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BLOW-UP OF WEAK SOLUTIONS FOR A HIGHER-ORDER HEAT EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This paper deal with the initial boundary value problem for a higher-order heat equation with logarithmic source term

$$u_t+(-\Delta)^m u-\Delta u_t=u^{k-2}u\ln|u|.$$

We obtain blow-up of weak solutions in the finite time, by employing potential well technique and concave technique. In addition, the upper bound of blow-up time is considered. This improves and extends some previous studies.

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1. Introduction

In this article, we consider the blow up of solutions for the higher-order heat equation with logarithmic nonlinearity

$$\begin{cases} u_t+Au-\Delta u_t=|u|^{k-2}u\ln|u|, & x\in\Omega, \quad t>0;\\ D^\gamma u(x,t)=0, \quad |\gamma|\le m-1, & x\in\partial\Omega, \quad t>0;\\ u(x,0)=u_0(x), & x\in\Omega; \end{cases} \tag{1.1}$$

where $A=(-\Delta)^m$, $m\ge 1$ a positive integer, Ω is a bound domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ is multi-index, γ_i $(i=1,2,\ldots,n)$ are nonnegative integers, $|\gamma|=\gamma_1+\gamma_2+\cdots+\gamma_n$, $D^\gamma=\frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\partial x_2^{\gamma_2}\ldots\partial x_n^{\gamma_n}}$ are multi-index derivative operator, $\Delta=\sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. The parameter k satisfies

$$\begin{cases} 2\le k\le+\infty, & n\le 2m,\\ 2\le k\le\frac{2n}{n-2m}, & n>2m. \end{cases}$$

Peng and Zhou [11] studied the following parabolic equation with logarithmic source term

$$u_t - \Delta u = u^{k-2} u \ln |u| .$$

They studied by employing energy technique and potential well technique, the global existence of solutions and blow-up in finite time. In addition, the upper bound of blow-up time is considered under appropriate conditions.

When $m = 1$, in the equation (1.1), becomes a heat equation as follows

$$u_t - \Delta u - \Delta u_t = u^{k-2} u \ln |u| ,$$

where $2 \leq k$, was considered by many authors [2–4, 7, 16]. In the case of $k = 2$, Chen and Tian [3] studied by employing the logarithmic Sobolev inequality and potential well method, the global existence of solutions and blow-up of solutions at $+\infty$. In the case of $2 < k$, Ding and Zhou [4] studied the blow-up of solutions in finite time, by using eigenfunction method. Also, the upper bound of the blow-up time is studied under appropriate conditions.

Recently many other authors investigated higher-order hyperbolic and parabolic type equation [5, 6, 8, 10, 13–15, 17, 18]. Ishige et al. [8] studied the Cauchy problem for nonlinear higher-order heat equation as follows

$$u_t + (-\Delta)^m u = |u|^k .$$

They obtained existence of solutions of the Cauchy problem by introducing a new majorizing kernel. In addition, they studied the local existence of solutions under the different conditions. Xiao and Li [15] considered the following initial boundary value problem for nonlinear higher-order heat equations

$$u_t + (-\Delta)^m u_t + (-\Delta)^m u = f(u).$$

They established the existence of weak solution to the static problem, via the potential well technique.

Motivated by the above studies, in this work, we investigate the finite time blow-up of weak solutions for the Eq. (1.1).

The remainder of our work is organized as follows. In Section2, some important Lemmas are given. In Section 3, the main result is proved.

2. Preliminaries

We material needed for proving the main result is introduced. Let $\|u\|_{H^m(\Omega)} = \left(\sum_{|\gamma| \leq m} \|D^\gamma u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$ denote $H^m(\Omega)$ norm, let $H_0^m(\Omega)$ denote the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. Let $\|.\|_r$ and $\|.\|$ denote the usual $L^r(\Omega)$ norm and $L^2(\Omega)$ norm (see [1, 12], for details).

For $u \in H_0^m(\Omega)\backslash\{0\}$, we define the energy functional

$$J(u) = \frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{1}{k}\int_\Omega |u|^k \ln|u|\,dx + \frac{1}{k^2}\|u\|_k^k, \tag{2.1}$$

and Nehari functional

$$I(u) = \left\|A^{\frac{1}{2}}u\right\|^2 - \int_\Omega |u|^k \ln|u|\,dx. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$J(u) = \frac{1}{k}I(u) + \left(\frac{1}{2} - \frac{1}{k}\right)\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{1}{k^2}\|u\|_k^k. \tag{2.3}$$

Let

$$\mathcal{N} = \{u \in H_0^m(\Omega)\backslash\{0\} : I(u) = 0\},$$

be the Nehari manifold. Also, we may define

$$d = \inf_{u\in\mathcal{N}} J(u), \tag{2.4}$$

and

$$V = \{u \in H_0^m(\Omega) \mid J(u) < d,\ I(u) < 0\}.$$

We refer to V as the potential well and d as the depth of the well.

Lemma 1 (Theorem 4.31 in [1]). *Let q be a number with $2 \le q < +\infty$, $n \le 2m$ and $2 \le q \le \frac{2n}{n-2m}$, $n > 2m$. Then there is a constant C depending*

$$\|u\|_q \le C\left\|A^{\frac{1}{2}}u\right\|, \quad \forall u \in H_0^m(\Omega).$$

Lemma 2. *$J(t)$ is a nonincreasing function for $t \ge 0$ and*

$$J'(u) = -\int_\Omega \left(u_t^2 + \nabla u_t^2\right) dx \le 0.$$

Proof. Multiplying the equation (1.1) by u_t and integrating on Ω, we get

$$\int_\Omega u_t^2 dx + \int_\Omega Auu_t dx + \int_\Omega \nabla u_t^2 dx = \int_\Omega u^{k-1}u_t \ln|u|\,dx.$$

By straightforward calculation, we obtain

$$\int_\Omega u_t^2 dx + \frac{1}{2}\frac{d}{dt}\left\|A^{\frac{1}{2}}u\right\|^2 + \int_\Omega \nabla u_t^2 dx = \frac{1}{k}\frac{d}{dt}\int_\Omega |u|^k \ln|u|\,dx - \frac{1}{k^2}\frac{d}{dt}\|u\|_k^k,$$

which yields that

$$\frac{1}{2}\frac{d}{dt}\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{1}{k}\frac{d}{dt}\int_\Omega |u|^k \ln|u|\,dx + \frac{1}{k^2}\frac{d}{dt}\|u\|_k^k = -\int_\Omega u_t^2 dx - \int_\Omega \nabla u_t^2 dx.$$

Thus we get

$$\frac{d}{dt}\left(\frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{1}{k}\int_\Omega |u|^k\ln|u|\,dx+\frac{1}{k^2}\|u\|_k^k\right)=-\int\limits_\Omega\left(u_t^2+\nabla u_t^2\right)dx. \tag{2.5}$$

By (2.1) and (2.5), we obtain

$$\frac{d}{dt}J(u)=-\int\limits_\Omega\left(u_t^2+\nabla u_t^2\right)dx. \tag{2.6}$$

Moreover, Integrating (2.6) with respect to t on $[0,t]$, we obtain

$$\int_0^t\|u_s(s)\|^2_{H_0^1(\Omega)}\,ds+J(u(t))=J(u_0).$$

□

Lemma 3. *Let $u\in H_0^m(\Omega)\backslash\{0\}$. We contract the function $h\colon\ \beta\to J(\beta u)$ for $\beta>0$. Then we get*

(i) $\lim_{\beta\to 0^+}h(\beta)=0$ *and* $\lim_{\beta\to+\infty}h(\beta)=-\infty$;
(ii) *there is a unique* $\overline{\beta}_1>0$ *such that* $h'(\overline{\beta}_1)=0$;
(iii) $h(\beta)$ *is increasing on* $(0,\overline{\beta}_1)$, *decreasing on* $(\overline{\beta}_1,+\infty)$ *and taking the maximum at* $\overline{\beta}_1$; $I(\beta u)=\beta h'(\beta)$ *and*

$$I(\beta u)\begin{cases}>0, & 0<\beta<\overline{\beta}_1,\\ =0, & \beta=\overline{\beta}_1,\\ <0, & \overline{\beta}_1<\beta<+\infty.\end{cases}$$

Proof. By the definition of h, for $u\in H_0^m(\Omega)\backslash\{0\}$, we have

$$\begin{aligned}h(\beta)&=\frac{1}{2}\left\|A^{\frac{1}{2}}(\beta u)\right\|^2-\frac{1}{k}\int_\Omega|\beta u|^k\ln|\beta u|\,dx+\frac{1}{k^2}\|\beta u\|_k^k\\&=\frac{\beta^2}{2}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{\beta^k}{k}\int_\Omega|u|^k\ln|u|\,dx-\frac{\beta^k}{k}\ln\beta\,\|u\|_k^k+\frac{\beta^k}{k^2}\|u\|_k^k.\end{aligned}$$

We see that *(i)* holds due to $\|u\|_k^k\neq 0$. We obtain

$$\begin{aligned}\frac{d}{d\beta}h(\beta)&=\beta\left\|A^{\frac{1}{2}}u\right\|^2-\beta^{k-1}\int_\Omega|u|^k\ln|u|\,dx\\&\quad-\beta^{k-1}\ln\beta\,\|u\|_k^k-\frac{\beta^{k-1}}{k}\|u\|_k^k+\frac{\beta^{k-1}}{k}\|u\|_k^k\\&=\beta\left(\left\|A^{\frac{1}{2}}u\right\|^2-\beta^{k-2}\int_\Omega|u|^k\ln|u|\,dx-\beta^{k-2}\ln\beta\,\|u\|_k^k\right).\end{aligned}$$

Let $\zeta(\beta)=\beta^{-1}h'(\beta)$, thus we get

$$\begin{aligned}\zeta(\beta)&=\beta^{-1}h'(\beta)\\&=\beta^{-1}\beta\left(\left\|A^{\frac{1}{2}}u\right\|^2-\beta^{k-2}\int_\Omega |u|^k\ln|u|\,dx-\beta^{k-2}\ln\beta\,\|u\|_k^k\right)\\&=\left\|A^{\frac{1}{2}}u\right\|^2-\beta^{k-2}\int_\Omega |u|^k\ln|u|\,dx-\beta^{k-2}\ln\beta\,\|u\|_k^k.\end{aligned}$$

Then

$$\zeta'(\beta)=-(k-2)\beta^{k-3}\int_\Omega |u|^k\ln|u|\,dx-(k-2)\beta^{k-3}\ln\beta\,\|u\|_k^k-\beta^{k-3}\,\|u\|_k^k,$$

which yields that there exists a $\overline{\beta}_1>0$ such that $\zeta'(\beta)>0$ on $(0,\overline{\beta}_1)$, $\zeta'(\beta)<0$ on $(\overline{\beta}_1,+\infty)$ and $\zeta'(\beta)=0$. So $\zeta(\beta)$ is increasing on $(0,\overline{\beta}_1)$, decreasing on $(\overline{\beta}_1,+\infty)$. Since $\lim_{\beta\to 0^+}\zeta(\beta)=\left\|A^{\frac{1}{2}}u\right\|^2>0$, $\lim_{\beta\to+\infty}\zeta(\beta)=-\infty$, there exists a unique $\overline{\beta}_1>0$ such that $\zeta(\overline{\beta}_1)=0$, i.e. $h'(\overline{\beta}_1)=0$. So *(ii)* holds. Then, $h'(\beta)=\beta\zeta(\beta)$ is positive on $(0,\overline{\beta}_1)$, negative on $(\overline{\beta}_1,+\infty)$. Thus $h(\beta)$ is increasing on $(0,\overline{\beta}_1)$, decreasing on $(\overline{\beta}_1,+\infty)$ and taking the maximum at $\overline{\beta}_1$. From (2.2), we get

$$\begin{aligned}I(\beta u)&=\left\|A^{\frac{1}{2}}(\beta u)\right\|^2-\int_\Omega|\beta u|^k\ln|\beta u|\,dx\\&=\beta^2\left\|A^{\frac{1}{2}}u\right\|^2-\beta^k\int_\Omega|u|^k\ln|u|\,dx-\beta^k\ln\beta\,\|u\|_k^k\\&=\beta\left(\beta\left\|A^{\frac{1}{2}}u\right\|^2-\beta^{k-1}\int_\Omega|u|^k\ln|u|\,dx-\beta^{k-1}\ln\beta\,\|u\|_k^k\right)\\&=\beta h'(\beta).\end{aligned}$$

Thus $I(\beta u)>0$ for $0<\beta<\overline{\beta}_1$, $I(\beta u)<0$ for $\overline{\beta}_1<\beta<+\infty$ and $I(\overline{\beta}_1u)=0$. So *(iii)* holds. For this reason, the proof is completed. □

Lemma 4. *d defined by (2.4) is positive and there exists a positive function $u\in\mathcal{N}$ such that $J(u)=d$.*

Proof. By (2.4), we suppose $\{u_r\}_r^\infty\subset N$ is a minimizing sequence of J. Since $\{u_r\}_r^\infty\subset N$ is also a minimizing sequence of J, we consider the case where $u_r>0$ a.e. in Ω, $r\in N$ without loss of generality. Thus

$$\lim_{r\to\infty}J(u_r)=d, \tag{2.7}$$

which implies that $\{J(u_r)\}_r^\infty$ is bounded, i.e. there exists a constant $C_1>0$ such that $|J(u_r)|\le C_1$. Using (2.3), $I(u_r)=d$ and $|J(u_r)|\le C_1$, we obtain

$$\left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}u_r\right\|^2+\frac{1}{k^2}\,\|u_r\|_k^k\le C_1. \tag{2.8}$$

From (2.8), we get

$$\left\|A^{\frac{1}{2}}u_r\right\|^2 \le \left(\frac{1}{2}-\frac{1}{k}\right)^{-1} C_1. \tag{2.9}$$

By (2.9) and Lemma 1, we obtain

$$\|u_r\|^2 \le C\left\|A^{\frac{1}{2}}u_r\right\|^2 \le \left(\frac{1}{2}-\frac{1}{k}\right)^{-1} C_1.$$

Moreover, we have already observed that J is coercive on $\mathcal{N}$ which satisfies that $\{u_r\}_r^\infty$ is bounded in $H_0^m(\Omega)$. Let $\mu > 0$ be small enough such that $k+\mu < \frac{2n}{n-2m}$. Since $H_0^m(\Omega) \hookrightarrow L^{k+\mu}(\Omega)$ is compact, so there exists a function u and a subsequence of $\{u_r\}_r^\infty$, still denote by $\{u_r\}_r^\infty$, such that

$$u_r \to u \text{ weakly in } H_0^m(\Omega),$$

$$u_r \to u \text{ strongly in } L^{k+\mu}(\Omega),$$

$$u_r(x) \to u(x) \text{ a.e. in } \Omega.$$

Also, $u \ge 0$ a.e. in Ω. First, we prove $u \ne 0$. From the dominated convergence theorem, we have

$$\int_\Omega |u|^k \ln|u|\,dx = \lim_{r\to\infty}\int_\Omega |u_r|^k \ln|u_r|\,dx \tag{2.10}$$

and

$$\int_\Omega |u|^k\,dx = \lim_{r\to\infty}\int_\Omega |u_r|^k\,dx. \tag{2.11}$$

From the weak lower semicontinuity of $H_0^m(\Omega)$, we get

$$\left\|A^{\frac{1}{2}}u\right\|^2 \le \liminf_{r\to\infty}\left\|A^{\frac{1}{2}}u_r\right\|^2. \tag{2.12}$$

Then it follows from (2.1), (2.7), (2.10), (2.11) and (2.12) that

$$\begin{aligned}
J(u) &= \frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{1}{k}\int_\Omega |u|^k\ln|u|\,dx + \frac{1}{k^2}\|u\|_k^k \\
&\le \liminf_{r\to\infty}\frac{1}{2}\left\|A^{\frac{1}{2}}u_r\right\|^2 - \lim_{r\to\infty}\frac{1}{k}\int_\Omega |u_r|^k\ln|u_r|\,dx + \lim_{r\to\infty}\frac{1}{k^2}\|u_r\|_k^k \\
&= \liminf_{r\to\infty}\left(\frac{1}{2}\left\|A^{\frac{1}{2}}u_r\right\|^2 - \frac{1}{k}\int_\Omega |u_r|^k\ln|u_r|\,dx + \frac{1}{k^2}\|u_r\|_k^k\right) \\
&= \liminf_{r\to\infty} J(u_r) = d.
\end{aligned} \tag{2.13}$$

Using (2.2), (2.10) and (2.12), we have

$$
\begin{aligned}
I(u) &= \left\|A^{\frac{1}{2}}u\right\|^2 - \int_\Omega |u|^k \ln|u|\,dx \\
&\le \liminf_{r\to\infty}\left\|A^{\frac{1}{2}}u_r\right\|^2 - \lim_{r\to\infty}\int_\Omega |u_r|^k \ln|u_r|\,dx \\
&= \liminf_{r\to\infty}\left(\left\|A^{\frac{1}{2}}u_r\right\|^2 - \int_\Omega |u_r|^k \ln|u_r|\,dx\right) \\
&= \liminf_{r\to\infty} I(u_r) = 0.
\end{aligned}
\tag{2.14}
$$

Since $u_r \in \mathcal{N}$, we have $I(u_r) = 0$. So by using Lemma 1 and the fact $x^{-\mu}\ln x \le (e\mu)^{-1}$ for $x \ge 1$, we get

$$
\begin{aligned}
\left\|A^{\frac{1}{2}}u_r\right\|^2 &= \int_\Omega |u_r|^k \ln|u_r|\,dx \\
&\le (e\mu)^{-1}\int_\Omega |u_r|^{k+\mu}\,dx \\
&= (e\mu)^{-1}\|u_r\|_{k+\mu}^{k+\mu} \\
&\le C\left\|A^{\frac{1}{2}}u_r\right\|_2^{k+\mu},
\end{aligned}
$$

where C is Sobolev embedding constant. This satisfies that

$$
\int_\Omega |u_r|^k \ln|u_r|\,dx = \left\|A^{\frac{1}{2}}u_r\right\|^2 \ge C. \tag{2.15}
$$

By (2.10) and (2.15), we have

$$
\int_\Omega |u|^k \ln|u|\,dx \ge C.
$$

Thus we have $u \in H_0^m(\Omega)\backslash\{0\}$.

If $I(u_r) < 0$, from Lemma 3, there exists a $\overline{\beta}_1$ such that $I(\overline{\beta}_1 u) = 0$ and $0 < \overline{\beta}_1 < 1$. Thus $\overline{\beta}_1 u \in \mathcal{N}$. It follows from (2.3), (2.4), (2.11) and (2.12) that

$$
\begin{aligned}
d &\le J(\overline{\beta}_1 u) \\
&= \frac{1}{k}I(\overline{\beta}_1 u) + \left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}\left(\overline{\beta}_1 u\right)\right\|^2 + \frac{1}{k^2}\left\|\overline{\beta}_1 u\right\|_k^k \\
&= \left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}\left(\overline{\beta}_1 u\right)\right\|^2 + \frac{1}{k^2}\left\|\overline{\beta}_1 u\right\|_k^k \\
&= (\overline{\beta}_1)^2\left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}u\right\|^2 + (\overline{\beta}_1)^k\left(\frac{1}{k^2}\right)\|u\|_k^k \\
&\le (\overline{\beta}_1)^2\left[\left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{1}{k^2}\|u\|_k^k\right]
\end{aligned}
$$

$$\leq (\overline{\beta}_1)^2 \liminf_{r\to\infty}\left[\left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}u_r\right\|^2+\frac{1}{k^2}\|u_r\|_k^k\right]$$
$$= (\overline{\beta}_1)^2 \liminf_{r\to\infty} J(u_r)$$
$$= (\overline{\beta}_1)^2 d,$$

which indicates $\overline{\beta}_1 \geq 1$ by $d > 0$. It contradicts $0 < \overline{\beta}_1 < 1$. By (2.14), we have $I(u) = 0$. For this reason, $u \in \mathcal{N}$. From (2.7), we have $J(u) \geq d$. From (2.13), we have $J(u) \leq d$. So $J(u) = d$. □

Lemma 5 (Theorem 2 in [9]). *Let $\phi(t)$ be a nonnegative function C^2, which satisfies, for $t > 0$, inequality*

$$\phi(t)\phi''(t) - (1+\gamma)\left[\phi'(t)\right]^2 \geq 0,$$

with some $\gamma > 0$. If $\phi(0) > 0$ and $\phi'(0) > 0$, then there exist a time

$$T \leq \frac{\phi(0)}{\beta\phi'(0)},$$

such that

$$\lim_{t\to T^-}\phi(t) = \infty.$$

3. Main results

Definition 1. (Weak Solution). We say that function $u(t)$ is weak solution of the problem (1.1) on $\Omega\times[0,T]$, if $u \in L^\infty(0,T;H_0^m(\Omega))$ with $u_t \in L^2(0,T;H_0^m(\Omega))$ and implies the initial condition $u(0) = u_0 \in H_0^m(\Omega)\backslash\{0\}$, and the follow equality

$$(u_t,w) + \left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w\right) + (\nabla u_t, \nabla w) = \left(|u|^{r-2}u\ln|u|, w\right),$$

for all $w \in H_0^m(\Omega)$ holds for a.e. $t \in [0,T]$, and $(.,.)$ means the inner product $(.,.)_{L^2(\Omega)}$, that is

$$(\eta,\xi) = \int_\Omega \eta(x)\xi(x)dx.$$

Theorem 1 (Blow up). *Suppose that $u_0 \in V$. Then $u(t)$ blows up at finite time in the sence of $T_* > 0$ and*

$$\lim_{t\to T_*}\|u(t)\|^2_{H_0^1(\Omega)} = \infty.$$

Furthermore, the upper bound for blow up time T_* is given by

$$T_* \leq \frac{4\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)^2(d-J(u_0))}.$$

Proof. Let $u(t) \in V$ for $t \in [0, T_{\max}]$. We prove that $u(t)$ blows up in the finite time. By employing contradiction, we assume that $u(t)$ is global. We consider a function $P\colon [0,T) \to \mathbb{R}^+$, and

$$P(t) = \int_0^t \|u(s)\|^2_{H_0^1(\Omega)} ds + (T_* - t)\|u_0\|^2_{H_0^1(\Omega)} + \phi(t+\psi)^2, \quad t \in [0,T), \tag{3.1}$$

where ϕ, ψ are two positive fixed which will be specified later.

Then, for any $t \in [0,T)$, a straightforward calculation gives

$$\begin{aligned} P'(t) &= \|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)} + 2\phi(t+\psi) \\ &= 2\int_0^t \int_\Omega (u_s u + \nabla u_s \nabla u)\, dxds + 2\phi(t+\psi), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} P''(t) &= 2\int_\Omega (u_s u + \nabla u_s \nabla u)\, dx + 2\phi \\ &= 2\int_\Omega u(u_s - \Delta u_s)\, dx + 2\phi \\ &= 2\int_\Omega |u|^k \ln|u| - 2\left\|A^{\frac12}u\right\|^2 + 2\phi \\ &= -2I(u) + 2\phi. \end{aligned} \tag{3.3}$$

By (3.3) and $I(u) < 0$, we obtain $P''(t) > 0$. From (2.3) and (3.3) that it follows

$$\begin{aligned} P''(t) &= -2I(u) + 2\phi \\ &\ge -2kJ(u) + (k-2)\left\|A^{\frac12}u\right\|^2 + \frac{2}{k}\|u\|_k^k \\ &\ge -2kJ(u_0) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + (k-2)\left\|A^{\frac12}u\right\|^2 + \frac{2}{k}\|u\|_k^k. \end{aligned} \tag{3.4}$$

Since $u(t) \in V$, $t \in [0,T]$, so $I(u) < 0$. By Lemma 3, there exists a $\overline{\beta}_1 \in (0,1)$ such that $I(\overline{\beta}_1 u(t)) = 0$. By (2.3) and the definition of d, we obtain

$$\begin{aligned} d &= \inf_{u \in \mathcal{N}} J(u) \le J(\overline{\beta}_1 u(t)) \\ &= \frac{1}{k} I(\overline{\beta}_1 u) + \left(\frac12 - \frac1k\right)\left\|A^{\frac12}\left(\overline{\beta}_1 u\right)\right\|^2 + \frac{1}{k^2}\left\|\overline{\beta}_1 u\right\|_k^k \\ &= \left(\frac12 - \frac1k\right)\left\|A^{\frac12}\left(\overline{\beta}_1 u\right)\right\|^2 + \frac{1}{k^2}\left\|\overline{\beta}_1 u\right\|_k^k \\ &= (\overline{\beta}_1)^2\left(\frac12 - \frac1k\right)\left\|A^{\frac12}u\right\|^2 + (\overline{\beta}_1)^k\left(\frac{1}{k^2}\right)\|u\|_k^k \\ &\le \left(\frac12 - \frac1k\right)\left\|A^{\frac12}u\right\|^2 + \frac{1}{k^2}\|u\|_k^k. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned} P''(t) &\geq -2kJ(u_0) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + (k-2)\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{2}{k}\|u\|_k^k \\ &= -2kJ(u_0) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + 2k\left[\frac{k-2}{2k}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{1}{k^2}\|u\|_k^k\right] \\ &\geq 2k(d - J(u_0)) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds. \end{aligned} \tag{3.6}$$

Thus we have

$$P(t) \geq P(0) > 0, \quad t \in [0, T_*].$$

Let

$$\rho(t) := \left(\int_0^t \|u(s)\|^2_{H_0^1(\Omega)} ds\right)^{\frac{1}{2}}, \quad \sigma(t) := \left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds\right)^{\frac{1}{2}}.$$

By employing Hölder's inequality, we get

$$\begin{aligned} &\left[\int_0^t \|u(s)\|^2_{H_0^1(\Omega)} ds + \phi(t+\psi)^2\right]\left[\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + \phi\right] \\ &- \left[\frac{1}{2}\left(\|u\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\right) + \phi(t+\psi)\right]^2 \\ &= \left[\rho^2(t) + \phi(t+\psi)^2\right]\left[\sigma^2(t) + \phi\right] - \left[\frac{1}{2}\int_0^t \frac{d}{ds}\|u\|^2_{H_0^1(\Omega)} ds + \phi(t+\psi)\right]^2 \\ &\geq \left[\rho^2(t) + \phi(t+\psi)^2\right]\left[\sigma^2(t) + \phi\right] - \left[\int_0^t \|u\|_{H_0^1(\Omega)}\|u_s\|_{H_0^1(\Omega)} ds + \phi(t+\psi)\right]^2 \\ &= \left[\sqrt{\phi}\rho(t)\right]^2 - 2\phi(t+\psi)\rho(t)\sigma(t) + \left[\sqrt{\phi}(t+\psi)\sigma(t)\right]^2 \\ &= \left[\sqrt{\phi}\rho(t) - \sqrt{\phi}(t+\psi)\sigma(t)\right]^2 \geq 0. \end{aligned}$$

Then, by (3.2), we get

$$\begin{aligned} \frac{1}{4}(P'(t))^2 &= \left[\frac{1}{2}\left(\|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\right) + \phi(t+\psi)\right]^2 \\ &= \left[\frac{1}{2}\left(\|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\right) + \phi(t+\psi)\right]^2 \\ &\quad + \left(\int_0^t \|u(\tau)\|^2_{H_0^1(\Omega)} d\tau + \phi(t+\psi)^2\right)\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + \phi\right) \\ &\quad - \left[P(t) - (T_* - t)\|u_0\|^2_{H_0^1(\Omega)}\right]\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + \phi\right) \\ &\leq P(t)\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)} ds + \phi\right). \end{aligned} \tag{3.7}$$

So it follows from (3.6) and (3.7) that

$$\begin{aligned} P(t)P''(t) - \frac{k}{2}\left(P'(t)\right)^2 &\geq P(t)\left[P''(t) - 2k\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds + \phi\right)\right] \\ &\geq P(t)\left[2k\left(d - J(u_0)\right) - 2k\phi\right]. \end{aligned}$$

We choose ϕ sufficiently small, such that

$$\phi \in \left(0, \frac{\mu}{2k}\right], \tag{3.8}$$

where

$$\mu := 2k\left(d - J(u_0)\right) > 0,$$

then it follows that

$$P(t)P''(t) - \frac{k}{2}\left(P'(t)\right)^2 \geq 0.$$

Let $\omega(t) = P(t)^{-\frac{k-2}{2}}$ for $t \in [t_0, T]$, then by $P(t) > 0$, $P'(t) > 0$, $k > 2$ and the definition of $\omega(t)$, we have

$$\omega'(t) = -\frac{k-2}{2}P(t)^{-\frac{k}{2}}P'(t). \tag{3.9}$$

By (3.9), we obtain

$$\begin{aligned} \omega''(t) &= \frac{k(k-2)}{4}P(t)^{-\frac{k+2}{2}}P'(t)^2 - \frac{k-2}{2}P(t)^{-\frac{k}{2}}P''(t) \\ &= \frac{k-2}{2}P(t)^{-\frac{k+2}{2}}\left[\frac{k}{2}P'(t)^2 - P(t)P''(t)\right] < 0 \quad \text{for all } t \in [t_0, T]. \end{aligned} \tag{3.10}$$

We see that, for any large enough $T > t_0$, $\omega(t)$ is a concave function in $[t_0, T]$. Since $\omega(t_0) > 0$ and $\omega''(t_0) < 0$, there exists a finite time T_* such that

$$\lim_{t \to T_*^-} \omega(t) = 0,$$

which yields

$$\lim_{t \to T_*^-} P(t) = \infty.$$

Moreover, we obtain

$$\lim_{t \to T_*^-} \|u(s)\|^2_{H_0^1(\Omega)} = \infty.$$

This is a contradiction to our assumption. Thus $u(t)$ blows up at finite time.

Now, we give an upper bound estimate of T_*. By (3.10) and $\omega''(t) \leq 0$, we obtain

$$\omega(T) - \omega(0) = T\int_0^1 \omega'(\theta T)\,d\theta \leq \omega'(0)T. \tag{3.11}$$

From (3.1) and the definition of $\omega(t)$, we obtain

$$\omega(0) = P(0)^{-\frac{k-2}{2}} > 0,$$

$$\omega(T) = P(T)^{-\frac{k-2}{2}} > 0,$$

$$\omega'(0) = -\frac{k-2}{2} P(0)^{-\frac{k}{2}} P'(0) < 0.$$

It follows from (3.11) that

$$T \le \frac{\omega(T) - \omega(0)}{\omega'(0)} < -\frac{\omega(0)}{\omega'(0)} = \frac{2P(0)}{(k-2)\,P'(0)}. \tag{3.12}$$

By (3.1) and (3.2), we obtain

$$P(0) = T\,\|u_0\|^2_{H_0^1(\Omega)} + \phi\psi^2$$

and

$$P'(0) = 2\phi\psi.$$

By Lemma 5 and (3.12), we get

$$T_* \le \frac{T\,\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\phi\psi} + \frac{\psi}{(k-2)} \qquad \text{for all } T \in [0, T_*). \tag{3.13}$$

Moreover, letting $T \to T_*$, we obtain

$$T_* \le \frac{\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\phi\psi} T_* + \frac{\psi}{(k-2)}.$$

Let ψ be sufficiently large such that

$$\psi \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\phi}, +\infty \right). \tag{3.14}$$

From (3.13), we obtain

$$T_* \le \frac{\phi\psi^2}{(k-2)\,\phi\psi - \|u_0\|^2_{H_0^1(\Omega)}}. \tag{3.15}$$

With respect to (3.8) and (3.14), we define

$$\begin{aligned} \varphi &= \left\{ (\phi, \psi) : \phi \in \left(0, \frac{\mu}{2k}\right], \psi \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\phi}, +\infty \right) \right\} \\ &= \left\{ (\psi, \phi) : \psi \in \left(\frac{2k\,\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\mu}, +\infty \right), \phi \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\psi}, \frac{\mu}{2k} \right] \right\} \end{aligned}$$

and then

$$T_* \le \inf \frac{\phi\psi^2}{(k-2)\,\phi\psi - \|u_0\|^2_{H_0^1(\Omega)}}.$$

Let $\eta = \phi\psi$ (see [11, Theorem 2.8]) and

$$f(\psi, \eta) := \frac{\eta\psi}{(k-2)\,\eta - \|u_0\|^2_{H_0^1(\Omega)}}.$$

We see that $f(\psi,\eta)$ is decreasing with η and we get

$$
\begin{aligned}
T_* &\le \inf_{\psi\in\left(\frac{2k\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\mu},+\infty\right)} f\left(\psi,\frac{\mu\psi}{2k}\right)\\
&= \inf_{\psi\in\left(\frac{2k\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\mu},+\infty\right)} \frac{\mu\psi^2}{(k-2)\mu\psi-2k\|u_0\|^2_{H_0^1(\Omega)}}\\
&= \frac{\mu\psi^2}{(k-2)\mu\psi-2k\|u_0\|^2_{H_0^1(\Omega)}}\Big|_{\psi=\frac{4k\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\mu}}\\
&= \frac{8k\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)^2\mu}.
\end{aligned}
\tag{3.16}
$$

Moreover, by (3.16) and the definition of μ, we obtain

$$T_* \le \frac{4\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)^2(d-J(u_0))}.$$

□

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