
(iv) $\|T(t)x - T(t)y\| \leq \|x - y\|$.

If $F(\mathcal{T}) \neq \varnothing$, it is known that $F(\mathcal{T})$ is closed and convex. An example of a one-parameter nonexpansive semigroup is given below.

Example 1 ([10]). Let $H = \mathbb{R}$ and $\mathcal{T} := \{T(t) : 0 \leq t < \infty\}$, where $T(t)x = (\frac{1}{10^t})x$ for all $x \in H$. Then $\mathcal{T}$ is a one-parameter nonexpansive semigroup.

The following lemma was proved in Shimizu and Takahashi [22]; see also [5, 21].

Lemma 7. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. Then, for any $h \geq 0$,*

$$\limsup_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x ds\right) \right\| = 0.$$

We next apply our main result to find the common fixed point of two nonexpansive semigroups.

Theorem 4. *Let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ and $\mathcal{U} := \{U(s) : 0 \leq s < \infty\}$ be two families of nonexpansive semigroups on C. Also let $f : C \to C$ be a contraction with* *coefficient* $\nu \in [0,1)$. *Assume* $\Omega := \mathrm{Fix}(\mathcal{T}) \cap \mathrm{Fix}(\mathcal{U}) \neq \emptyset$. *Choose* $x_0 \in C$ *and let* $\{x_n\}$ *be the sequence generated by the algorithm*

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ x_{n+1} = \beta_n f(x_n) + \gamma_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \xi_n \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds, \quad n \geq 1, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\} \subset (0,1)$ *are sequences such that*

(i) $\beta_n + \gamma_n + \xi_n = 1$;
(ii) $\liminf_{n\to\infty} \gamma_n \xi_n > 0$, $\liminf_{n\to\infty} \xi_n \alpha_n (1-\alpha_n) > 0$;
(iii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iv) $\{t_n\}$ *is a sequence of positive numbers such that* $t_n \to \infty$ *as* $n \to \infty$.

Then $x_n \to p \in \Omega$, *where* p *is the unique solution of the following VIP* (4.2)*: Find* $p \in \Omega$ *such that*

$$\langle (I-f)p, x-p \rangle \geq 0 \qquad \forall\, x \in \Omega. \tag{4.2}$$

Proof. Since f is ν-contraction mapping, it follows that $I-f$ is $2(1+\nu^2)$-Lipschitzian and $(1-\nu)$-strongly monotone. Therefore by Lemma 1, it follows that the VIP (4.2) has a unique solution $p \in \Omega$.

We divide the remaining part of the proof into two steps.

Step 1: The sequence $\{x_n\}$ is bounded.

To see this, let $p \in \Omega$. Then from (4.1)

$$\begin{aligned} \|u_n - p\| &= \left\| (1-\alpha_n)x_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p \right\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p \right\| \\ &= (1-\alpha_n)\|x_n - p\| + \alpha_n \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)p ds \right\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \frac{1}{t_n}\int_0^{t_n} \|T(s)x_n - T(s)p\| ds \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{4.3}$$

Thus from (4.1) and (4.3) we get

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| \beta_n f(x_n) + \gamma_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \xi_n \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - p \right\| \\ &\leq \beta_n \|f(x_n) - p\| + \gamma_n \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p \right\| \\ &\quad + \xi_n \left\| \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - p \right\| \\ &\leq \beta_n \nu \|x_n - p\| + \beta_n \|f(p) - p\| + \gamma_n \|x_n - p\| + \xi_n \|u_n - p\| \end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n \nu \|x_n - p\| + \beta_n \|f(p) - p\| + \gamma_n \|x_n - p\| + \xi_n \|x_n - p\| \\
&\quad = [1 - \beta_n(1-\nu)]\|x_n - p\| + \beta_n \|f(p) - p\| \\
&\leq \max\left\{\|x_n - p\|, \frac{\|f(p)-p\|}{1-\nu}\right\} \\
&\quad \vdots \\
&\leq \max\left\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-\nu}\right\}.
\end{aligned}
$$

Which shows that $\{\|x_n - p\|\}$ is bounded and hence $\{x_n\}$ is bounded.

Step 2: We claim that $x_n \to p \in \Omega$. Indeed, following similar procedure as in the derivation of (3.7), we have that

$$
\Gamma_{n+1}(p) \leq (1-\lambda_n)\Gamma_n(p) + \lambda_n \eta_n,
$$

where $\lambda_n = \beta_n(1-(1-\beta_n)\nu^2)$, $\Gamma_n(p) = \|x_n - p\|$,

$$
\eta_n = \frac{[\beta_n \|f(x_n) - p\|^2 + 2(1-\beta_n)\langle f(p) - p, M(x_n, u_n) - p\rangle]}{1-(1-\beta_n)\nu^2}
$$

and

$$
M(x_n, u_n) = \frac{\gamma_n}{(1-\beta_n)t_n}\int_0^{t_n} T(s)x_n ds + \frac{\xi_n}{(1-\beta_n)t_n}\int_0^{t_n} U(s)u_n ds.
$$

Similar to the proof of Theorem 3, we consider two cases.

Case 1: Suppose the sequence $\{\Gamma_n(p)\}$ is monotonically non-increasing for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. From (4.1), (4.3) and by applying Lemma 3 (ii), we get that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\beta_n f(x_n) + \gamma_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \xi_n \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - p\right\|^2 \\
&\leq \beta_n \|f(x_n) - p\|^2 + \gamma_n \left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p\right\|^2 \\
&\quad + \xi_n \left\|\frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - p\right\|^2 \\
&\quad - \gamma_n \xi_n \left\|\frac{1}{t_n}\int_0^{t_n} (T(s)x_n - U(s)u_n) ds\right\|^2 \\
&\leq \beta_n \|f(x_n) - p\|^2 + \gamma_n \|x_n - p\|^2 + \xi_n \|u_n - p\|^2 \\
&\quad - \gamma_n \xi_n \left\|\frac{1}{t_n}\int_0^{t_n} (T(s)x_n - U(s)u_n) ds\right\|^2 \\
&\quad - \xi_n \alpha_n (1-\alpha_n)\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n\right\|^2
\end{aligned}
$$

$$\leq \beta_n \|f(x_n) - p\|^2 + (1-\beta_n)\|x_n - p\|^2$$
$$- \gamma_n \xi_n \left\| \frac{1}{t_n} \int_0^{t_n} (T(s)x_n - U(s)u_n) ds \right\|^2$$
$$- \xi_n \alpha_n (1-\alpha_n) \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - x_n \right\|^2. \tag{4.4}$$

From (4.4), we then have that

$$\xi_n \alpha_n (1-\alpha_n) \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - x_n \right\|^2 + \gamma_n \xi_n \left\| \frac{1}{t_n} \int_0^{t_n} (T(s)x_n - U(s)u_n) ds \right\|^2$$
$$\leq \beta_n \|f(x_n) - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{4.5}$$

Taking the limit as $n \to \infty$ in (4.5) gives

$$\left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - x_n \right\| \to 0 \tag{4.6}$$

and

$$\left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - \frac{1}{t_n} \int_0^{t_n} U(s)u_n ds \right\| \to 0 \tag{4.7}$$

as $n \to \infty$. Therefore from (4.1) and (4.6) we get

$$\|u_n - x_n\| = \alpha_n \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - x_n \right\| \to 0,$$

from (4.6) and (4.7), we get

$$\left\| \frac{1}{t_n} \int_0^{t_n} U(s)u_n ds - x_n \right\| \leq \left\| \frac{1}{t_n} \int_0^{t_n} U(s)u_n ds - \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds \right\|$$
$$+ \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - x_n \right\| \to 0, \tag{4.8}$$

and from (4.1) and (4.8), we get

$$\|x_{n+1} - x_n\| \leq \beta_n \|f(x_n) - x_n\| + \gamma_n \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - x_n \right\|$$
$$+ \xi_n \left\| \frac{1}{t_n} \int_0^{t_n} U(s)u_n ds - x_n \right\| \to 0$$

as $n \to \infty$. Moreover, for all $h \in [0, \infty)$, we obtain that

$$\|T(h)x_n - x_n\| \leq \left\| T(h)x_n - T(h)(\frac{1}{t_n} \int_0^{t_n} T(s)x_n ds) \right\|$$
$$+ \left\| T(h)(\frac{1}{t_n} \int_0^{t_n} T(s)x_n ds) - \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds \right\|$$

$$+\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds - x_n\right\|$$
$$\leq 2\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds - x_n\right\|$$
$$+\left\|T(h)(\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds) - \frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\right\|. \tag{4.9}$$

By applying Lemma 7, (4.6) and taking the limit as $n \to \infty$ of (4.9) we derive that

$$\|T(h)x_n - x_n\| \to 0 \text{ as } n \to \infty.$$

Furthermore, using Lemma 7, (4.6) and (4.7), we get

$$\|U(h)x_n - x_n\| \leq \left\|U(h)x_n - U(h)(\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds)\right\|$$
$$+\left\|U(h)(\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds) - U(h)(\frac{1}{t_n}\int_0^{t_n}U(s)u_n ds)\right\|$$
$$+\left\|U(h)(\frac{1}{t_n}\int_0^{t_n}U(s)u_n ds) - \frac{1}{t_n}\int_0^{t_n}U(s)u_n ds\right\|$$
$$+\left\|\frac{1}{t_n}\int_0^{t_n}U(s)u_n ds - \frac{1}{t_n}\int_0^{t_n}T(s)x_n ds\right\| + \left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds - x_n\right\|$$
$$\leq 2\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds - x_n\right\|$$
$$+2\left\|\frac{1}{t_n}\int_0^{t_n}T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n}U(s)u_n ds\right\|$$
$$+\left\|U(h)(\frac{1}{t_n}\int_0^{t_n}U(s)u_n ds) - \frac{1}{t_n}\int_0^{t_n}U(s)u_n ds\right\| \to 0 \text{ as } n \to \infty.$$

Since $\{x_n\}$ is bounded, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in C$. Therefore by the demiclosedness of $T(h)$ and $U(h)$ (see Lemma 2), we have that $x^* \in \mathrm{Fix}(\mathcal{T}) \cap \mathrm{Fix}(\mathcal{U})$. Note also that from (4.6) and (4.8),

$$M(x_{n_k}, u_{n_k}) = \frac{\gamma_{n_k}}{(1-\beta_{n_k})t_{n_k}}\int_0^{t_{n_k}}T(s)x_{n_k} ds + \frac{\xi_{n_k}}{(1-\beta_{n_k})t_{n_k}}\int_0^{t_{n_k}}U(s)u_{n_k} ds$$
$$\to \frac{\gamma_{n_k}}{(1-\beta_{n_k})}x_{n_k} + \frac{\xi_{n_k}}{(1-\beta_{n_k})}x_{n_k}$$
$$\rightharpoonup x^* \quad \text{as} \quad k \to \infty. \tag{4.10}$$

We will next show that $\limsup_{n\to\infty}\eta_n \leq 0$ and establish the strong convergence of the sequence $\{x_n\}$. Indeed, using (4.10)

$$\limsup_{n\to\infty}\eta_n \leq \limsup_{k\to\infty}\langle f(p) - p, M(x_{n_k}, u_{n_k}) - p\rangle$$

$$\leq \langle f(p)-p, x^*-p\rangle$$
$$\leq 0, \tag{4.11}$$

where (4.11) follows from (4.2). Furthermore, it is easy to see that $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$. Hence, we conclude by Lemma 4 that $\lim_{n\to\infty}\Gamma_n(p) = 0$. Therefore $x_n \to p$ as $n \to \infty$.

Case 2: Suppose $\{\Gamma_n(p)\}$ is not monotonically decreasing. It is easy to conclude from the proof of Case 2 of Theorem 3 and Case 1 of Theorem 4 that $x_n \to p$ as $n \to \infty$.

So in both cases, we obtain that $x_n \to p \in \Omega$.

□

Taking $f(x) = u$ for all $x \in C$, we obtain the following Halpern S-iteration as a consequence of Theorem 4.

Corollary 2. *Let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ and $\mathcal{U} := \{U(s) : 0 \leq s < \infty\}$ be two families of nonexpansive semigroups on C and $u \in C$ be arbitrary. Assume $\Omega := \mathrm{Fix}(\mathcal{T}) \cap \mathrm{Fix}(\mathcal{U}) \neq \varnothing$. Let $\{x_n\}$ be the sequence generated by the algorithm*

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ x_{n+1} = \beta_n u + \gamma_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \xi_n \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\xi_n\} \subset (0,1)$ are sequences such that

(i) $\beta_n + \gamma_n + \xi_n = 1$*;*
(ii) $\liminf_{n\to\infty}\gamma_n\xi_n > 0$, $\liminf_{n\to\infty}\xi_n\alpha_n(1-\alpha_n) > 0$*;*
(iii) $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$*;*
(iv) $\{t_n\}$ *is a sequence of positive numbers such that* $t_n \to \infty$ *as* $n \to \infty$.

Then $x_n \to p \in \Omega$, where $p = P_\Omega u$.

5. Numerical Examples

Example 2. Let $H = \ell_2(\mathbb{R})$, where $\ell_2(\mathbb{R}) := \{\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n,\ldots), \sigma_i \in \mathbb{R} : \sum_{i=1}^{\infty}|\sigma_i|^2 < \infty\}$, $\|\sigma\| = (\sum_{i=1}^{\infty}|\sigma_i|^2)^{\frac{1}{2}}$, $\forall\ \sigma \in H$. Let $C := \{\sigma \in H : \|\sigma\| \leq 3\}$. We define $T, U : C \to C$ by

$$Tx = \frac{x}{2} + (1,0,0,\cdots) \qquad \forall\, x \in C$$

and

$$Ux = \frac{x}{4} + (\frac{3}{2},0,0,\cdots) \qquad \forall\, x \in C,$$

respectively. It is easy to see that $T(2,0,0,\cdots) = U(2,0,0,\cdots) = (2,0,0,\cdots)$. Therefore $\mathrm{Fix}(T)\cap\mathrm{Fix}(U) \neq \varnothing$. We choose $f(x) = \frac{x}{1.02}$ for every $x \in C$, $\beta_n = \frac{1}{n+1}$,

$\gamma_n = \frac{2n}{3(n+1)}$, $\xi_n = \frac{n}{3(n+1)}$ and $\alpha_n = \frac{n+1}{2n-1}$. In this case, (3.1) and (1.4) gives

$$\begin{cases} u_n = \frac{n-2}{2n-1}x_n + \frac{n+1}{2n-1}Tx_n, \\ x_{n+1} = \frac{x_n}{1.02(n+1)} + \frac{2n}{3(n+1)}Tx_n + \frac{n}{3(n+1)}Uu_n, \quad n \geq 1, \end{cases}$$

and

$$x_{n+1} = \frac{x_n}{1.02(n+1)} + \frac{2n}{3(n+1)}Tx_n + \frac{n}{3(n+1)}Ux_n, n \geq 1,$$

respectively.

We choose different initial values as follows:

Case Ia: $x_1 = (0.3, 0.8, -0.4, -0.2, 0, 0, 0, \dots)$;
Case Ib: $x_1 = (-2, 1, 1, -0.2, 0, 0, 0, \dots)$;
Case Ic: $x_1 = (1.3, 0.4, 0, -0.2, 0, 0, 0, \dots)$;
Case Id: $x_1 = (0.4, 0, 1, -0.8, 0, 0, 0, \dots)$.

Using MATLAB 2017(b), we compare the performance of Algorithm 2 with Algorithm (1.4) of Ahmad et al. [2]. The stopping criterion used for our computation is $\frac{\|x_{n+1}-x_n\|^2}{\|x_2-x_1\|^2} < 10^{-7}$. We plot the graphs of errors against the number of iterations in each case.

TABLE 1. Numerical results for Example 2.

		Alg. (1.4)	Alg. 3.1
Case Ia	CPU time (sec)	0.0014	9.5022e-4
	No of Iter.	43	37
Case Ib	CPU time (sec)	0.0016	9.3929e-4
	No. of Iter.	45	39
Case Ic	CPU time (sec)	0.0014	9.3664e-4
	No of Iter.	41	35
Case Id	CPU time (sec)	0.0013	9.5949-4
	No of Iter.	43	37

The next example is inspired by Vong and Liu [25] and He et al. [14].

Example 3. Let $\mathbb{R}^2$ be the two dimensional Euclidean space with the usual inner product $\langle x, y\rangle = x_1y_1 + x_2y_2$ for all $x = (x_1, x_2)^\intercal, y = (y_1, y_2)^\intercal \in \mathbb{R}^2$ and the norm $\|x\| = \sqrt{x_1^2 + x_2^2}$. Let $C := [-1, 1] \times [-1, 1]$ and define $T : C \to C$ by

$$T : x = (x_1, x_2)^\intercal \mapsto \left(\frac{1}{2}\sin(\frac{x_1+x_2}{\sqrt{2}}), \frac{1}{2}(\cos(\frac{x_1+x_2}{\sqrt{2}}) - 1)\right).$$

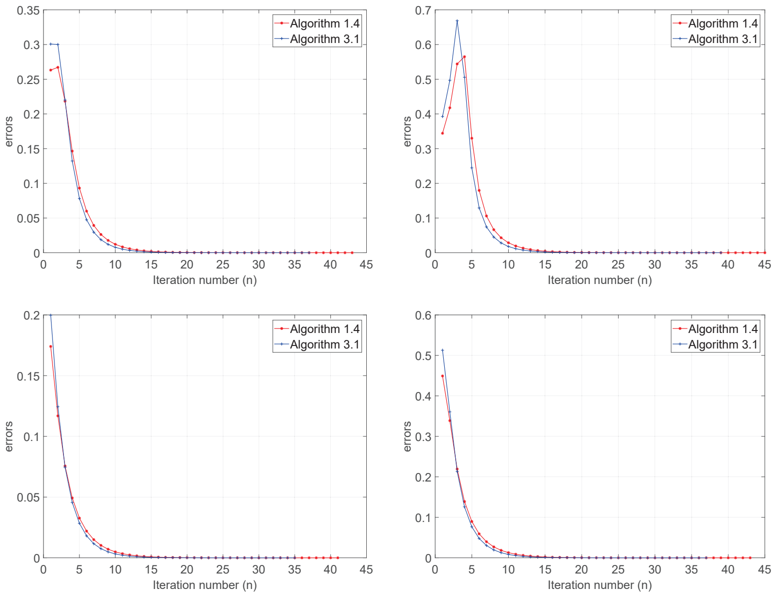

FIGURE 1. Example 2: Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

For each $x = (x_1, x_2) \in \mathbb{R}^2$, T is a Fréchet differentiable and

$$T'(x) = \frac{1}{2\sqrt{2}} \begin{pmatrix} \cos(\frac{x_1+x_2}{\sqrt{2}}) & \cos(\frac{x_1+x_2}{\sqrt{2}}) \\ -\sin(\frac{x_1+x_2}{\sqrt{2}}) & -\sin(\frac{x_1+x_2}{\sqrt{2}}) \end{pmatrix}. \tag{5.1}$$

It is known that $T'(x)$ is a bounded linear operator from $\mathbb{R}^2$ to itself. The norm of $T'(x)$ can be derived by the formula:

$$\|T'(x)\| = \sqrt{\lambda} \qquad \forall\, x \in \mathbb{R}^2,$$

where λ is the maximal eigenvalue of the matrix $(T'(x))^\intercal T'(x)$. From (5.1), it is easy to see that

$$(T'(x))^\intercal = \frac{1}{2\sqrt{2}} \begin{pmatrix} \cos(\frac{x_1+x_2}{\sqrt{2}}) & -\sin(\frac{x_1+x_2}{\sqrt{2}}) \\ \cos(\frac{x_1+x_2}{\sqrt{2}}) & -\sin(\frac{x_1+x_2}{\sqrt{2}}) \end{pmatrix} \tag{5.2}$$

and therefore from (5.1) and (5.2)

$$(T'(x))^\intercal T'(x) = \frac{1}{8} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5.3}$$

It therefore follows from (5.3) that $\|T'(x)\| = \frac{1}{2}$, $\forall\, x \in \mathbb{R}^2$. Then, using Mean Value Theorem, for any $x, y \in \mathbb{R}^2$, there is a constant $\tau \in (0,1)$ such that

$$\|Tx - Ty\| = \|T'(\tau x + (1-\tau)y)(x-y)\|$$

$$\le \|T'(\tau x + (1-\tau)y)\| \|(x-y)\|$$
$$\le \|x-y\|.$$

Which shows that T is a nonexpansive mapping. We define $U : C \to C$ by $Ux = \frac{x}{2}$ for all $x \in C$. Then it is easy to see that $(0,0) \in \mathrm{Fix}(T) \cap \mathrm{Fix}(U)$. We choose $f(x) = \frac{x}{1.05}$ for every $x \in C$, $\beta_n = \frac{1}{n+1}$, $\gamma_n = \frac{2n}{3(n+1)}$, $\xi_n = \frac{n}{3(n+1)}$ and $\alpha_n = \frac{n+1}{2n-1}$. In this case, (3.1) and (1.4) gives

$$\begin{cases} u_n = \frac{n-2}{2n-1}x_n + \frac{n+1}{2n-1}Tx_n, \\ x_{n+1} = \frac{x_n}{1.05(n+1)} + \frac{2n}{3(n+1)}Tx_n + \frac{n}{3(n+1)}Uu_n, \quad n \ge 1, \end{cases}$$

and

$$x_{n+1} = \frac{x_n}{1.05(n+1)} + \frac{2n}{3(n+1)}Tx_n + \frac{n}{3(n+1)}Ux_n, \quad n \ge 1,$$

respectively.

We choose different initial values as follows:

Case IIa: $x_1 = (0.05, -0.4)$;
Case IIb: $x_1 = (-0.9, 0)$;
Case IIc: $x_1 = (0.9, 0.4)$;
Case IId: $x_1 = (-0.5, -0.8)$.

Using MATLAB 2017(b), we compare the performance of Algorithm 2 with Algorithm (1.4) of Ahmad et al. [2]. The stopping criterion used for our computation is $\|x_{n+1} - x_n\|^2 < 10^{-7}$. We plot the graphs of $||x_{n+1} - x_n||^2$ against the number of iterations in each case.

TABLE 2. Numerical results for Example 2.

		Alg. (1.4)	Alg. 3.1
Case IIa	CPU time (sec)	0.0013	8.6347e-4
	No of Iter.	26	17
Case IIb	CPU time (sec)	0.0014	8.7407e-4
	No. of Iter.	27	18
Case IIc	CPU time (sec)	0.0027	7.9229e-4
	No of Iter.	28	18
Case IId	CPU time (sec)	0.0011	9.3366e-4
	No of Iter.	27	18

Remark 1. From the computational results, it can be inferred that our algorithm performs better in both number of iterations and computation time taken. In addition, the choices of different initial values do not have significant effects on the output of the Algorithm (3.1) in terms of the number of iterations and computation time taken.

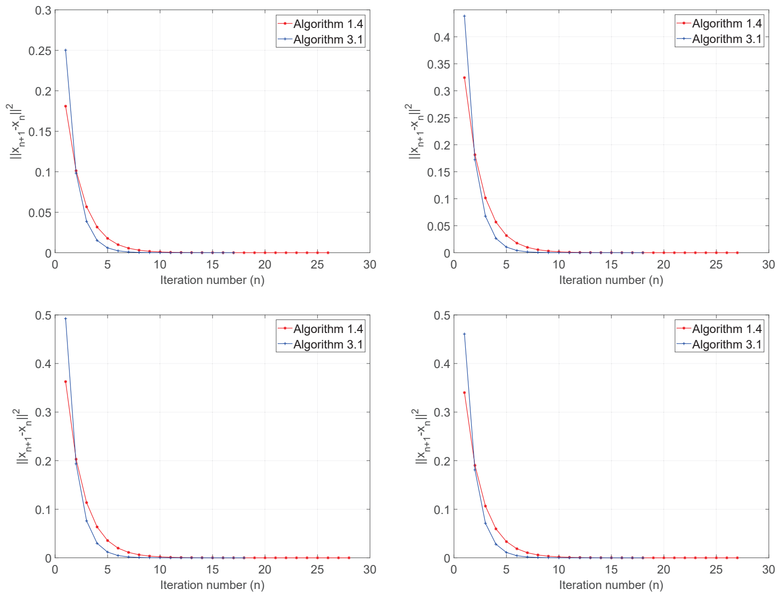

FIGURE 2. Example 3: Top left: Case IIa; Top right: Case IIb; Bottom left: Case IIc; Bottom right: Case IId.

6. CONCLUSION

In this paper, we present a viscosity-S iteration for finding the common fixed points of two nonexpansive mappings in real Hilbert spaces. We apply our results to nonexpansive semigroup of operators. We also illustrate the efficiency and effectiveness of our algorithm using examples from a finite dimensional Hilbert space and an infinite dimensional Hilbert space and by comparing with a similar existing algorithm in literature.

ACKNOWLEDGEMENTS

The first author acknowledges with thanks the International Mathematical Union Breakout Graduate Fellowship (IMU-BGF) Award for his doctoral study. The third author acknowledges with thanks the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. The fourth author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS, IMU and NRF.

Declaration

The authors declare that they have no competing interests.

REFERENCES

[1] R. P. Agarwal, D. O'Regan, and D. R. Sahu, "Iterative construction of fixed points of nearly asymptotically nonexpansive mappings," *J. Nonlinear Convex Anal.*, vol. 8, no. 1, pp. 61–79, 2007.

[2] M. Ahmad, W. Nazeer, S. Kang, and S. Naqvi, "On iterative approach to common fixed points of nonexpansive mappings in hilbert spaces." *J. Comp. Anal. and Appl.*, vol. 26, no. 7, pp. 1302–1308, 2019.

[3] T. O. Alakoya, L. O. Jolaoso, A. Taiwo, and O. T. Mewomo, "Inertial algorithm with self-adaptive stepsize for split common null point and common fixed point problems for multivalued mappings in banach spaces." *Optimization*, 2021, doi: 10.1080/02331934.2021.1895154.

[4] T. O. Alakoya, A. Taiwo, O. T. Mewomo, and Y. J. Cho, "An iterative algorithm for solving variational inequality, generalized mixed equilibrium, convex minimization and zeros problems for a class of nonexpansive-type mappings." *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, vol. 67, no. 1, pp. 1–31, 2021, doi: 10.1007/s11565-020-00354-2.

[5] S. Atsushiba and W. Takahashi, "A weak convergence theorem for nonexpansive semigroups by the Mann iteration process in Banach spaces," in *Nonlinear analysis and convex analysis (Niigata, 1998)*. World Sci. Publ., River Edge, NJ, 1999, pp. 102–109.

[6] H. H. Bauschke and J. M. Borwein, "On projection algorithms for solving convex feasibility problems," *SIAM Rev.*, vol. 38, no. 3, pp. 367–426, 1996, doi: 10.1137/S0036144593251710. [Online]. Available: https://doi.org/10.1137/S0036144593251710

[7] H. H. Bauschke and P. L. Combettes, "A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces," *Math. Oper. Res.*, vol. 26, no. 2, pp. 248–264, 2001, doi: 10.1287/moor.26.2.248.10558. [Online]. Available: https://doi.org/10.1287/moor.26.2.248.10558

[8] H. H. Bauschke and P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, ser. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011, with a foreword by Hédy Attouch. [Online]. Available: https://doi.org/10.1007/978-1-4419-9467-7. doi: 10.1007/978-1-4419-9467-7

[9] L. Bussaban and A. Kettapun, "Common fixed points of an iterative method for Berinde nonexpansive mappings," *Thai J. Math.*, vol. 16, no. 1, pp. 49–60, 2018.

[10] H. Che and M. Li, "Solving split variational inclusion problem and fixed point problem for nonexpansive semigroup without prior knowledge of operator norms," *Math. Probl. Eng.*, pp. Art. ID 408 165, 9, 2015, doi: 10.1155/2015/408165. [Online]. Available: https://doi.org/10.1155/2015/408165

[11] C. Chidume, *Geometric properties of Banach spaces and nonlinear iterations*, ser. Lecture Notes in Mathematics. Springer-Verlag London, Ltd., London, 2009, vol. 1965.

[12] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990, vol. 28. [Online]. Available: https://doi.org/10.1017/CBO9780511526152. doi: 10.1017/CBO9780511526152

[13] B. Halpern, "Fixed points of nonexpanding maps," *Bull. Amer. Math. Soc.*, vol. 73, pp. 957–961, 1967, doi: 10.1090/S0002-9904-1967-11864-0. [Online]. Available: https://doi.org/10.1090/S0002-9904-1967-11864-0

[14] S. He, C. Yang, and P. Duan, "Realization of the hybrid method for Mann iterations," *Appl. Math. Comput.*, vol. 217, no. 8, pp. 4239–4247, 2010, doi: 10.1016/j.amc.2010.10.039. [Online]. Available: https://doi.org/10.1016/j.amc.2010.10.039

[15] L. O. Jolaoso, A. Taiwo, T. O. Alakoya, and O. T. Mewomo, "A strong convergence theorem for solving pseudo-monotone variational inequalities using projection methods," *J. Optim. Theory Appl.*, vol. 185, no. 3, pp. 744–766, 2020, doi: 10.1007/s10957-020-01672-3. [Online]. Available: https://doi.org/10.1007/s10957-020-01672-3

[16] Y. Ke and C. Ma, "The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces," *Fixed Point Theory Appl.*, pp. 2015:190, 21, 2015, doi: 10.1186/s13663-015-0439-6. [Online]. Available: https://doi.org/10.1186/s13663-015-0439-6

[17] P.-L. Lions, "Approximation de points fixes de contractions," *C. R. Acad. Sci. Paris Sér. A-B*, vol. 284, no. 21, pp. A1357–A1359, 1977.

[18] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Anal.*, vol. 16, no. 7-8, pp. 899–912, 2008, doi: 10.1007/s11228-008-0102-z. [Online]. Available: https://doi.org/10.1007/s11228-008-0102-z

[19] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *J. Math. Anal. Appl.*, vol. 241, no. 1, pp. 46–55, 2000, doi: 10.1006/jmaa.1999.6615. [Online]. Available: https://doi.org/10.1006/jmaa.1999.6615

[20] G. N. Ogwo, C. Izuchukwu, and O. T. Mewomo, "Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity," *Numer. Algorithms*, vol. 88, no. 3, pp. 1419–1456, 2021, doi: 10.1007/s11075-021-01081-1. [Online]. Available: https://doi.org/10.1007/s11075-021-01081-1

[21] S. Reich, "A note on the mean ergodic theorem for nonlinear semigroups," *J. Math. Anal. Appl.*, vol. 91, no. 2, pp. 547–551, 1983, doi: 10.1016/0022-247X(83)90168-3. [Online]. Available: https://doi.org/10.1016/0022-247X(83)90168-3

[22] T. Shimizu and W. Takahashi, "Strong convergence to common fixed points of families of nonexpansive mappings," *J. Math. Anal. Appl.*, vol. 211, no. 1, pp. 71–83, 1997, doi: 10.1006/jmaa.1997.5398. [Online]. Available: https://doi.org/10.1006/jmaa.1997.5398

[23] R. Suparatulatorn, W. Cholamjiak, and S. Suantai, "A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs," *Numer. Algorithms*, vol. 77, no. 2, pp. 479–490, 2018, doi: 10.1007/s11075-017-0324-y. [Online]. Available: https://doi.org/10.1007/s11075-017-0324-y

[24] A. Taiwo, T. O. Alakoya, and O. T. Mewomo, "Halpern-type iterative process for solving split common fixed point and monotone variational inclusion problem between Banach spaces," *Numer. Algorithms*, vol. 86, no. 4, pp. 1359–1389, 2021, doi: 10.1007/s11075-020-00937-2. [Online]. Available: https://doi.org/10.1007/s11075-020-00937-2

[25] S. Vong and D. Liu, "An inertial Mann algorithm for nonexpansive mappings," *J. Fixed Point Theory Appl.*, vol. 20, no. 3, pp. Paper No. 102, 16, 2018, doi: 10.1007/s11784-018-0583-9. [Online]. Available: https://doi.org/10.1007/s11784-018-0583-9

[26] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Arch. Math. (Basel)*, vol. 58, no. 5, pp. 486–491, 1992, doi: 10.1007/BF01190119. [Online]. Available: https://doi.org/10.1007/BF01190119

[27] H.-K. Xu, "Another control condition in an iterative method for nonexpansive mappings," *Bull. Austral. Math. Soc.*, vol. 65, no. 1, pp. 109–113, 2002, doi: 10.1017/S0004972700020116. [Online]. Available: https://doi.org/10.1017/S0004972700020116

[28] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in *Inherently parallel algorithms in feasibility and optimization and their applications (Haifa, 2000)*, ser. Stud. Comput. Math. North-Holland, Amsterdam, 2001, vol. 8, pp. 473–504. [Online]. Available: https://doi.org/10.1016/S1570-579X(01)80028-8. doi: 10.1016/S1570-579X(01)80028-8

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