MONOTONICITY PROPERTIES AND FUNCTIONAL INEQUALITIES FOR THE BARNES MITTAG-LEFFLER FUNCTION

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Abstract. In this paper, our main focus is to establish complete monotonicity properties, convexity properties and some interesting inequalities for the Barnes Mittag-Leffler function. Furthermore, monotonicity properties of ratios of Barnes Mittag-Leffler functions are derived. Moreover, Turán type inequalities and several new inequalities are obtained for this function as application. Results obtained in this work are new and their importance is illustrated by several attractive consequences and corollaries.

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1. INTRODUCTION

Mittag-Leffler function is one of the important special functions, which plays vital role in mathematical physics, fractional calculus, approximation theory and various fields of science and engineering. It appears in the differential equations of fractional order and study of complex system. Mittag-Leffler function was introduced by Gosta Mittag-Leffler in 1903 [15], defined as

$$E_\kappa(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n + 1)}, \quad z \in \mathbb{C}, \quad \kappa \geq 0,$$

where $\Gamma$ is the well-known Euler’s gamma function. From (1.1), it can be observed that $E_\kappa(z)$ interpolates between $\exp(z)$ and $1/(1-z)$ for $0 < \kappa < 1$. In 1905, Wiman [24] studied a generalization of $E_\kappa(z)$, defined as

$$E_{\kappa,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n + \nu)}, \quad \kappa, \nu \in \mathbb{C}, \quad \Re(\kappa) > 0, \Re(\nu) > 0.$$

The generalized Mittag-Leffler function $E_{\kappa,\nu}(z)$ is also known as Wiman’s function. There are several generalizations of Mittag-Leffler function available in the literature. One of the most important generalizations of Mittag-Leffler function is the Barnes
Mittag-Leffler function $E_{\kappa,v}^{(a)}(s;z)$, which is defined \[6\] as

$$E_{\kappa,v}^{(a)}(s;z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(kn+v)(a+n)z}, \quad a,v \in \mathbb{C} \setminus \mathbb{Z}_0^+, \Re(k) > 0, s,z \in \mathbb{C}. \quad (1.3)$$

It can be noted that $E_{\kappa,v}^{(a)}(0;z) = E_{\kappa,v}(z)$, $E_{\kappa,1}(z) = E_{\kappa}(z)$ and $E_1(z) = \exp(z)$. For further information on Mittag-Leffler function, we refer to \[7\] and references cited therein.

In terms of the extended Hurwitz-Lerch zeta function $\Phi_{p,q;\kappa}(z,s,a)$, defined by \[19, p. 503, Eq. (6.2)\]

$$\Phi_{(p,q;\kappa)}^{(z,s,a)}(z,s,a) = \Phi_{\lambda_1,\ldots,\lambda_q;\mu_1,\ldots,\mu_q}^{(z,s,a)}(z,s,a) = \prod_{j=1}^{q} \frac{\Gamma(\mu_j)}{\Gamma(\lambda_j)} \sum_{k=0}^{\infty} \prod_{j=1}^{q} \frac{\Gamma(\lambda_j + k\rho_j)}{\Gamma(\mu_j + k\sigma_j)} \frac{z^k}{k!(k+a)^s}, \quad (1.4)$$

we see that

$$E_{\kappa,v}^{(a)}(s;z) = \frac{1}{\Gamma(s)} \Phi_{(1,1);\{v,k,1\}}^{(1,1)}(z,s,a).$$

From \[19, Theorem 8\], we deduce that the Barnes Mittag-Leffler function $E_{\kappa,v}^{(a)}(s;z)$ possesses the following integral representation

$$E_{\kappa,v}^{(a)}(s;z) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-at} E_{\kappa,v}(ze^{-t}) dt, \quad (\min(\Re(a) , \Re(s), \kappa) > 0). \quad (1.5)$$

Monotonicity properties and functional inequalities play vital role in the theory of various inequalities and equilibrium problems \[2, 3, 5, 23, 25\]. Nowadays most of the researchers use Turán-type inequalities to find monotonicity properties of special functions. Turán’s inequality was first introduced by the Hungarian mathematician Paul Turán \[21\] in 1950 as follows:

$$\frac{P_n(x)P_{n+2}(x)}{P_{n+1}^2(x)} \leq 1, \quad x \in [-1,1], \quad n \in \mathbb{N} \cup \{0\}, \quad (1.6)$$

where $P_n(x)$ are Legendre polynomials. Several Turán-type inequalities for various special functions can be found in \[3, 4, 11, 12\] and references cited therein. It will be interesting to see from \(1.6\) that stronger results on monotonicity of ratios of functions of the form \(1.6\) with upper or lower constants as unity can be obtained using these types of inequalities. These inequalities have applications in the theory of transmutation operators for estimating transmulation kernels and norms \[18\] and also in the problems of function expansions by system of integer shifts of Gaussians \[10\].

This paper is organized as follows. In the next Section, we present some useful lemmas which are required to complete the proofs of the main results. Using the
integral representation (1.5) of the Barnes Mittag-Leffler function, complete monotonicity property and several interesting inequalities and their consequences are established in Section 3. Turán type inequalities for $E_{k,v}^{(a)}(s;z)$ are obtained in Section 4. Moreover, monotonicity, convexity and log-convexity properties for the ratios of Barnes Mittag-Leffler function are also studied in this section. Section 5 is devoted to derive further interesting inequalities for $E_{k,v}^{(a)}(s;z)$.

2. USEFUL LEMMAS

Before proving the main results in this paper, we need the following preliminary lemmas.

**Lemma 1** ([12]). Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences of real numbers. If $b_n > 0$ for $n \geq 0$ and if the sequence $(a_n/b_n)$ is increasing (decreasing), then $(a_1 + \cdots + a_n/b_1 + \cdots + b_n)_{n \geq 0}$ is increasing (decreasing).

**Lemma 2** ([17]). Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences of real numbers and let the power series $f(x) = \sum_{n=0}^\infty a_n x^n$ and $g(x) = \sum_{n=0}^\infty b_n x^n$ be convergent for all $|x| < r$. If $b_n > 0$ for $n \geq 0$ and if the sequence $(a_n/b_n)$ is increasing (decreasing), then the function $x \mapsto f(x)/g(x)$ is increasing (decreasing) on $(0,r)$.

**Lemma 3** ([11]). Suppose that $f, g : [a,b] \to \mathbb{R}$ are continuous functions, which are differentiable on $(a,b)$. Moreover, assume that $g'(x) \neq 0$ on $(a,b)$. If $f'/g'$ is increasing (or decreasing) on $(a,b)$, then the following ratios

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}$$

are also increasing (or decreasing) on $(a,b)$.

3. COMPLETE MONOTONICITY PROPERTY AND RELATED INEQUALITIES

In the next theorem, we will study complete monotonicity property of the function $E_{k,v}^{(a)}(s;-z)$. A $C^\infty$ function $f(x)$ is said to be completely monotonic on an interval $I$ if

$$(-1)^{n-1} f^{(n-1)}(x) \geq 0,$$

for any $n \in \mathbb{N}$ and $x \in I$.

**Theorem 1.** Assume that $\min(\Re(a), \Re(s)) > 0$, $\kappa > 0$. Under the following conditions

$$(\kappa, \nu \in (0,1) \; \nu \geq \kappa) \text{ or } (\kappa \in (0,1] \text{ and } \nu \geq \kappa),$$

the function $z \mapsto E_{k,v}^{(a)}(s;-z)$ is completely monotonic on $(0,\infty)$. Moreover, the following inequalities

$$E_{k,v}^{(a)}(s;-z_1)E_{k,v}^{(a)}(s;-z_2) \leq E_{k,v}^{(a)}(s;-z_1-z_2), \quad (\min(z_1,z_2) > 0), \quad (3.1)$$
Using an elementary property of Gamma-function:

\[
E_{\kappa}^{(a)}(s;-(z_1+z_2)/2) \leq \sqrt{E_{\kappa}^{(a)}(s;-z_1)E_{\kappa}^{(a)}(s;-z_2)}, \quad (\min(z_1,z_2) > 0)
\]  

(3.2)

\[
\frac{1}{a} \Gamma(v) \exp \left( -\frac{a^2 \Gamma(v)}{(a+1)^2 \Gamma(v+x)} z \right) \leq E_{\kappa}^{(a)}(s;-z), \quad z > 0,
\]

(3.3)

hold true under the given hypothesis.

**Proof.** By using (1.5) with [20, Corollary 3] and [8, Theorem 2.1], we establish that \( E_{\kappa}^{(a)}(s;-z) \) is completely monotonic on \((0, \infty)\), under the given hypothesis. However, we see that the function \( z \mapsto a^2 \Gamma(v) E_{\kappa}^{(a)}(s;-z) \) is completely monotonic on \((0, \infty)\) and maps \((0, \infty)\) to \((0, 1)\). According to [9, Theorem 3], we get the result (3.1). Let us now focus on the inequality (3.2). By using the fact that every completely monotonic function is log-convex [22, p. 167], we deduce that the function \( E_{\kappa}^{(a)}(s;-z) \) is log-convex and consequently the inequality (3.2) holds true. It remains to prove the inequality (3.3). Let

\[
F(z) = \log \left( a^2 \Gamma(v) E_{\kappa}^{(a)}(s;-z) \right) = \log E_{\kappa}^{(a)}(s;z) \quad \text{and} \quad G(z) = z.
\]

Observe that the function \( z \mapsto (E_{\kappa}^{(a)}(s;z))'/\tilde{E}_{\kappa}^{(a)}(s;z) \) is increasing on \((0, \infty)\) and consequently the function

\[
z \mapsto \frac{F(z) - F(0)}{G(z) - G(0)} =: \Delta(z)
\]

is also increasing on \((0, \infty)\) by Lemma 3. Furthermore,

\[
\lim_{z \to 0} \Delta(z) = \lim_{z \to 0} \frac{(E_{\kappa}^{(a)}(s;z))'}{E_{\kappa}^{(a)}(s;z)} = -\frac{a^2 \Gamma(v)}{(a+1)^2 \Gamma(v + \kappa)},
\]

which completes the proof of Theorem 1. \(\square\)

**Theorem 2.** Under the assumptions of Theorem 1, the following inequality holds true:

\[
E_{\kappa}^{(a)}(e^{-z/2}) \leq E_{\kappa}^{(a)}(s;z) \leq \frac{(a+1)^2 - \Gamma(s)}{(a+1)^2 \Gamma(v + \kappa)} + \frac{\Gamma(s)}{(a+1)^2} E_{\kappa}^{(a)}(v).
\]

(3.4)

**Proof.** We recall the Jensen’s integral inequality [14, Chap. I, Eq. (7.15)],

\[
\Phi \left( \int_a^b f(s) d\sigma(s) / \int_a^b d\sigma(s) \right) \leq \int_a^b \Phi(f(s)) d\sigma(s) / \int_a^b d\sigma(s),
\]

(3.5)

where, \( \Phi \) is convex and \( f \) is integrable with respect to a probability measure \( \sigma \). Let

\[
\Phi_{\epsilon}(t) = E_{\kappa}^{(a)}(ze^{-t}), \quad f(t) = t \quad \text{and} \quad d\sigma(t) = \frac{1}{\Gamma(s)} t^{s-1} e^{-at} dt.
\]

Using an elementary property of Gamma-function:

\[
\frac{\Gamma(s)}{a^s} = \int_0^\infty t^{s-1} e^{-at} dt, \quad (\Re(a) > 0, \Re(a) > 0),
\]

(3.6)
and the representation (1.5), we obtain
\[ \int_a^b d\sigma(t) = \frac{1}{a'}, \quad \int_a^b f(t)d\sigma(t) = \frac{s}{a'^{a+1}} \quad \text{and} \quad \int_a^b \phi(f(t))d\sigma(t) = E_{\kappa}^{(a)}(s; z). \]
Again, from (3.5), we have
\[ E_{\kappa}^{(a)}(e^{-\frac{s}{z}}z) \leq E_{\kappa}^{(a)}(s; z), \]
which proves the lower bound of (3.4). In order to demonstrate the upper bound, we will apply the converse Jensen inequality, due to Lah and Ribarić, which reads as follows. Set
\[ A(f) = \int_m^M f(s)d\sigma(s)/\int_m^M d\sigma(s), \]
where \( \sigma \) is a non-negative measure and \( f \) is a continuous function. If \(-\infty < m < M < \infty \) and \( \phi \) is convex on \([m, M]\), then according to [16, Theorem 3.37], we have
\[ (M - m)A(\phi(f)) \leq (M - A(f))\phi(m) + (A(f) - m)\phi(M). \quad (3.7) \]
Rewriting the representation (1.5) as follows:
\[ E_{\kappa}(s; z) = \frac{1}{\Gamma(s)} \int_0^1 \log^{s-1}(1/u)u^{a-1}E_{\kappa}(zu)du, \quad (3.8) \]
and taking
\[ [m, M] = [0, 1], \varphi(u) = E_{\kappa}(zu), f(u) = u \quad \text{and} \quad d\sigma(u) = \frac{\log^{s-1}(1/u)u^{a-1}du}{\Gamma(s)}, \]
in (3.7), in view of (3.6) and (3.8), we obtain
\[ \frac{E_{\kappa}(s; z)}{a'} \leq \frac{(a(a+1))^s - \Gamma(s)}{(a(a+1))^{(a+1)}s^{a+1}} + \frac{\Gamma(s)}{(a(a+1))^{a+1}}E_{\kappa}(z). \]

**Theorem 3.** The following inequalities hold true:

(a) Let \( a, \nu, s, z \) be positive real numbers such that \( s \leq 1 \) and \( a \geq \nu \). Then the following inequality
\[ \frac{E_{1,\nu+1}(z)}{(a - \nu + 1)^s} \leq E_{1,\nu}^{(a)}(s, z) \quad (3.9) \]
holds true. Furthermore, the above inequality is reversed if \( s \geq 1 \) and \( \nu - 1 < a \leq \nu \).

(b) Let \( a, \nu, s, z > 0 \) be such that \( a > \nu \) and \( 0 < s < 1 \). Then the inequality
\[ E_{1,\nu}^{(a)}(s, z)E_{1,\nu}^{(a)}(z + 2, z) \leq \frac{2^{2s}s\Gamma(s + 1/2)}{\sqrt{\pi}\Gamma(s + 2)}E_{1,\nu+1}E_{1,\nu}^{(2s-\nu)}(2s + 1, z) \quad (3.10) \]
is valid.
Proof. (a) First of all, we recall the Chebyshev integral inequality [13, p. 40]. If \( f, g : [a, b] \to \mathbb{R} \) are integrable functions, both increasing or both decreasing, and \( p : [a, b] \to \mathbb{R} \) is a positive integrable function, then

\[
\int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt.
\]

(3.11)

It can be noted that if one of the functions \( f \) or \( g \) is decreasing and the other is increasing, then (3.11) is reversed. We will use (3.8) and (3.11) to prove (3.9). We set

\[
p(t) = 1, \quad f(t) = t^{a-v} \log^{s-1}(1/t) \quad \text{and} \quad g(t) = t^{\nu-1} E_{1,\nu}(zt).
\]

From the differentiation formula [7, Eq. (4.3.1), p. 51]

\[
\left( \frac{d}{dz} \right)^m [z^{\nu-1} E_{\kappa,\nu}(z^\kappa)] = z^{\nu-m-1} E_{\kappa,\nu-m}(z^\kappa), \quad (m \geq 1),
\]

we see that the function \( g \) is increasing on \((0,1]\). Moreover, the function \( f \) is decreasing (increasing) if \( s \geq 1 \) and \( \nu-1 < a \leq \nu \) \((0 < s \leq 1 \) and \( a \geq \nu \)) on \((0,1] \). Furthermore, we have

\[
\int_0^1 p(t)f(t)dt = \frac{t^{a-v}}{\Gamma(s)} \int_0^1 \log^{s-1}(1/t)dt
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty t^{a-\nu+1}e^{-(a-\nu+1)t}dt = \frac{1}{(a-\nu+1)^s}, \quad \text{(by (3.6)).}
\]

Taking into account [7, Eq. (4.4.4), p. 61]

\[
\int_0^z t^{\nu-1} E_{\kappa,\nu}(\lambda t^\kappa)dt = \frac{z^\kappa E_{\kappa,\nu+1}(\lambda z^\kappa)}{\lambda^\nu} \quad (\nu > 0),
\]

we obtain

\[
\int_0^1 p(t)g(t)dt = E_{1,\nu+1}(z).
\]

By applying (3.11), we obtain the desired bound (3.9).

(b) We set

\[
p(t) = t^{\nu-1} E_{1,\nu}(zt), \quad f(t) = \frac{t^{a-v}}{\Gamma(s)} \log^{s-1}(1/t) \quad \text{and} \quad g(t) = \frac{t^{a-v}}{\Gamma(s+2)} \log^{s+1}(1/t).
\]

Moreover, by (3.8) we obtain

\[
\int_0^1 p(t)f(t)dt = E_{1,\nu}^{(a)}(s; z), \quad \int_0^1 p(t)g(t)dt = E_{1,\nu}^{(a)}(s+2; z),
\]

and

\[
\int_0^1 p(t)g(t)f(t)dt = \frac{\Gamma(2s+1)}{\Gamma(s)\Gamma(s+2)} E_{1,\nu}^{(2a-v)}(s+2; z).
\]

Observe that the function \( f \) and \( g \) are decreasing if \( 0 < s < 1 \) and \( \nu-1 < a < \nu \). Then, using (3.11) and the Legendre duplication formula, we obtain (3.10). \( \square \)
On setting $\nu = 1$ in the above theorem, in view of the fact that
\[ E_{1,2}(z) = \frac{e^z - 1}{z}, \]
we obtain the following result.

**Corollary 1.**
(a) Let $a, s, z$ be positive real numbers such that $s \leq 1$ and $a \geq 1$. Then
\[ \frac{e^z - 1}{za^s} \leq E_{1,1}^{(a)}(z). \]  
Moreover, the inequality (3.12) is reversed if $a \leq 1$ and $s \geq 1$.
(b) Let $a, s, z > 0$ be such that $a > 1$ and $0 < s < 1$. Then the following inequality holds:
\[ E_{1,1}^{(a)}(s, z) \leq \frac{2^{2s} \Gamma(s + 1/2)(e^z - 1)}{\sqrt{\pi \Gamma(s + 2)}} E_{1,1}^{(2a - 1)}(2s + 1, z). \]  

**Theorem 4.** Suppose that $p, q, r, s, \kappa$ and $\nu$ are positive real numbers such that $\frac{1}{r} + \frac{1}{s} = 1$ with $r > 1$. Then the following inequality holds true:
\[ E_{\kappa, \nu}^{(a)} \left( \frac{p + q}{s} + 1 ; z \right) \leq \frac{\Gamma^{1/r}(p + 1)\Gamma^{1/s}(q + 1)}{\Gamma \left( \frac{p}{r} + \frac{q}{s} + 1 \right)} \left[ E_{\kappa, \nu}^{(a)}(p + 1 ; z) \right]^{1/r} \left[ E_{\kappa, \nu}^{(a)}(q + 1 ; z) \right]^{1/s}. \]  

**Proof.** Using Hölder’s inequality and (1.5), we have
\[
E_{\kappa, \nu}^{(a)} \left( \frac{p + q}{s} + 1 ; z \right) = \frac{1}{\Gamma \left( \frac{p}{r} + \frac{q}{s} + 1 \right)} \int_0^\infty \left( t^p e^{-at} E_{\kappa, \nu}(ze^{-t}) \right) \left( t^q e^{-at} E_{\kappa, \nu}(ze^{-t}) \right)^{1/s} dt \\
\leq \frac{1}{\Gamma \left( \frac{p}{r} + \frac{q}{s} + 1 \right)} \left( \int_0^\infty t^p e^{-at} E_{\kappa, \nu}(ze^{-t}) dt \right)^{1/r} \left( \int_0^\infty t^q e^{-at} E_{\kappa, \nu}(ze^{-t}) dt \right)^{1/s} \\
= \frac{\Gamma^{1/r}(p + 1)\Gamma^{1/s}(q + 1)}{\Gamma \left( \frac{p}{r} + \frac{q}{s} + 1 \right)} \left[ E_{\kappa, \nu}^{(a)}(p + 1 ; z) \right]^{1/r} \left[ E_{\kappa, \nu}^{(a)}(q + 1 ; z) \right]^{1/s},
\]
which completes the proof of the theorem.  

**Remark 1.** Setting $r = s = 2$ in (3.14) of Theorem 4, we obtain the following inequality:
\[
\left[ \Gamma \left( \frac{p + q + 2}{2} \right) E_{\kappa, \nu}^{(a)} \left( \frac{p + q + 2}{2} ; z \right) \right]^2 \leq \Gamma(p + 1)\Gamma(q + 1)E_{\kappa, \nu}^{(a)}(p + 1 ; z)E_{\kappa, \nu}^{(a)}(q + 1 ; z).
\]
Theorem 5. Let \(a, b, p, q, \kappa\) and \(\nu\) be positive real numbers satisfying the conditions \(p > 1\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Then the following inequality holds true:

\[
E_{\kappa \nu}^{\left(\frac{a}{b} + \frac{b}{a}\right)}(s; z) \leq \left[ E_{\kappa \nu}^{(a)}(s; z) \right]^{1/p} \left[ E_{\kappa \nu}^{(b)}(s; z) \right]^{1/q}.
\]

(3.15)

Proof. Using Hölder’s inequality and (1.5), obtain

\[
E_{\kappa \nu}^{\left(\frac{a}{b} + \frac{b}{a}\right)}(s; z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{(s-1)} e^{-t} E_{\kappa \nu}(ze^{-t}) dt
\]

\[
= \int_0 ^\infty \left( \frac{t^{(s-1)} e^{-t} E_{\kappa \nu}(ze^{-t})}{\Gamma(s)} \right)^{1/p} \left( \frac{t^{(s-1)} e^{-bt} E_{\kappa \nu}(ze^{-t})}{\Gamma(s)} \right)^{1/q} dt
\]

\[
\leq \left( \frac{1}{\Gamma(s)} \int_0 ^\infty t^{(s-1)} e^{-at} E_{\kappa \nu}(ze^{-t}) dt \right)^{1/p} \left( \frac{1}{\Gamma(s)} \int_0 ^\infty t^{(s-1)} e^{-bt} E_{\kappa \nu}(ze^{-t}) dt \right)^{1/q}
\]

\[
= \left[ E_{\kappa \nu}^{(a)}(s; z) \right]^{1/p} \left[ E_{\kappa \nu}^{(b)}(s; z) \right]^{1/q},
\]

which proves the theorem. \(\square\)

Remark 2. Setting \(p = q = 2\) in (3.15) of Theorem 5, we obtain the following inequality:

\[
\left[ E_{\kappa \nu}^{\left(\frac{a}{b} + \frac{b}{a}\right)}(s; z) \right]^2 \leq E_{\kappa \nu}^{(a)}(s; z) E_{\kappa \nu}^{(b)}(s; z).
\]

4. Turán type inequalities for Barnes Mittag-Leffler function

In this section, we discuss the monotonicity property of the following Turán ratio, which will be useful to establish the sharp Turán type inequality for \(E_{\kappa \nu}^{(a)}(s; z)\):

\[
\mathcal{I}_{\kappa \nu}(s; z) = \frac{E_{\kappa \nu}^{(a)}(s; z) \Gamma(\nu)}{E_{\kappa \nu}^{(a)}(s; z) \Gamma(\nu + \mu + 2)}.
\]

(4.1)

Theorem 6. Let \(\nu, \mu, \kappa\) be three distinct positive real numbers and \(s \geq 0\). If \(\max(\kappa, \nu) < \mu\), then the function \(z \mapsto \mathcal{I}_{\kappa \nu}(s; z)\) defined by (4.1) is increasing on \((0, \infty)\). Furthermore, the following sharp Turán type inequality

\[
\left[ E_{\kappa \nu + 1}^{(a)}(s; z) \right]^2 - E_{\kappa \nu}^{(a)}(s; z) E_{\kappa \nu + 2}^{(a)}(s; z) \leq \left( \frac{1}{s + 1} \right) \left[ E_{\kappa \nu + 1}^{(a)}(s; z) \right]^2,
\]

holds for all \(\nu > 0\).

Proof. By means of the Cauchy product, we have

\[
E_{\kappa \nu}^{(a)}(s; z) E_{\kappa \mu}^{(a)}(s; z) = \sum_{k=0}^m \sum_{j=0}^k A_{j,k} z^k,
\]
Combining (4.3), (4.5) and (4.6), we deduce that the sequence

\[ (S_k) \]

and consequently the sequence

\[ (S_k) \]

is increasing too, by means of Lemma 1. Further, using Lemma 2, we deduce that the

\[ (S_k) \]

completes the proof of the theorem. \( \square \)

Now, we define the sequence \((C_{j,k})\) by \(C_{j,k} = A_{j,k}/B_{j,k}\). Therefore,

\[
C_{j+1,k} = \frac{C_{j,k} + (v + \mu)/2 + k}{\Gamma(k - j + v + \mu/2)} \cdot \frac{\Gamma(k - j + \mu) - \Gamma(k - j + v + \mu/2)}{\Gamma(k - j + \mu - k)} \cdot \frac{\Gamma(k - j + v + \mu/2)}{\Gamma(k - j + v + k)}.
\]

In virtue of the following inequality for the gamma function

\[ \Gamma(x+a)\Gamma(x+b) \leq \Gamma(x)\Gamma(x+a+b), \quad a > 0, b > 0, x > 0, \]

and choosing \(x = k(k-j) + (v+\mu)/2 - k, a = (\mu - v)/2\) and \(b = k\), we obtain

\[
\Gamma(k-j + \mu)\Gamma(k-j + (v+\mu)/2) \geq 1.
\]

On the other hand, upon setting \(x = k+j, a = (\mu - v)/2\) and \(b = k\) in (4.4), we get

\[
\Gamma(k+j + (v+\mu)/2) \Gamma(k+j+v) \geq 1.
\]

Combining (4.3), (4.5) and (4.6), we deduce that the sequence \((C_{j,k})\) is increasing and consequently the sequence \((S_k)\) defined by \(S_k = \sum_{j=0}^{k} A_{j,k}/\sum_{j=0}^{k} B_{j,k}\) is increasing too, by means of Lemma 1. Further, using Lemma 2, we deduce that the function \(z \mapsto E_{k,v,\mu}^{(a)}(s;z)\) is increasing on \((0,\infty)\). Moreover,

\[
E_{k,v,\mu}^{(a)}(s;z) \geq \frac{E_{k+v+1}(s;\mu)}{E_{k+v}(s;\mu)}.
\]

Finally, putting \(\mu = v + 2\) in the above inequality, we get (4.2) for all \(v > 0\), which completes the proof of the theorem. \( \square \)

**Corollary 2.** The following assertions are true.

(i) Let \(\kappa > 0, s \geq 0, a > 0\) and \(z > 0\), then the function \(v \mapsto E_{k,v,\mu}^{(a)}(s;z)\), is convex on \((0,\infty)\).
(ii) Under the hypothesis of Theorem 6, the function $\nu \mapsto \frac{E_{\kappa,\nu}(s,z_1)}{E_{\kappa,\nu}(s,z_2)}$ is log-convex on $(0, \infty)$, for all $0 < z_2 < z_1$.

Proof. (i) From Theorem 6, we deduce that the function $z \mapsto \log E_{\kappa,\nu}(s;z)$ is increasing on $(0, \infty)$. We have,

$$\frac{\partial}{\partial z} \log E_{\kappa,\nu}(s;z) = \frac{\partial}{\partial z} E_{\kappa,\nu}(s;z) + \frac{\partial}{\partial z} E_{\kappa,\nu}(s;z) + 2 \frac{\partial}{\partial z} E_{\kappa,(\nu+\mu)/2}(s;z) \geq 0,$$

and consequently, the function $\nu \mapsto \frac{\partial}{\partial z} E_{\kappa,\nu}(s;z)$ is convex on $(0, \infty)$. In view of (4.9), we get

$$\frac{\partial}{\partial z} E_{\kappa,\nu+1}(s;z) \geq \frac{E_{\kappa,\nu}(s,z)}{\kappa E_{\kappa,\nu+1}(s,z)} - \frac{\nu}{\kappa z},$$

which implies that the function $\nu \mapsto \frac{E_{\kappa,\nu}(s;z)}{E_{\kappa,\nu+1}(s;z)}$ is convex on $(0, \infty)$.

(ii) Again, by means of Theorem 6, we have $E_{\kappa,\nu}(s;z_1) > E_{\kappa,\nu}(s;z_2)$ for $0 < z_2 < z_1$, which yields

$$\frac{E_{\kappa,\nu}(s;z_1)}{E_{\kappa,\nu}(s;z_2)} E_{\kappa,\nu+1}(s;z_2) E_{\kappa,\nu+1}(s;z_1) \geq \frac{E_{\kappa,(\nu+\mu)/2}(s;z_1)}{E_{\kappa,(\nu+\mu)/2}(s;z_2)} \left(1 + \frac{\nu}{\kappa z} \right),$$

which implies that the function $\nu \mapsto \frac{E_{\kappa,\nu}(s;z)}{E_{\kappa,\nu+1}(s;z)}$ is log-convex on $(0, \infty)$. \hfill \square

Theorem 7. Suppose that $\mu, \nu_1, \nu_2$ and $a$ are positive real numbers and $s \geq 0$. If $\nu_1 < \nu_2$, then the function $z \mapsto \frac{E_{\kappa,\nu_1}(s;z)}{E_{\kappa,\nu_2}(s;z)}$ is decreasing on $(0, \infty)$. Moreover, the following Turán type inequality

$$[E_{\kappa,\nu_1}(s;z)]^2 - E_{\kappa,\nu}(s;z)E_{\kappa,\nu+2}(s;z) \leq (\kappa + 1)[E_{\kappa,\nu_1}(s;z)]^2,$$

holds true for all $z > 0$.

Proof. From (1.3), we have

$$\frac{E_{\kappa,\nu_1}(s;z)}{E_{\kappa,\nu_2}(s;z)} = \sum_{n=0}^{\infty} \frac{a_n z^n}{\Gamma(\kappa n + \nu_1)(n+a)^s} = \sum_{n=0}^{\infty} \frac{b_n z^n}{\Gamma(\kappa n + \nu_2)(n+a)^s},$$

where

$$a_n = \frac{1}{\Gamma(\kappa n + \nu_1)(n+a)^s}, \quad b_n = \frac{1}{\Gamma(\kappa n + \nu_2)(n+a)^s}.$$
Now, we define the sequence \((c_n)_{n \geq 0}\) by \(c_n = a_n / b_n\). Therefore,
\[
c_{n+1} - c_n = \frac{\Gamma(\kappa n + v_2 + \kappa) \Gamma(\kappa n + v_1) - \Gamma(\kappa n + v_1 + \kappa) \Gamma(\kappa n + v_2)}{\Gamma(\kappa n + v_1 + \kappa) \Gamma(\kappa n + v_1)}.
\]

Setting \(z = \kappa n + v_1\), \(a = \kappa\) and \(b = v_2 - v_1\) in (4.4), we get
\[
\Gamma(\kappa n + v_2 + \kappa) \Gamma(\kappa n + v_1) \geq \Gamma(\kappa n + v_1 + \kappa) \Gamma(\kappa n + v_2).
\]

This yields that the sequence \((c_n)_{n \geq 0}\) is increasing, and consequently the function \(z \mapsto \frac{E_{\kappa n}^{(a)}(s; z)}{E_{\kappa n+1}^{(a)}(s; z)}\) is decreasing on \((0, \infty)\) if \(v_2 > v_1\). Now, setting \(v_2 = v + 2\) and \(v_1 = v + 1\), we deduce that the function \(z \mapsto \frac{z E_{\kappa n+1}^{(a)}(s; z)}{E_{\kappa n+2}^{(a)}(s; z)}\) is increasing on \((0, \infty)\). This implies that
\[
\frac{\partial}{\partial z} \frac{z E_{\kappa n+1}^{(a)}(s; z)}{E_{\kappa n+2}^{(a)}(s; z)} \geq 0.
\]

Using the recurrence formula
\[
\frac{\partial}{\partial z} E_{\kappa n+1}^{(a)}(s; z) = \frac{E_{\kappa n}^{(a)}(s; z) - \kappa E_{\kappa n+1}^{(a)}(s; z)}{\kappa s}, \quad (4.9)
\]

we obtain
\[
\left[E_{\kappa n+2}^{(a)}(s; z)\right]^2 \frac{\partial}{\partial z} \frac{E_{\kappa n+1}^{(a)}(s; z)}{E_{\kappa n+2}^{(a)}(s; z)} = \left(1 + \frac{1}{\kappa}\right) E_{\kappa n+1}^{(a)}(s; z) E_{\kappa n+2}^{(a)}(s; z)
\]
\[
+ \frac{E_{\kappa n}^{(a)}(s; z) E_{\kappa n+1}^{(a)}(s; z) - \left[E_{\kappa n+1}^{(a)}(s; z)\right]^2}{\kappa} \geq 0,
\]

which yields
\[
\left[E_{\kappa n+1}^{(a)}(s; z)\right]^2 - E_{\kappa n}^{(a)}(s; z) E_{\kappa n+2}^{(a)}(s; z) \leq (\kappa + 1) E_{\kappa n+1}^{(a)}(s; z) E_{\kappa n+2}^{(a)}(s; z).
\]

Moreover, we observe that
\[
E_{\kappa n+2}^{(a)}(s; z) \leq E_{\kappa n+1}^{(a)}(s; z) \text{ for all } z > 0.
\]

Combining the above inequality with (4.11), we obtain (4.7).

\[
5. \text{ Further inequalities for Barnes Mittag-Leffler function}
\]

This section is devoted to study further new inequalities for Barnes Mittag-Leffler function.
Theorem 8. Let $\nu > 0$, $\mu > 0$ and $s \geq 0$. If $\mu < \nu$, then the following inequality
\[
\left[ E_{\nu, \nu+1}^{(a)}(s; z) \right]^\frac{1}{\nu} \leq a \frac{\log \Gamma(\nu)}{\Gamma(\nu + 1)} \left[ E_{\nu, \nu+1}^{(a)}(s; z) \right]^\frac{1}{\nu},
\] (5.1)
holds true for all $z > 0$.

Proof. Suppose that $0 < \mu < \nu$ and $s \geq 0$ and the function $F_{\nu, \nu+1}^{(a)} : [0, \infty) \to \mathbb{R}$ is defined by
\[
F_{\nu, \nu+1}^{(a)}(s; z) = \frac{\mu}{\nu} \log E_{\nu, \nu+1}^{(a)}(s; z) - \log E_{\nu, \nu+1}^{(a)}(s; z),
\]
where
\[
\log E_{\nu, \nu+1}^{(a)}(s; z) = a \Gamma(\nu) E_{\nu, \nu+1}^{(a)}(s; z).
\]
In view of the recurrence relation (4.9), we have
\[
\frac{\partial}{\partial z} F_{\nu, \nu+1}^{(a)}(s; z) = \frac{1}{\kappa z} \left[ \frac{\mu}{\nu} \frac{E_{\nu, \nu+1}^{(a)}(s; z)}{E_{\nu, \nu+1}^{(a)}(s; z)} - \frac{E_{\nu, \nu+1}^{(a)}(s; z)}{E_{\nu, \nu+1}^{(a)}(s; z)} \right] = \frac{\mu}{\kappa z} \left[ \frac{E_{\nu, \nu+1}^{(a)}(s; z)}{E_{\nu, \nu+1}^{(a)}(s; z)} - \frac{E_{\nu, \nu+1}^{(a)}(s; z)}{E_{\nu, \nu+1}^{(a)}(s; z)} \right].
\] (5.2)
Moreover, the function $v \mapsto E_{\nu, \nu+1}^{(a)}(s; z)$ is log-convex on $(0, \infty)$. Indeed, for convenience, let us write
\[
E_{\nu, \nu+1}^{(a)}(s; z) = \sum_{n=0}^{\infty} a_n(v) z^n, \text{ where } a_n(v) = \frac{a \Gamma(\nu)}{\Gamma(\nu + \nu)(n + a)}, n \geq 0.
\]
Using the fact that the sum of log-convex functions is log-convex too, we just need to prove the log-convexity of $a_n(v)$. Hence, for all $n \geq 0$, we have
\[
\frac{\partial^2}{\partial v^2} \log(a_n(v)) = \psi'(v) - \psi'(\kappa n + v),
\]
where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. Since $\psi$ is concave, we deduce that
\[
\frac{\partial^2}{\partial v^2} \log(a_n(v)) \geq 0,
\]
and consequently $v \mapsto a_n(v)$ is log-convex on $(0, \infty)$, which implies that the function $v \mapsto E_{\nu, \nu+1}^{(a)}(s; z)$ is log-convex on $(0, \infty)$. Therefore, the function $v \mapsto \log E_{\nu, \nu+1}^{(a)}(s; z) - \log E_{\nu, \nu+1}^{(a)}(s; z)$ is increasing on $(0, \infty)$ and consequently the function $v \mapsto \frac{E_{\nu, \nu+1}^{(a)}(s; z)}{E_{\nu, \nu+1}^{(a)}(s; z)}$ is increasing on $(0, \infty)$. Combining this fact with (5.2), we obtain
\[
\frac{\partial}{\partial z} F_{\nu, \nu+1}^{(a)}(s; z) \leq 0.
\]
Therefore, $F_{\nu, \nu+1}^{(a)}(s; z) \leq F_{\nu, \nu+1}^{(a)}(s; 0) = 0$, which proves the desired result. \qed
Corollary 3. Under the assumption of Theorem 8, the following inequality
\[ \frac{a^{\nu - \mu}}{\Gamma(\nu + 1)} \left[ E_{\kappa \mu + 1}^{(a)}(s; z) \right]^{\frac{\nu - \mu}{\nu}} + \frac{E_{\kappa \nu + 1}^{(a)}(s; z)}{E_{\kappa \nu + 1}^{(a)}(s; z)} \geq 2, \] (5.3)
holds true for all \( z > 0 \).

Proof. Using (5.1), we obtain
\[ \frac{a^{\nu - \mu}}{\Gamma(\nu + 1)} \left[ E_{\kappa \mu + 1}^{(a)}(s; z) \right]^{\frac{\nu - \mu}{\nu}} \geq 1. \]
With the help of the above inequality and the arithmetic-geometric mean inequality, we get
\[ \frac{1}{2} \left( \frac{a^{\nu - \mu}}{\Gamma(\nu + 1)} \left[ E_{\kappa \mu + 1}^{(a)}(s; z) \right]^{\frac{\nu - \mu}{\nu}} + \frac{E_{\kappa \nu + 1}^{(a)}(s; z)}{E_{\kappa \nu + 1}^{(a)}(s; z)} \right) \geq 1, \]
which proves (5.3). \( \Box \)

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