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SOLUTIONS OF CERTAIN CLASS OF NON-LINEAR TIME-FRACTIONAL DIFFUSION EQUATIONS VIA THE FRACTIONAL DIFFERENTIAL TRANSFORM METHOD

MEHMET BOZER, MEHMET ALI ÖZARSLAN, AND HÜLYA DEMEZ

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Abstract. In this paper, the generalized differential transform method (GDTM) is used to solve certain class of nonlinear time-fractional diffusion equation. An efficient recurrence relation is obtained to solve this problem. Some numerical examples are given for different class of derivative orders which are analyzed numerically for the specified examples.

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1. INTRODUCTION

Fractional Differential Equations have been the subject of intensive research of recent years because of their applications in physics and engineering especially in electromagnetic, electrochemistry, material science, bioengineering, quantum mechanics, finance applied problems including diffusion equations (see [22], [9], [5], [10], [13], [14], [15], [25], [2], [11], [23] and [26]).

Several numerical methods have been generalized to solve this kind of applied problems such as Variational Iteration Method [8], Adomian Decomposition Method [17], [27], [20],[16], Fractional Differential Transform Method [3], [21] etc. The basic idea of differential transform method was first proposed by Zhou [28]. This method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. It's different from the traditional higher order Taylor series method because it requires symbolic computation. In 2007, Arikoglu and Ozkol [4] introduced Fractional Differential Transform Method (FDTM) as a new analytical technique for solving fractional type differential equations. In [19], the authors developed a new generalization of the two dimensional differential transform method which is called the Generalized Differential Transform Method (GDTM) (also see [7], [24], [18] and [8]).

It is pointed out by Bervillier [6] that the limited improvements of the DTM over the well-known Taylor-series solution of ODE's is because of their great similarity. Beside this, in problems involving fractional derivatives major contribution of the DTM can be found.

Now, let us give the following necessary definitions and properties which is needed to summarize the FDTM.

Let $\Omega = [a,b]$ be a finite interval on the real axis $\mathbb{R}$ and the Riemann-Liouville fractional integral [12] of order α is defined by

$$(I_{a^+}^{\alpha} f)(x) : = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \ (x > a; \ \mathrm{Re}(\alpha) > 0), \tag{1.1}$$

where $\Gamma(\alpha)$ is the gamma function defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{1.2}$$

The Riemann-Liouville fractional derivative [12] of order α is defined by

$$(D_{a^+}^{\alpha} f)(x) := \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{n-\alpha} f\right)(x), \ n = [\mathrm{Re}(\alpha)] + 1; \ x > a. \tag{1.3}$$

The following relation holds true for the power function (see p. 418 of [12])

$$\left(D_{a^+}^{\alpha} (t-a)^{\beta}\right)(x) := \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}, \ x > a, \mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta) > -1. \tag{1.4}$$

The Caputo fractional derivative [12] is defined by

$$(C_{a^+}^{\alpha} f)(x) = (I_{a^+}^{n-\alpha} f^{(n)})(x), \ x > a, \ n-1 < \alpha \leq n, \ n \in \mathbb{N}. \tag{1.5}$$

The sequential fractional derivative [1] for a sufficiently smooth function $f(t)$ due to Miller-Ross [15] is defined by

$$D^{\delta} f(t) = D^{\delta_1} D^{\delta_2} ... D^{\delta_k} f(t), \tag{1.6}$$

where $\delta = (\delta_1 ... \delta_k)$ is a multi-index.

In general, the operator D^{δ} in (1.6) can either be Riemann-Liouville or Caputo or any other kind of integro-differential operator.

In [24] Rida, et al. considered the reaction diffusion equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = K \frac{\partial^2 u}{\partial x^2} + f(u), 0 < \alpha \leq 1, t > 0, x \in \mathbb{R} \tag{1.7}$$

where K is the diffusion coefficient and they provide the extension of the GDTM in order to give its numerical solution. It was noteworthy that for $f(u) = 6u(1-u)$, Eq. (1.7) reduces to the time-fractional Fisher equation.

Subsequently, in the year 2013, Cetinkaya and Kiymaz [7], considered the following time-fractional diffusion equation

$$D_t^{\beta} u = \lambda \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(F(x)u(x,t)\right), 0 < \beta \leq 1, x, t > 0 \tag{1.8}$$

with initial condition $u(x,0) = f(x)$. Here λ is a positive constant, $F(x)$ is the external force, $u(x,t)$ represents the probability density function of finding a particle at the point x in the time t and $D_t^{\beta} u(x,t) = I_0^{1-\beta}\left[\frac{\partial}{\partial t} u(x,t)\right]$. A general recurrence relation for (1.8) was obtained with the GDTM.

Our main concern in this paper is the following nonlinear time-fractional diffusion equation. Through the above definitions

$$D_t^{\beta} u(x,t) = \lambda C_x^{(2\alpha} u(x,t) - C_x^{\alpha}\left[F(x)u(x,t)\left(1 - \mu u(x,t)\right)\right], 0 < \alpha, \beta \leq 1, x, t > 0 \tag{1.9}$$

with initial condition $u(x,0) = f(x)$. Here $C_x^{(2\alpha}$ represent the Caputo sequential fractional derivatives of order 2α. Also λ and μ is a positive constant, $F(x)$ is the external force, $u(x,t)$ represents the probability density function of finding a particle at the point x in the time t and $D_t^{\beta} u(x,t) = \frac{\partial}{\partial t}\left[I_0^{1-\beta} u(x,t)\right]$.

Remark 1. If we set $\alpha = 1$ and $\mu = 0$ in (1.9), we have the time-fractional diffusion equation which is given by (1.8).

The paper is organized as follows: In Section 2, the GDTM is summarized. In Section 3, the solution of the general problem is derived. In Section 4, the method is implemented to a couple of examples and some special cases of the examples are obtained numerically and graphically. Conclusion is given in the last section.

2. Generalized Differential Transform

In this section we shall summarize the GDTM. Consider a function of two variables $u(x,t)$ and suppose that it can be represented as a product of two single-variable function, i.e.,

$$u(x,t) = f(x)g(t). \tag{2.1}$$

Based on the properties of generalized two-dimensional differential transform, the function $u(x,t)$ can be represented as

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{\infty} F_{\alpha}(i)(x-x_0)^{i\alpha} \sum_{j=0}^{\infty} G_{\beta}(j)(t-t_0)^{j\beta} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} U_{\alpha,\beta}(i,j)(x-x_0)^{i\alpha}(t-t_0)^{j\beta} \end{aligned} \tag{2.2}$$

where $0 < \alpha, \beta \leq 1$ and $U_{\alpha,\beta}(i,j) = F_{\alpha}(i)G_{\beta}(j)$ is the spectrum of $u(x,t)$.

If the function $u(x,t)$ is analytic and differentiable continuously with respect to time t, then the GDT of the function $u(x,t)$ is defined as follows:

$$U_{\alpha,\beta}(i,j) = \frac{1}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}\left[\left(C_{x_0}^{(i\alpha)}\right)\left(C_{t_0}^{(j\beta)}\right)u(x,t)\right]_{(x_0,t_0)} \tag{2.3}$$

where $\left(C_{x_0}^{(i\alpha)}\right) = C_{x_0}^{\alpha}C_{x_0}^{\alpha}...C_{x_0}^{\alpha},\ i-times$. Besides, equation (2.2) is called the generalized inverse differential transform of $U_{\alpha,\beta}(i,j)$. In the case of $\alpha=\beta=1$, the generalized two-dimensional differential transform (2.3) reduces to the classical two-dimensional transform.

Theorem 1. *[19] (see also [7]) Suppose that $U_{\alpha,\beta}(i,j)$, $V_{\alpha,\beta}(i,j)$ and $W_{\alpha,\beta}(i,j)$ are the differential transformations of the function $u(x,t)$, $v(x,t)$ and $w(x,t)$, respectively. Then the following statements hold true:*

(1) *If $u(x,t) = v(x,t) \pm w(x,t)$ then $U_{\alpha,\beta}(i,j) = V_{\alpha,\beta}(i,j) \pm W_{\alpha,\beta}(i,j)$.*
(2) *If $u(x,t) = \lambda v(x,t), \lambda \in \mathbb{R}$ then $U_{\alpha,\beta}(i,j) = \lambda V_{\alpha,\beta}(i,j)$.*
(3) *If $u(x,t) = v(x,t)w(x,t)$ then*

$$U_{\alpha,\beta}(i,j) = \sum_{r=0}^{i}\sum_{s=0}^{j} V_{\alpha,\beta}(r,j-s)W_{\alpha,\beta}(i-r,s).$$

(4) *If $u(x,t) = (x-x_0)^{n\alpha}(t-t_0)^{m\beta}$ then $U_{\alpha,\beta}(i,j) = \delta(i-n)\delta(j-m)$.*
(5) *If $u(x,t) = f(x)g(t)$ and the function $f(x) = x^{\lambda}h(x)$ where $\lambda > -1$, $h(x)$ has the generalized Taylor series expansion*

$$h(x) = \sum_{m=0}^{\infty} a_m(x-x_0)^{m\alpha}$$

and

(a): *$\beta < \lambda+1$ and α arbitrary or*
(b): *$\beta \geq \lambda+1$, α arbitrary and $a_m = 0$ for $m = 0,1,\ldots,n-1$ where $(n-1) < \beta < n$, then the generalized differential transform (2.3) becomes*

$$U_{\alpha,\beta}(i,j) = \frac{1}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}\left[C_{x_0}^{(i\alpha}\left(C_{t_0}^{(j\beta)}\right)u(x,t)\right]_{(x_0,t_0)}.$$

(6) *If $u(x,t) = C_{x_0}^{\gamma}v(x,t), m-1 < \gamma \leq m$ and $v(x,t) = f(x)g(t)$ then*

$$U_{\alpha,\beta}(i,j) = \frac{\Gamma(i\alpha+\gamma+1)}{\Gamma(i\alpha+1)}V_{\alpha,\beta}(i+\frac{\gamma}{\alpha},j).$$

(7) *If $u(x,t) = C_{t_0}^{\gamma}v(x,t), m-1 < \gamma \leq m$ and $v(x,t) = f(x)g(t)$ then*

$$U_{\alpha,\beta}(i,j) = \frac{\Gamma(j\beta+\gamma+1)}{\Gamma(j\beta+1)}V_{\alpha,\beta}(i,j+\frac{\gamma}{\beta}).$$

(8) *If* $u(x,t) = C_{x_0}^{\gamma} C_{t_0}^{\mu} v(x,t), m-1 < \gamma \le m, n-1 < \mu \le n$ *and* $v(x,t) = f(x)g(t)$ *where the function* $f(x)$ *and* $g(t)$ *satisfy the condition given in* 5 *above then*

$$U_{\alpha,\beta}(i,j) = \frac{\Gamma(i\alpha+\gamma+1)}{\Gamma(i\alpha+1)} \frac{\Gamma(j\beta+\gamma+1)}{\Gamma(j\beta+1)} V_{\alpha,\beta}(i+\frac{\gamma}{\alpha}, j+\frac{\mu}{\beta}).$$

Proof. The proof is straighthforward using (2.2) and (2.3) and it was given in detail in [19]. □

3. The Solution of the Main Problem

In this section, we obtain a recurrence relation for the coefficients of the solution $u(x,t)$ of the Eq. (1.9).

Theorem 2. *If the function* $F(x)$ *has the series expansion* $F(x) = \sum_{n=0}^{\infty} a_n (x-x_o)^{n\alpha}$ *with a radius of convergence* $R > 0$ *where* $a_n = \frac{1}{\Gamma(n\alpha+1)} \left[C_{x_0}^{(n\alpha} F(x_0) \right]$ *for* $n = 0,1,2,\ldots,$ *then the GDT of the solution of Eq.* (1.9) *satisfies the following recurrence relation*

$$\begin{aligned} U_{\alpha,\beta}(i,j+1) = \frac{\Gamma(j\beta+1)}{\Gamma((j+1)\beta+1)} \Bigg[& \lambda \frac{\Gamma((i+2)\alpha+1)}{\Gamma(i\alpha+1)} U_{\alpha,\beta}(i+2,j) \\ & - \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)} \sum_{m=0}^{i+1} a_{i+1-m} U_{\alpha,\beta}(m,j) \\ & + \mu \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)} \sum_{m=0}^{i+1} \sum_{r=0}^{m} \sum_{s=0}^{j} a_{i+1-m} U_{\alpha,\beta}(r,j-s) U_{\alpha,\beta}(m-r,s) \Bigg]. \end{aligned} \tag{3.1}$$

Proof. Suppose that the solution $u(x,t)$ can be represented as a product of two single-variable functions. Applying the generalized two-dimensional differential transform to both sides of Eq. (1.9) and using theorem (1), Eq. (1.9) transforms to

$$\begin{aligned} \frac{\Gamma((j+1)\beta+1)}{\Gamma(j\beta+1)} U_{\alpha,\beta}(i,j+1) & \\ = \lambda & \frac{\Gamma((i+2)\alpha+1)}{\Gamma(i\alpha+1)} U_{\alpha,\beta}(i+2,j) \\ & - \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)} \sum_{m=0}^{i+1} a_{i+1-m} U_{\alpha,\beta}(m,j) \\ & + \mu \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)} \sum_{m=0}^{i+1} \sum_{r=0}^{m} \sum_{s=0}^{j} a_{i+1-m} U_{\alpha,\beta}(r,j-s) U_{\alpha,\beta}(m-r,s) \end{aligned}$$

which can be written as

$$U_{\alpha,\beta}(i,j+1) = \frac{\Gamma(j\beta+1)}{\Gamma((j+1)\beta+1)} \Bigg[\lambda \frac{\Gamma((i+2)\alpha+1)}{\Gamma(i\alpha+1)} U_{\alpha,\beta}(i+2,j)$$

$$-\frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)}\sum_{m=0}^{i+1}a_{i+1-m}U_{\alpha,\beta}(m,j)$$
$$\left.+\mu\frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)}\sum_{m=0}^{i+1}\sum_{r=0}^{m}\sum_{s=0}^{j}a_{i+1-m}U_{\alpha,\beta}(r,j-s)U_{\alpha,\beta}(m-r,s)\right].$$

□

We should note that the generalized two-dimensional transform of the initial condition $u(x,0)$ is given by

$$U_{\alpha,\beta}(i,0)=\frac{1}{\Gamma(i\alpha+1)}\left[\left(C_{x_0}^{(i\alpha}\right)u(x,0)\right],i=0,1,2,\ldots. \tag{3.2}$$

4. Numerical Examples

In this section, we have selected two examples which show the simplicity and effectiveness of the proposed general recurrence relation (3.1) in order to give the solution of the main problem. We also present the numerical results for each example.

Example 1. Consider the following time-fractional diffusion equation

$$D_t^{\beta}u(x,t)=\lambda C_x^{(2\alpha}u(x,t)-C_x^{\alpha}\left[-(x^{\alpha})u(x,t)\left(1-\mu u(x,t)\right)\right],x,t>0 \tag{4.1}$$

where $0<\alpha,\beta\leq 1$, $\lambda=1$, $\mu=1$ and subject to the initial condition

$$u(x,0)=x^{\alpha}. \tag{4.2}$$

Since $F(x)=-(x)^{\alpha}$, we have $a_1=-1$ and $a_n=0$ for $n\neq 1$. In addition to this, substituting $f(x)=x^{\alpha}$ in (3.2), we have

$$U_{\alpha,\beta}(1,0)=1,U_{\alpha,\beta}(i,0)=0 \text{ for } i\neq 1. \tag{4.3}$$

Applying the recurrence relation (3.1) and using the transformed initial condition (4.3), the first few components of $U_{\alpha,\beta}(i,j)$ can be calculated as follows:

$$U_{\alpha,\beta}(1,1)=\frac{\Gamma(2\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)},\ U_{\alpha,\beta}(i,1)=0 \text{ for } i\neq 1, \tag{4.4}$$
$$U_{\alpha,\beta}(1,2)=\frac{(\Gamma(2\alpha+1))^2}{\Gamma(2\beta+1)\left(\Gamma(\alpha+1)\right)^2},\ U_{\alpha,\beta}(i,2)=0 \text{ for } i\neq 1,$$
$$U_{\alpha,\beta}(1,3)=\frac{(\Gamma(2\alpha+1))^3}{\Gamma(3\beta+1)\left(\Gamma(\alpha+1)\right)^3},\ U_{\alpha,\beta}(i,3)=0 \text{ for } i\neq 1,$$
$$U_{\alpha,\beta}(1,4)=\frac{(\Gamma(2\alpha+1))^4}{\Gamma(4\beta+1)\left(\Gamma(\alpha+1)\right)^4},\ U_{\alpha,\beta}(i,4)=0 \text{ for } i\neq 1,$$
$$\vdots$$

$$U_{\alpha,\beta}(1,j)=\frac{(\Gamma(2\alpha+1))^{j}}{\Gamma(j\beta+1)\,(\Gamma(\alpha+1))^{j}},\; U_{\alpha,\beta}(i,j)=0 \text{ for } i\neq 1.$$

So the solution $u(x,t)$ of Eq. (4.1) is obtained by

$$\begin{aligned}
u(x,t)&=x^{\alpha}\left[1+\frac{\Gamma(2\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)}t^{\beta}+\frac{(\Gamma(2\alpha+1))^{2}}{\Gamma(2\beta+1)\,(\Gamma(\alpha+1))^{2}}t^{2\beta}\right.\\
&\left.+\frac{(\Gamma(2\alpha+1))^{3}}{\Gamma(3\beta+1)\,(\Gamma(\alpha+1))^{3}}t^{3\beta}+\frac{(\Gamma(2\alpha+1))^{4}}{\Gamma(4\beta+1)\,(\Gamma(\alpha+1))^{4}}t^{4\beta}+\dots\right]\\
&=x^{\alpha}\sum_{k=0}^{\infty}\frac{(\Gamma(2\alpha+1))^{k}}{\Gamma(k\beta+1)\,(\Gamma(\alpha+1))^{k}}t^{k\beta}\\
&=x^{\alpha}E_{\beta}\left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}t^{\beta}\right),
\end{aligned}$$

		$u(x,t)$			
t	x	$\alpha=0.85$, $\beta=0.9$	$\alpha=0.9$, $\beta=0.95$	$\alpha=0.95$ $\beta=0.99$	$\alpha=1$, $\beta=1$
0.00	0.25	0.3078	0.2872	0.2679	0.2500
	0.50	0.5548	0.5359	0.5176	0.5000
	0.75	0.7831	0.7719	0.7609	0.7500
	1.00	1.0000	1.0000	1.000	1.0000
0.25	0.25	0.5072	0.4651	0.4312	0.4122
	0.50	0.9142	0.8678	0.8330	0.8244
	0.75	1.2904	1.2500	1.2244	1.2365
	1.00	1.6479	1.6194	1.6092	1.6487
0.50	0.25	0.7953	0.7352	0.6906	0.6796
	0.50	1.4336	1.3719	1.3341	1.3591
	0.75	2.0235	1.9760	1.9610	2.0387
	1.00	2.5841	2.5600	2.5773	2.7183
0.75	0.25	1.2347	1.1563	1.1049	1.1204
	0.50	2.2255	2.1577	2.1345	2.2408
	0.75	3.1413	3.1079	3.1374	3.3613
	1.00	4.0115	4.0264	4.1235	4.4817
1.00	0.25	1.9088	1.8148	1.7670	1.8473
	0.50	3.4406	3.3866	3.4136	3.6945
	0.75	4.8564	4.8780	5.0176	5.5418
	1.00	6.2018	6.3196	6.5946	7.3891

TABLE 1. Approximate solution for u(x,t).

where

$$E_{\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)}$$

is the usual Mittag-Leffler function.

It is seen from Figure 1 that $u(x,t)$ increase as t increases for all α and β. However for fixed t, $u(x,t)$ is found to decrease with the increase in α and β.

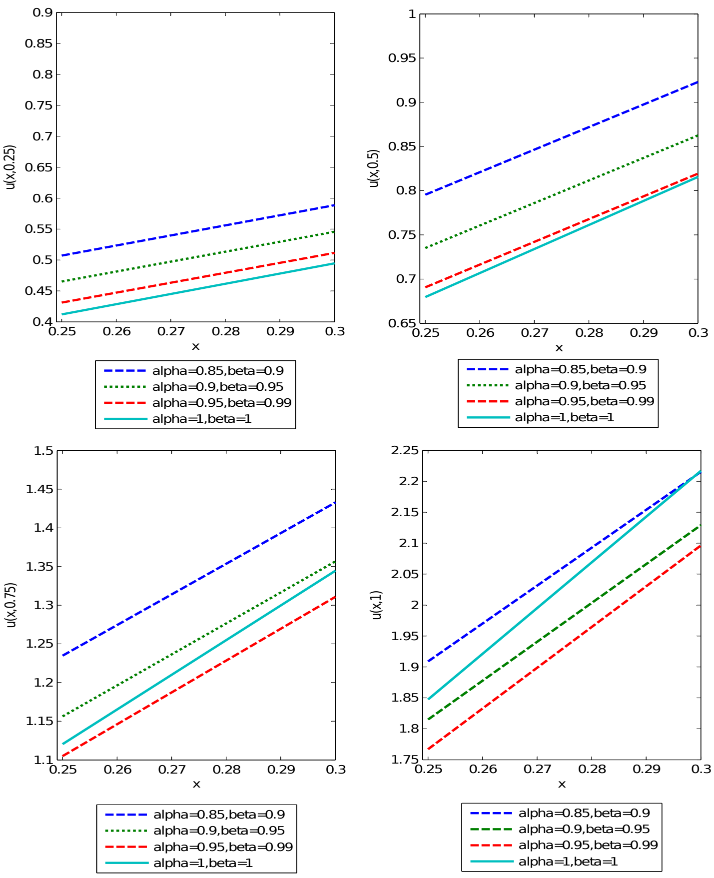

FIGURE 1. The solution u(x,t) vs. point x.

It is also seen from the $3-D$ figures which are described through Figure 2 that the variations of $u(x,t)$ are linear with x. However it becomes exponential with t for different values of α and β as stated in the caption of the figures.

Remark 2. Setting $\mu = 0$ and $\alpha = 1$ in (4.1) and (4.2) we get the same results given in [7].

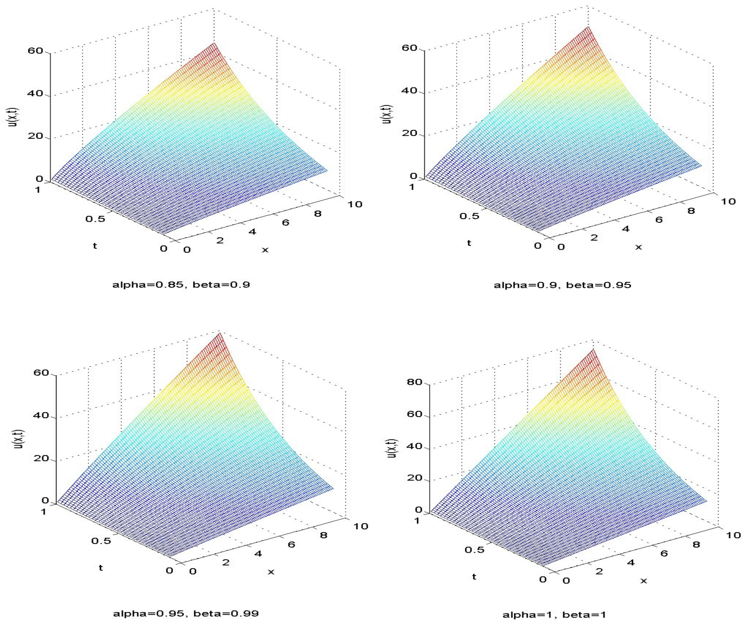

FIGURE 2. The solution u(x,t) with respect to x and time t.

Example 2. Consider the following time-fractional reaction diffusion equation

$$D_t^{\beta}u(x,t)=\lambda C_x^{(2\alpha}u(x,t)-C_x^{\alpha}\left[E_{\alpha}(-x^{\alpha})u(x,t)\left(1-\mu u(x,t)\right)\right],x,t>0 \tag{4.5}$$

where $0<\alpha,\beta\leq 1$, $\lambda=2$, $\mu=-1$ subject to the initial condition

$$u(x,0)=1+\sum_{k=1}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}=E_{\alpha}(x^{\alpha}). \tag{4.6}$$

Since $F(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(x)^{k\alpha}}{\Gamma(k\alpha+1)}$, then $a_0=1,\ a_1=\frac{-1}{\Gamma(\alpha+1)},\ a_2=\frac{1}{\Gamma(2\alpha+1)},\ldots$ In addition to this, substituting $f(x)=\sum_{k=0}^{\infty}\frac{(x)^{k\alpha}}{\Gamma(k\alpha+1)}$ in (3.2), we have

$$U_{\alpha,\beta}(i,0)=\frac{1}{\Gamma(i\alpha+1)},\ i=0,1,2,\ldots. \tag{4.7}$$

Applying the recurrence relation (3.1) and using the transformed initial condition (4.7), the first few components of $U_{\alpha,\beta}(i,j)$ can be calculated as follows:

$$U_{\alpha,\beta}(0,0)=1,\ U_{\alpha,\beta}(1,0)=\frac{1}{\Gamma(\alpha+1)},\ U_{\alpha,\beta}(2,0)=\frac{1}{\Gamma(2\alpha+1)},\ldots$$

$$U_{\alpha,\beta}(0,1)=\frac{1}{\Gamma(\beta+1)},U_{\alpha,\beta}(0,2)=\frac{1}{\Gamma(2\beta+1)},\dots$$

$$\vdots$$

$$U_{\alpha,\beta}(i,j)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{\Gamma(i\alpha+1)\Gamma(j\beta+1)},\ i,j=0,1,2,\dots$$

So the solution $u(x,t)$ of Eq. (4.5) is obtained by

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}x^{i\alpha}t^{j\beta}=E_{\alpha}(x^{\alpha})E_{\beta}(t^{\beta}).$$

Table 2 shows the approximate solution for Eq. (4.5) obtained for different values of α and β.

		$u(x,t)$			
t	x	$\alpha=0.85$, $\beta=0.9$	$\alpha=0.9$, $\beta=0.95$	$\alpha=0.95$, $\beta=0.99$	$\alpha=1$, $\beta=1$
0.00	0.25	1.3961	1.3540	1.3169	1.2840
	0.50	1.8456	1.7725	1.7073	1.6487
	0.75	2.4092	2.3010	2.2042	2.1170
	1.00	3.1255	2.9749	2.8400	2.7183
0.25	0.25	1.8903	1.7831	1.6992	1.6487
	0.50	2.4990	2.3342	2.2029	2.1170
	0.75	3.2620	3.0302	2.8441	2.7183
	1.00	4.2319	3.9177	3.6644	3.4903
0.50	0.25	2.4746	2.3116	2.1860	2.1170
	0.50	3.2714	3.0261	2.8339	2.7183
	0.75	4.2703	3.9284	3.6588	3.4903
	1.00	5.5399	5.0790	4.7142	4.4817
0.75	0.25	3.2124	2.9845	2.8099	2.7183
	0.50	4.2468	3.9070	3.6428	3.4903
	0.75	5.5436	5.0719	4.7032	4.4817
	1.00	7.1919	6.5574	6.0598	5.7546
1.00	0.25	4.1533	3.8454	3.6105	3.4903
	0.50	5.4906	5.0339	4.6807	4.4817
	0.75	7.1671	6.5349	6.0432	5.7546
	1.00	9.2982	8.4488	7.7863	7.3891

TABLE 2. Approximate solution for u(x,t).

It is seen from Figure 3 that $u(x,t)$ increase with the increase in t for all α and β. However it is found to decrease with the increase in α and β.

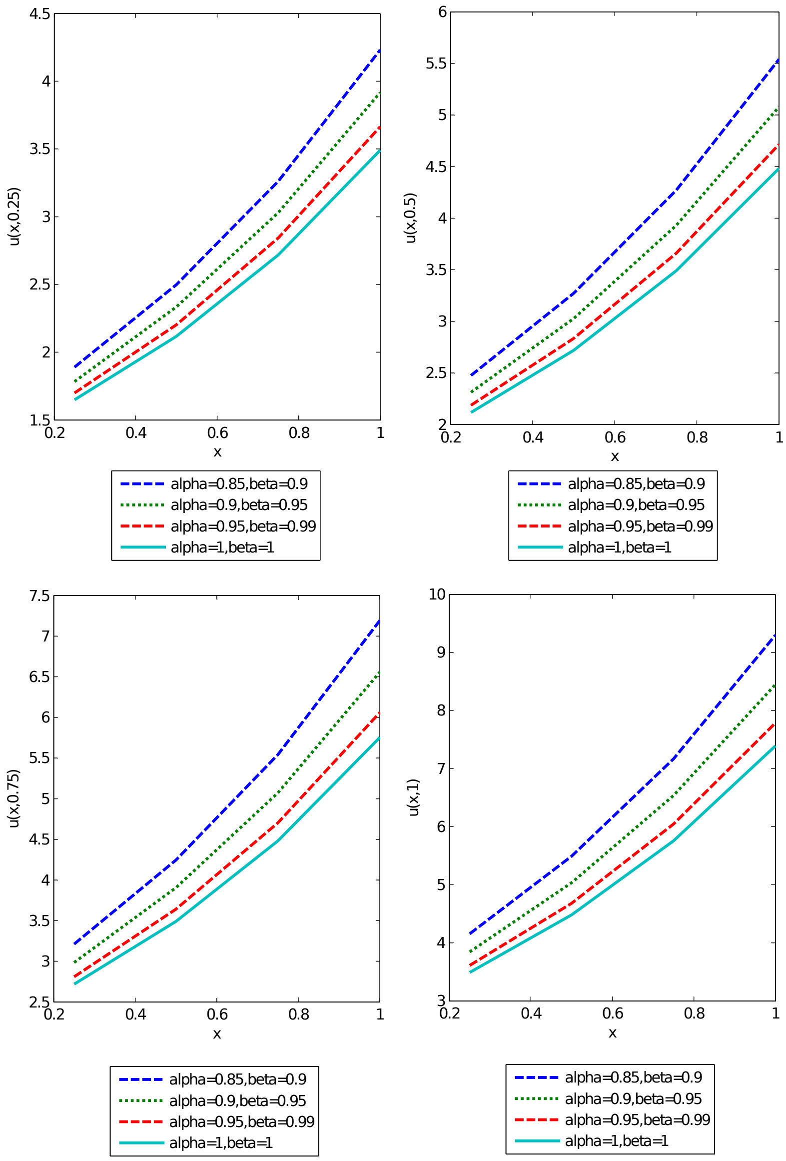

FIGURE 3. The solution u(x,t) vs. point x.

It is also seen from the $3-D$ figures which are described through Figure 2 that the variations of $u(x,t)$ becomes exponential with x and t for different values of α and β as stated in the caption of the figures.

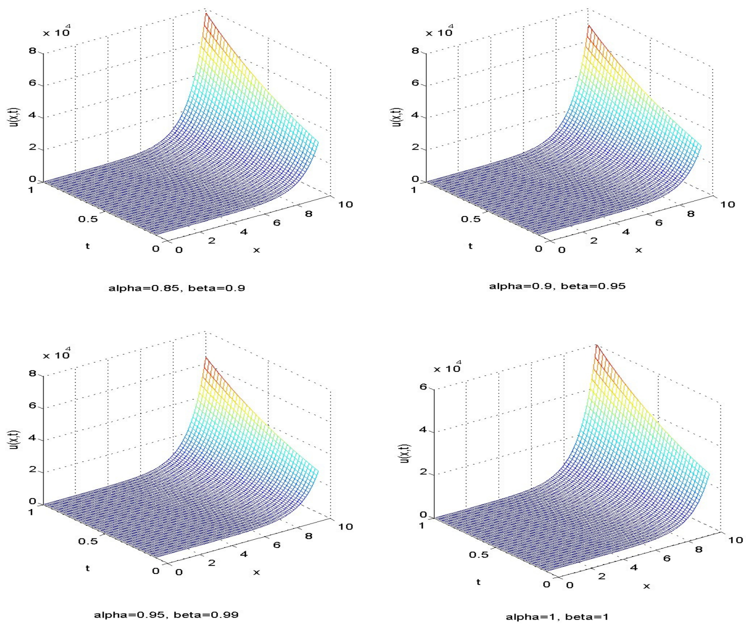

FIGURE 4. The solution u(x,t) with respect to x and time t.

Remark 3. Setting $\mu = 0$, $\alpha = 1$ and $\lambda = 1$ in (4.5) and (4.6) we get the same results given in [7].

5. CONCLUSION

In this paper, we consider a non-linear time-fractional diffusion equation given in (1.9) and we applied the GDTM for its solution. The approximate solutions resulting from the method are shown graphically. Results also show that the numerical scheme is very effective and convenient for solving non-linear time-fractional diffusion equation of fractional order. Numerical computations associated with the two examples discussed above were performed by using Matlab. Another important contribution of this paper is the observation that, the presence of external force increases the rate of diffusion in the nonlinear system but the rate becomes slower in the presence of nonlinear term.

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Authors' addresses

Mehmet Bozer

(**Corresponding author**) Eastern Mediterranean University, Department of Mathematics, Gazimagusa, TRNC, Mersin 10, Turkey

E-mail address: `mehmet.bozer@emu.edu.tr`

Mehmet Ali Özarslan

Eastern Mediterranean University, Department of Mathematics, Gazimagusa, TRNC, Mersin 10, Turkey

E-mail address: `mehmetali.ozarslan@emu.edu.tr`

Hülya Demez

Eastern Mediterranean University, Department of Mathematics, Gazimagusa, TRNC, Mersin 10, Turkey

E-mail address: `hulya.demez@emu.edu.tr`