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AN ENGEL CONDITION WITH b-GENERALIZED DERIVATIONS IN PRIME RINGS

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Abstract. Let $\mathcal{R}$ be a prime ring, I be a nonzero ideal of $\mathcal{R}$, Q be its maximal right ring of quotients and C be its extended centroid. The aim of this paper is to show that if $\mathcal{R}$ admits a nonzero b-generalized derivation $\mathcal{F}$ such that $[\mathcal{F}(x^m)x^n + x^n\mathcal{F}(x^m), x^r]_k = 0$ for all $x \in I$, where m,n,r,k are fixed positive integers, then there exists $\lambda \in C$ such that $\mathcal{F}(x) = \lambda x$ unless $\mathcal{R} \cong \mathrm{M}_2(\mathrm{GF}(2))$, the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements.

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1. Introduction

In all that follows, unless specially stated, $\mathcal{R}$ always denotes an associative ring with center $\mathrm{Z}(\mathcal{R})$. A ring $\mathcal{R}$ is called prime if $a\mathcal{R}b = (0)$ (where $a,b \in \mathcal{R}$) implies $a = 0$ or $b = 0$. We denote by Q maximal right ring of quotients of $\mathcal{R}$ and C is the center of Q which is called the extended centroid of $\mathcal{R}$ see [5, Chapter 2] for more details. As usual the symbol $[x,y]$ will denote the commutator $xy - yx$. Given $x,y \in \mathcal{R}$ set $[x,y]_1 = xy - yx$ and inductively $[x,y]_k = [[x,y]_{k-1}, y]$ for $k > 1$. Note that Engel condition is a polynomial $[x,y]_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} y^i x y^{k-i}$ in noncommutative indeterminates x,y and $[x+y,z]_k = [x,z]_k + [y,z]_k$.

An additive mapping $d\colon \mathcal{R} \to \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ for all $x,y \in \mathcal{R}$. An additive mapping $G\colon \mathcal{R} \to \mathcal{R}$ is called a generalized derivation of $\mathcal{R}$ if there exists a derivation d of $\mathcal{R}$ such that $G(xy) = G(x)y + xd(y)$ for all $x,y \in \mathcal{R}$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significative example is a map of the form $G(x) = ax + xb$ for some $a,b \in \mathcal{R}$; such generalized derivations are called inner. Over the last few decades, several authors have studied on rings with generalized derivations (viz.; [1, 2, 6–8, 10, 15] and references therein).

In a recent paper [12], Koşan and Lee proposed the following new definition. Let $d\colon \mathcal{R} \to Q$ be an additive mapping and $b \in Q$. An additive map $\mathcal{F}\colon \mathcal{R} \to Q$ is called a left b-generalized derivation, with associated mapping d, if $\mathcal{F}(xy) = \mathcal{F}(x)y + bxd(y)$, for all $x, y \in \mathcal{R}$. In the same paper, it is proved that, if $\mathcal{R}$ is a prime ring, then d is a derivation of $\mathcal{R}$. For simplicity of notation, this mapping $\mathcal{F}$ will be called a b-generalized derivation with associated pair (b,d). Clearly, any generalized derivation with associated derivation d is a b-generalized derivation with associated pair $(1,d)$. Similarly, the mapping $x \mapsto ax + b[x,c]$, for $a,b,c \in Q$, is a b-generalized derivation with associated pair $(b, ad(c))$, where $ad(c)(x) = [x,c]$ denotes the inner derivation of $\mathcal{R}$ induced by the element c. More generally, the mapping $x \mapsto ax + qxc$, for $a,q,c \in Q$, is a b-generalized derivation with associated pair $(q, ad(c))$. This mapping is called inner b-generalized derivation.

Recently, Alahmadi et al. [1] proved the following result:

Theorem 1 ([1, Theorem 1.1]). *Let $\mathcal{R}$ be a noncommutative prime ring with extended centroid C and k,m,n,r be fixed positive integers. If there exists a generalized derivation G of $\mathcal{R}$ such that $[G(x^m)x^n + x^n G(x^m), x^r]_k = 0$ for all $x \in \mathcal{R}$, then there exists $\lambda \in C$ such that $G(x) = \lambda x$ for all $x \in \mathcal{R}$.*

In this paper, we investigate the above result for b-generalized derivation.

Theorem 2. *Let $\mathcal{R}$ be a noncommutative prime ring and I be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a nonzero b-generalized derivation $\mathcal{F}$ associated with the map d such that $[\mathcal{F}(x^m)x^n + x^n\mathcal{F}(x^m), x^r]_k = 0$ for all $x \in I$, where m,n,k,r are fixed positive integers, then there exists $\lambda \in C$ such that $\mathcal{F}(x) = \lambda x$ for all $x \in \mathcal{R}$ unless $\mathcal{R} \cong \mathrm{M}_2(\mathrm{GF}(2))$, the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements.*

The following example shows that any b-generalized derivation $\mathcal{F}$ may satisfy all conditions of Theorem 2 on the ring $\mathcal{R}$ which is isomorphic to the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements, but $\mathcal{F}$ may not be of the form in Theorem 2.

Example 1. *Let $\mathcal{R} = \mathrm{M}_2(\mathrm{GF}(2))$ be the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements. The set of matrix units in $\mathcal{R}$ will be denoted by $\{e_{ij} \mid 1 \le i,j \le 2\}$. Define a mapping $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ such that $\mathcal{F}(X) = aX + bXc$ for all $X \in \mathcal{R}$, where $a = e_{11} + e_{21}, b = e_{21}$ and $c = e_{11} + e_{12}$. Clearly, $\mathcal{F}$ is a b-generalized derivation of $\mathcal{R}$. A simple calculation gives that $\mathcal{F}(X)X + X\mathcal{F}(X) = 0$ for $X \in \{0, e_{11}, e_{21}, e_{22}, e_{11}+e_{22}, e_{11}+e_{21}, e_{11}+e_{12}, e_{12}+e_{22}, e_{21}+e_{22}\}$ and $X^6 \in \mathrm{Z}(\mathcal{R})$ for the remaining elements of $\mathcal{R}$. Thus, for any positive integers r and k, it can be easily verified that $[\mathcal{F}(X)X + X\mathcal{F}(X), X^{6r}]_k = 0$ for all $X \in \mathcal{R}$. However, $\mathcal{F}$ is not of the form described in Theorem 2.*

Let ρ be an automorphism of $\mathcal{R}$. It is well known that any automorphism ρ of $\mathcal{R}$ can be uniquely extended to an automorphism of Q. The automorphism ρ of $\mathcal{R}$ is said to be X-inner if there exists a unit $a \in Q$ such that $\rho(x) = axa^{-1}$ for all $x \in \mathcal{R}$. An additive mapping $d\colon \mathcal{R} \to \mathcal{R}$ is called a ρ-derivation of $\mathcal{R}$ if $d(xy) = d(x)y + \rho(x)d(y)$ for all $x,y \in \mathcal{R}$. An additive mapping $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ is said to be a generalized ρ-derivation of $\mathcal{R}$ if there exists a ρ-derivation d of $\mathcal{R}$ such that $\mathcal{F}(xy) = \mathcal{F}(x)y + \rho(x)d(y)$ for all $x, y \in \mathcal{R}$. In particular, for X-inner automorphism ρ induced by $a \in Q$, any generalized ρ-derivation $\mathcal{F}$ of $\mathcal{R}$ becomes an a-generalized derivation of $\mathcal{R}$ with associated map $a^{-1}d$. Because of the above observations, we have the following result, which is an application of Theorem 2.

Corollary 1. *Let $\mathcal{R}$ be a noncommutative prime ring and I be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a nonzero ρ-generalized derivation $\mathcal{F}$ associated with an X-inner automorphism ρ of $\mathcal{R}$ such that $[\mathcal{F}(x^m)x^n + x^n\mathcal{F}(x^m), x^r]_k = 0$ for all $x \in I$, where m, n, k, r are fixed positive integers, then there exists $\lambda \in C$ such that $\mathcal{F}(x) = \lambda x$ for all $x \in \mathcal{R}$ unless $\mathcal{R} \cong \mathrm{M}_2(\mathrm{GF}(2))$, the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements.*

2. The lemmas

To prove Theorem 2, we prove the following sequence of lemmas

Lemma 1. *Let V be an infinite dimensional vector space over a field F and let $\mathcal{R}$ be a dense subring of $End({}_FV)$. Suppose that $a, b, c \in End({}_FV)$ such that $[(ax^m + bx^mc)x^n + x^n(ax^m + bx^mc), x^r]_k = 0$ for all $x \in \mathcal{R}$, where m, n, r, k are fixed positive integers. If $b \neq 0$, then $c \in F \cdot I_V$ and $a + bc \in F \cdot I_V$, where I_V denotes the identity transformation of V.*

Proof. We have

$$[(ax^m + bx^mc)x^n + x^n(ax^m + bx^mc), x^r]_k = 0 \text{ for all } x \in \mathcal{R}. \tag{2.1}$$

Claim 1: $c \in F \cdot I_V$. Assume on the contrary that $c \notin F \cdot I_V$. By [4, Lemma 7.1], there is $v \in V$ such that v and cv are linearly independent over F. We divide the proof into two cases.

Case 1: $bV \neq F \cdot v$. Since $b \neq 0$ and $\dim_F V = \infty$, there exists $u \in V$ such that u, v, cv are linearly independent over F and bu, v are linearly independent over F. Write $bu = \alpha cv + \beta v + \gamma u + \delta w$, where $\alpha, \beta, \gamma, \delta \in F$ and $w \notin F \cdot cv + F \cdot v + F \cdot u$. Clearly, α, γ, δ are not all zero. Choose $v_{-1}, v_0, v_1, \ldots, v_{m+n+rk+1} \in V$ such that $v_{-1}, v_0, v_1, \ldots, v_{m+n+rk+1}$ are linearly independent over F and $v_{-1} = v, v_0 = cv, v_m = u, v_{m+1} = w$. Then $v = v_{-1}, cv_{-1} = v_0$ and $bv_m = \alpha v_0 + \beta v_{-1} + \gamma v_m + \delta v_{m+1}$. By the density of $\mathcal{R}$, there exist $x, r \in \mathcal{R}$ such that $xv_{-1} = 0$ and $xv_i = v_{i+1}$ for all $i = 0, 1, 2, \ldots, m + n + rk$. From (2.1), we have

$$\begin{aligned}
0 &= [(ax^m + bx^mc)x^n + x^n(ax^m + bx^mc), x^r]_k v \\
&= \sum_{i=0}^{k} {}^kC_i(-1)^i x^{ri}\{(ax^m + bx^mc)x^n + x^n(ax^m + bx^mc)\}x^{rk-ri}v_{-1} \\
&= (-1)^k x^{kr+n}bx^mcv_{-1} = (-1)^k x^{kr+n}bx^mcv
\end{aligned}$$

$$= (-1)^k x^{kr+n} b x^m v_0 = (-1)^k x^{kr+n} b v_m$$
$$= (-1)^k x^{kr+n} (\alpha v_0 + \beta v_{-1} + \gamma v_m + \delta v_{m+1})$$
$$= (-1)^k (\alpha v_{kr+n} + \gamma v_{m+n+kr} + \delta v_{m+n+kr+1}),$$

which gives a contradiction. Thus, $c \in F \cdot I_V$.

Case 2: $bV = F \cdot v$. Choose $w \in V$ such that $w \notin F \cdot v + F \cdot cv$. Then w, v, cv are linearly independent over F. Suppose first that w and cw are linearly independent over F. Clearly, $bV = F \cdot v \neq F \cdot w$. By the same proof of Case 1, there exists $x \in \mathcal{R}$ such that $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k w \neq 0$, a contradiction. Suppose next that w and cw are linearly dependent over F. Write $cw = \alpha w$, for some $\alpha \in F$. Then $c(v + w) = cv + cw = cv + \alpha w$. Hence, $c(v+w)$ and $v+w$ are linearly independent over F as w, v, cv are linearly independent over F. Clearly, $bV = F \cdot v \neq F \cdot (w + v)$. By the same proof of Case 1, there exists $x \in \mathcal{R}$ such that $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k (w + v) \neq 0$, a contradiction. This proves Claim 1.

Claim 2: $a + bc \in F \cdot I_V$. By Claim 1, (2.1) reduces to

$$[(a+bc)x^{m+n} + x^n(a+bc)x^m, x^r]_k = 0 \text{ for all } x \in \mathcal{R}. \tag{2.2}$$

Suppose on contrary that $a + bc \notin F \cdot I_V$. Then by [4, Lemma 7.1], $v, (a+bc)v$ are linearly independent over F for some $v \in V$. Choose $v_1, \ldots, v_{m+2} \in V$ such that $v_1, \ldots, v_{m+2}$ are linearly independent over F and $v_{m+1} = v$, $v_{m+2} = (a + bc)v$. By the density of $\mathcal{R}$ there exists $x \in \mathcal{R}$ such that $xv_i = v_{i+1}$ for all $i = 1, 2, \ldots, m$, $xv_{m+1} = 0$, $xv_{m+2} = v_{m+2}$. Therefore from (2.2), we have

$$0 = [(a+bc)x^{m+n} + x^n(a+bc)x^m, x^r]_k v_1$$
$$= \sum_{i=0}^{k} {}^kC_i (-1)^i x^{ri} \left\{(a+bc)x^{m+n} + x^n(a+bc)x^m\right\} x^{r(k-i)} v_1$$
$$= (-1)^k x^{rk+n}(a+bc)x^m v_1 = (-1)^k x^{rk+n}(a+bc) v_{m+1}$$
$$= (-1)^k x^{rk+n}(a+bc)v = (-1)^k x^{rk+n} v_{m+2} = (-1)^k v_{m+2}$$
$$= (-1)^k (a+bc)v,$$

which gives a contradiction, and hence $a + bc \in F \cdot I_V$.

□

Lemma 2. *Let $\mathcal{R} = \mathrm{M}_s(F)$ be the $s \times s$ matrix ring over a field F, where $s \geq 3$ is an integer. Suppose $a, b, c \in \mathcal{R}$ such that $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0$ for all $x \in \mathcal{R}$, where m, n, r, k are fixed positive integers. If $b \neq 0$, then $c \in F \cdot I_s$ and $a + bc \in F \cdot I_s$, where I_s denotes the identity matrix of $\mathcal{R}$.*

Proof. By the assumption we have

$$[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0 \text{ for all } x \in \mathcal{R}. \tag{2.3}$$

Let ψ be an F-linear automorphism of $\mathcal{R}$. Then

$$[(\psi(a)y^m+\psi(b)y^m\psi(c))y^n+y^n(\psi(a)y^m+\psi(b)y^m\psi(c)),y^r]_k=0 \text{ for all } y\in\mathcal{R}. \tag{2.4}$$

Claim 1: $c\in F\cdot I_s$. Write $c=\Sigma_{i,j=1}^{s}c_{ij}e_{ij}$ and $b=\Sigma_{i,j=1}^{s}b_{ij}e_{ij}$, where $b_{ij},c_{ij}\in F$. It can be easily conclude that $[(ax^m+bx^mc)x^n+x^n(ax^m+bx^mc),x^r]_{2k+1}=0$ for all $x\in\mathcal{R}$. So we may assume that k is an odd integer. Also, for any idempotent e of $\mathcal{R}$, $[x,e]_3=[x,e]$ for all $x\in\mathcal{R}$ Therefore $[x,e]_{2k+1}=[x,e]$ for all $x\in\mathcal{R}$. Putting $x=e_{ii}$, where i is an integer with $1\le i\le s$ in (2.3) and using the fact that $[x,e]_{2k+1}=[x,e]$, we get

$$\begin{aligned}0&=[(ae_{ii}^m+be_{ii}^mc)e_{ii}^n+e_{ii}^n(ae_{ii}^m+be_{ii}^mc),e_{ii}^r]_k\\&=[(ae_{ii}+be_{ii}c)e_{ii}+e_{ii}(ae_{ii}+be_{ii}c),e_{ii}]_k\\&=[ae_{ii}+be_{ii}ce_{ii}+e_{ii}ae_{ii}+e_{ii}be_{ii}c,e_{ii}]\\&=ae_{ii}+be_{ii}ce_{ii}-e_{ii}ae_{ii}-e_{ii}be_{ii}c.\end{aligned}$$

Now, multiplying by e_{jj} from right, we get

$$e_{ii}be_{ii}ce_{jj}=0. \tag{2.5}$$

This implies that

$$b_{ii}c_{ij}=0 \text{ for all } j\neq i \text{ and } 1\le i,j\le s. \tag{2.6}$$

Again, putting $x=e_{ii}+e_{ji}$, where i,j are distinct integers with $1\le i,j\le s$ in (2.3), we obtain

$$\begin{aligned}0=&[(a(e_{ii}+e_{ji})^m+b(e_{ii}+e_{ji})^mc)(e_{ii}+e_{ji})^n\\&+(e_{ii}+e_{ji})^n(a(e_{ii}+e_{ji})^m+b(e_{ii}+e_{ji})^mc),(e_{ii}+e_{ji})^r]_k\\=&[(a(e_{ii}+e_{ji})+b(e_{ii}+e_{ji})c)(e_{ii}+e_{ji})\\&+(e_{ii}+e_{ji})(a(e_{ii}+e_{ji})+b(e_{ii}+e_{ji})c),(e_{ii}+e_{ji})]\\=&a(e_{ii}+e_{ji})+b(e_{ii}+e_{ji})c(e_{ii}+e_{ji})\\&-(e_{ii}+e_{ji})a(e_{ii}+e_{ji})-(e_{ii}+e_{ji})b(e_{ii}+e_{ji})c\\=&a(e_{ii}+e_{ji})+b(e_{ii}+e_{ji})c(e_{ii}+e_{ji})\\&-(e_{ii}+e_{ji})a(e_{ii}+e_{ji})-(e_{ii}+e_{ji})b(e_{ii}+e_{ji})c.\end{aligned}$$

Multiplying by e_{ll} from right, we get

$$(e_{ii}+e_{ji})b(e_{ii}+e_{ji})ce_{ll}=0.$$

Again multiply by e_{ii} from left and using (2.5), we conclude that

$$0=e_{ii}be_{ji}ce_{ll}=b_{ij}c_{il}e_{jl} \text{ for all } j\neq i.$$

This implies that

$$b_{ij}c_{il}=0 \text{ for all } j\neq i. \tag{2.7}$$

Thus, from (2.6) and (2.7), we conclude that

$$\text{if } c_{il} \neq 0 \text{ for some } i \neq l, \text{ then } b_{ij} = 0 \text{ for all } j = 1, 2, \ldots s. \tag{2.8}$$

First we need to show that c is a diagonal matrix. Suppose c is not a diagonal matrix and assume that $c_{21} \neq 0$. Then by [16, Lemma 2.1], there exists an inner automorphism ψ of $\mathcal{R}$ induced by q such that $\psi(c) = qcq^{-1} = \Sigma_{i,j=1}^{s} c_{ij}^{\psi} e_{ij}$, where $c_{ij}^{\psi} \in F$ and

$$c_{21}^{\psi} \neq 0,\ c_{31}^{\psi} \neq 0, \ldots, c_{s1}^{\psi} \neq 0,\ c_{1s}^{\psi} \neq 0. \tag{2.9}$$

Write $\psi(b) = \Sigma_{i,j=1}^{s} b_{ij}^{\psi} e_{ij}$, where $b_{ij}^{\psi} \in F$. Combining (2.4), (2.8) and (2.9), we find that $b_{ij}^{\psi} = 0$ for all i, j with $1 \leq i, j \leq s$. Therefore $\psi(b) = 0$ and hence $b = 0$, which is a contradiction. Thus c is a diagonal matrix that is, $c = \Sigma_{i=1}^{s} c_{ii} e_{ii}$. Let j be an integer with $2 \leq j \leq s$ and let ϕ be an F-linear automorphism of $\mathcal{R}$ defined by $\phi(x) = (I_s + e_{1j}) x (I_s - e_{1j})$ for all $x \in \mathcal{R}$. Then $\phi(c) = (c_{jj} - c_{11}) e_{1j} + \Sigma_{i=1}^{s} c_{ii} e_{ii}$. Since $b \neq 0$, so $\phi(b) \neq 0$. In view of (2.4) and using the same arguments as we have used above, we find that $\phi(c)$ is a diagonal matrix. Thus $c_{jj} - c_{11} = 0$ for all $2 \leq j \leq s$. This implies that $c \in F \cdot I_s$.

Claim 2: $a + bc \in F \cdot I_s$. From (2.3) and Claim 1, we have $[(a+bc)x^{m+n} + x^n(a+bc)x^m, x^r]_k = 0$ for all $x \in \mathcal{R}$. This implies that

$$\begin{aligned} 0 &= [(x^{m+n})^t (a+bc)^t + (x^m)^t (a+bc)^t (x^n)^t, (x^r)^t]_k \\ &= [(x^t)^{m+n} (a+bc)^t + (x^t)^m (a+bc)^t (x^t)^n, (x^t)^r]_k, \end{aligned}$$

where x^t denotes the usual matrix transpose of x in $\mathcal{R}$. Substituting x^t for x and using the fact that $(x^t)^t = x$, we get $[b' x^m c' x^n + x^n b' x^m c', x^r]_k = 0$ for all $x \in \mathcal{R}$, where $c' = (a+bc)^t$ and $b' = I_s$, the identity matrix of $\mathcal{R}$. Again from (2.3) and by the same arguments as above we have used, we get $(a+bc)^t \in F \cdot I_s$. This implies that $a + bc \in F \cdot I_s$.

□

Lemma 3. *Let $\mathcal{R} = \mathrm{M}_2(F)$ be the 2×2 matrix ring over a field F and $a, b, c \in \mathcal{R}$. Suppose that $b \neq 0$ and $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0$ for all $x \in \mathcal{R}$, where m, n, r, k are fixed positive integers. If c is not a diagonal matrix, then b is not an invertible matrix and $F = \{0, 1\}$.*

Proof. We have

$$[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0 \tag{2.10}$$

for all $x \in \mathcal{R}$. Let ϕ be a F-linear automorphism of $\mathcal{R}$. Then from (2.10), we have

$$[(\psi(a)y^m + \psi(b)y^m \psi(c))y^n + y^n(\psi(a)y^m + \psi(b)y^m \psi(c)), y^r]_k = 0$$

for all $y \in \mathcal{R}$. It follows from (2.6) that

$$\text{if } c_{il} \neq 0 \text{ for some } i \neq l, \text{ then } b_{ij} = 0 \text{ for all } i, j \in \{1,2\}. \tag{2.11}$$

Let $c, b \in \mathrm{M}_2(F)$. Then $c = \Sigma_{i,j=1}^{s} c_{ij}e_{ij}$ and $b = \Sigma_{i,j=1}^{s} b_{ij}e_{ij}$, where $b_{ij}, c_{ij} \in F$ and $i, j \in \{1,2\}$. By hypothesis, c is not a diagonal matrix, we may assume $c_{12} \neq 0$. Thus, from (2.11), we have $b_{11} = b_{12} = 0$. If $c_{21} \neq 0$, then $b_{21} = b_{22} = 0$. This implies $b = 0$, a contradiction. Hence, we have $c_{21} = 0$. So

$$c_{12} \neq 0,\ c_{21} = 0,\ b_{11} = b_{12} = 0. \tag{2.12}$$

It is clear from above that b is not an invertible matrix. Let us define, for $\alpha \in F$, ϕ_α be an F-linear automorphism of $\mathcal{R}$ such that $\phi_\alpha(x) = (I_2 + \alpha e_{21})x(I_2 - \alpha e_{21})$ for all $x \in \mathcal{R}$. If $\phi_\alpha(b) = \Sigma_{i,j=1}^{s} b_{ij}^{\alpha} e_{ij}$ and $\phi_\alpha(c) = \Sigma_{i,j=1}^{s} c_{ij}^{\alpha} e_{ij}$, where $b_{ij}^{\alpha}, c_{ij}^{\alpha} \in F$, then it follows from above that $b_{21}^{\alpha} = b_{21} - \alpha b_{22}$ and $c_{21}^{\alpha} = \alpha(c_{11} - c_{22}) - \alpha^2 c_{12}$. If $c_{21}^{\alpha} \neq 0$, then we see from (2.11) that $b_{21}^{\alpha} = b_{21} - \alpha b_{22} = 0$ and $b_{22}^{\alpha} = b_{22} = 0$. Thus, $b_{21} = b_{22} = 0$. Now, from (2.12), we have $b = 0$. This leads to a contradiction. Therefore,

$$c_{21}^{\alpha} = \alpha(c_{11} - c_{22}) - \alpha^2 c_{12} = 0 \tag{2.13}$$

for all $\alpha \in F$. Suppose, if F has more than two elements then from (2.13), we can conclude that $c_{12} = 0$, which gives a contradiction $b = 0$. Therefore, F can not have more than two elements i.e., $F = \{0, 1\}$. This completes the proof of Lemma. □

Lemma 4. *Let $\mathcal{R} = \mathrm{M}_2(F)$ be the 2×2 matrix ring over a field F and $a, b, c \in \mathcal{R}$ such that $b \neq 0$ and $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0$ for all $x \in \mathcal{R}$, where m, n, r, k are fixed positive integers. If $b \neq 0$, then $c \in F.I_2$ and $a + cb \in F.I_2$, unless $F \cong \mathrm{GF}(2)$, the Galois field of two elements.*

Proof. Assume that $F \ncong \mathrm{GF}(2)$. In view of Lemma 3, we have c is a diagonal matrix, and hence $c = \Sigma_{i=1}^{2} c_{ii}e_{ii}$, where $c_{ii} \in F$. Let ϕ be a F-linear automorphism of $\mathcal{R}$, defined by $\phi(x) = (I_2 + e_{12})x(I_2 - e_{12})$ for all $x \in \mathcal{R}$. Hence, $\phi(c) = \Sigma_{i=1}^{2} c_{ii}e_{ii} + (c_{22} - c_{11})e_{12}$. Obviously, $\phi(b) \neq 0$ and $[(\phi(a)x^m + \phi(b)x^m\phi(c))x^n + x^n(\phi(a)x^m + \phi(b)x^m\phi(c)), x^r]_k = 0$ for all $x \in \mathcal{R}$. It follows from Lemma 3 that $\phi(c)$ is a diagonal matrix. This gives $c_{22} - c_{11} = 0$, and hence $c = c_{11}I_2 \in F.I_2$. Now, by the assumption, we have $[(a + bc)x^{m+n} + x^n(a + bc)x^m, x^r]_k = 0$ for all $x \in \mathcal{R}$. This implies that

$$\begin{aligned} 0 &= [(x^{m+n})^t(a+bc)^t + (x^m)^t(a+bc)^t(x^n)^t, (x^r)^t]_k \\ &= [(x^t)^{m+n}(a+bc)^t + (x^t)^m(a+bc)^t(x^t)^n, (x^t)^r]_k, \end{aligned}$$

where x^t denotes the usual matrix transpose of x in $\mathcal{R}$. Substituting x^t for x and using the fact that $(x^t)^t = x$, we get $[b'x^m c'x^n + x^n b'x^m c', x^r]_k = 0$ for all $x \in \mathcal{R}$, where $c' = (a + bc)^t$ and $b' = I_2$, the identity matrix of $\mathcal{R}$. Again by using the same arguments as we have used in the above, we get $c' = (a + bc)^t \in F \cdot I_2$. This implies that $a + bc \in F \cdot I_2$. □

3. Proof of Theorem 2

Now, we are in position to prove our theorem.
Suppose first that $b=0$. Then $\mathcal{F}(xy)=\mathcal{F}(x)y$ for all $x,y\in\mathcal{R}$. In view of [3, Lemma 2.3], there is $a\in Q$ such that $\mathcal{F}(x)=ax$ for all $x\in\mathcal{R}$. In this case, by the hypothesis, we have

$$[ax^{m+n}+x^nax^m,x^r]_k=0$$

for all $x\in I$ and hence for all $x\in\mathcal{R}$. In view of [1, Corollary 1.7], we get $a\in C$, which gives the required result.
Now, we assume that $b\neq 0$. By [12, Theorem 2.3], $d\colon\mathcal{R}\to Q$ is a derivation and there exists $a'\in Q$ such that $\mathcal{F}(x)=a'x+bd(x)$ for all $x\in\mathcal{R}$. It is known that d can be uniquely extended to a derivation of Q [14, Lemma 2]. By the assumption we have

$$[\mathcal{F}(x^m)x^n+x^n\mathcal{F}(x^m),x^r]_k=0 \text{ for all } x\in I. \tag{3.1}$$

We divide the proof into two cases.

Case 1: d is Q-inner. That is, there exists $c'\in Q$ such that $d(x)=[c',x]$ for all $x\in\mathcal{R}$. So $\mathcal{F}(x)=a'x+bd(x)=a'x+b[c',x]=ax+bxc$ for all $x\in\mathcal{R}$, where $a=a'+bc'$ and $c=-c'$. By (3.1), we have

$$[(ax^m+bx^mc)x^n+x^n(ax^m+bx^mc),x^r]_k=0 \text{ for all } x\in I. \tag{3.2}$$

Since I, $\mathcal{R}$ and Q satisfies the same polynomial identities by [5, Theorem 6.4.4]. Therefore, $[(ax^m+bx^mc)x^n+x^n(ax^m+bx^mc),x^r]_k=0$ for all $x\in Q$. If $c\in C$ and $a+cb\in C$, then $g(x)=\lambda x$ for all $x\in\mathcal{R}$, where $\lambda=a+cb$ and $d=0$ as $c=-c'\in C$, proving the theorem. Now we assume that $c\notin C$ or $a+cb\notin C$. Let $\varphi(x)=[(ax^m+bx^mc)x^n+x^n(ax^m+bx^mc),x^r]_k$ Clearly, if $c\in C$ and $a+cb\notin C$, then $\varphi(x)$ is a nonzero GPI of Q by (3.2). Suppose $c\notin C$. Then by (3.2) $\varphi(x)$ is a nonzero GPI of Q as $b\neq 0$. So we conclude that $\varphi(x)$ is a nonzero GPI of Q. Let F be the algebraic closure of C if C is infinite and set $F=C$ for C finite. Clearly, the map $r\in Q\mapsto r\otimes 1\in Q\otimes_C F$ gives a ring embedding. So we may assume Q is a subring of $Q\otimes_C F$. By [13, Proposition], $\varphi(x)$ is also a nonzero GPI of $Q\otimes_C F$. Moreover, in view of [9, Theorem 3.5], $Q\otimes_C F$ is a prime ring with F as its extended centroid. Thus, $\overline{Q}=Q\otimes_C F$ is a prime ring satisfies a nonzero GPI $\varphi(x)$ and its extended centroid F is either an algebraically closed field or a finite field. By Martindale's Theorem [5, Theorem 6.1.6], $\overline{Q}$ is a primitive ring having nonzero socle with F as its associated division ring. Moreover, $\overline{Q}$ is a dense subring of $\mathrm{End}({}_FV)$, where V is a vector space over F. If $\dim_F V=1$, then $\overline{Q}$ is commutative and hence $\mathcal{R}$ is commutative, a contradiction. So $\dim_F V\geq 2$. Firstly, we suppose $\dim_F V=2$. Then $\overline{Q}=\mathrm{End}({}_FV)\cong \mathrm{M}_2(F)$. Application of Lemma 4 yields that $c\in F$ and $a+bc\in F$ or $F\cong \mathrm{GF}(2)$. In first case, we have $\mathcal{F}(x)=\lambda x$, where $\lambda=a+bc\in C$ in the other case, we have $C=F\cong\mathrm{GF}(2)$ and hence $\mathcal{R}=Q\cong \mathrm{M}_2(\mathrm{GF}(2))$. This proves the theorem. Next, if $\dim_F V\geq 3$, then by

Lemma 2 and Lemma 1, we have $c \in F$ and $a+bc \in F$. Therefore, $-c' \in C$ and $a+bc \in C$ and hence $\mathcal{F}(x) = \lambda x$ for all $x \in \mathcal{R}$, where $\lambda = a+bc \in C$, which is the required form of $\mathcal{F}$.

Case 2: d is not Q-inner. From (3.1), we have

$$[a'x^{m+n} + bd(x^m)x^n + x^n a'x^m + x^n bd(x^m), x^r]_k = 0$$

for all $x,y \in I$ and thus for all $x,y \in \mathcal{R}$ [14, Theorem 2]. Now, we set $h(Y,X) = \sum_{i=0}^{m-1} X^i Y X^{m-1-i}$, a polynomial in two non commuting variables X and Y. Note that $d(x^m) = h(d(x),x)$. Then the above expression becomes as $[a'x^{m+n} + bh(d(x),x)x^n + x^n a'x^m + x^n bh(d(x),x), x^r]_k = 0$ for all $x \in \mathcal{R}$.

Applying Kharchenko's theorem [11], we obtain

$$[a'x^{m+n} + bh(y,x)x^n + x^n a'x^m + x^n bh(y,x), x^r]_k = 0$$

for all $x,y \in \mathcal{R}$ and hence for all $x,y \in Q$ by [5, Theorem 6.4.4]. In particular choose $u \notin C$ and take $y = [u,x]$ and using the fact that $h([u,x],x) = [u,x^m]$ in the above expression we get In particular for $y = 0$, we have

$$[a'x^{m+n} + b[u,x^m]x^n + x^n a'x^m + x^n b[u,x^m], x^r]_k = 0 \text{ for all } x \in Q.$$

This can be rewritten as

$$[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0 \text{ for all } x \in \mathcal{R}, \tag{3.3}$$

where $a = a' + bu$ and $c = -u$. Since $c = -u \notin C$, so in view of (3.2) and (3.3), we obtain the required result from Case 1. Thereby the proof is completed. □

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