Miskolc Mathematical Notes
Vol. 24 (2023), No. 2, pp. 893–907

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3997

MONOTONICITY PROPERTIES AND FUNCTIONAL INEQUALITIES FOR THE BARNES MITTAG-LEFFLER FUNCTION

KHALED MEHREZ, SOURAV DAS, AND ANISH KUMAR

Received 17 November, 2021

Abstract. In this paper, our main focus is to establish complete monotonicity properties, convexity properties and some interesting inequalities for the Barnes Mittag-Leffler function. Furthermore, monotonicity properties of ratios of Barnes Mittag-Leffler functions are derived. Moreover, Turán type inequalities and several new inequalities are obtained for this function as application. Results obtained in this work are new and their importance is illustrated by several attractive consequences and corollaries.

2010 *Mathematics Subject Classification:* 33C20; 33E12; 26D07

Keywords: monotonicity properties, Turán-type inequalities, Barnes Mittag-Leffler function

1. INTRODUCTION

Mittag-Leffler function is one of the important special functions, which plays vital role in mathematical physics, fractional calculus, approximation theory and various fields of science and engineering. It appears in the differential equations of fractional order and study of complex system. Mittag-Leffler function was introduced by Gosta Mittag-Leffler in 1903 [15], defined as

$$E_\kappa(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n + 1)}, \quad z \in \mathbb{C},\ \kappa \geq 0, \tag{1.1}$$

where Γ is the well-known Euler's gamma function. From (1.1), it can be observed that $E_\kappa(z)$ interpolates between $\exp(z)$ and $1/(1-z)$ for $0 < \kappa < 1$. In 1905, Wiman [24] studied a generalization of $E_\kappa(z)$, defined as

$$E_{\kappa,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n + \nu)}, \quad \kappa, \nu \in \mathbb{C},\ \Re(\kappa) > 0, \Re(\nu) > 0. \tag{1.2}$$

The generalized Mittag-Leffler function $E_{\kappa,\nu}(z)$ is also known as Wiman's function. There are several generalizations of Mittag-Leffler function available in the literature. One of the most important generalizations of Mittag-Leffler function is the Barnes

Mittag-Leffler function $E^{(a)}_{\kappa,\nu}(s;z)$, which is defined [6] as

$$E^{(a)}_{\kappa,\nu}(s;z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n+\nu)(n+a)^s}, \quad a,\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-, \Re(\kappa)>0, s,z\in\mathbb{C}. \tag{1.3}$$

It can be noted that $E^{(a)}_{\kappa,\nu}(0;z) = E_{\kappa,\nu}(z)$, $E_{\kappa,1}(z) = E_{\kappa}(z)$ and $E_1(z) = \exp(z)$. For further information on Mittag-Leffler function, we refer to [7] and references cited therein.

In terms of the extended Hurwitz-Lerch zeta function $\Phi^{\rho,\sigma,\kappa}_{\lambda,\mu,\nu}(z,s,a)$, defined by [19, p. 503, Eq. (6.2)]

$$\begin{aligned} \Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a) &= \Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,a) \\ &= \frac{\prod_{j=1}^q \Gamma(\mu_j)}{\prod_{j=1}^p \Gamma(\lambda_j)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(\lambda_j+k\rho_j)}{\prod_{j=1}^q \Gamma(\mu_j+k\sigma_j)} \frac{z^k}{k!(k+a)^s}, \end{aligned} \tag{1.4}$$

we see that

$$E^{(a)}_{\kappa,\nu}(s;z) = \frac{1}{\Gamma(\nu)} \Phi^{(1,1;1)}_{(\nu,\kappa;1)}(z,s,a).$$

From [19, Theorem 8], we deduce that the Barnes Mittag-Leffler function $E^{(a)}_{\kappa,\nu}(s;z)$ possesses the following integral representation

$$E^{(a)}_{\kappa,\nu}(s;z) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} E_{\kappa,\nu}(ze^{-t})dt, \ \ (\min(\Re(a),\Re(s),\kappa)>0). \tag{1.5}$$

Monotonicity properties and functional inequalities play vital role in the theory of various inequalities and equilibrium problems [2, 3, 5, 23, 25]. Nowadays most of the researchers use Turán-type inequalities to find monotonicity properties of special functions. Turán's inequality was first introduced by the Hungarian mathematician Paul Turán [21] in 1950 as follows:

$$\frac{P_n(x)P_{n+2}(x)}{P^2_{n+1}(x)} \le 1, \quad x\in[-1,1], \quad n\in\mathbb{N}\cup\{0\}, \tag{1.6}$$

where $P_n(x)$ are Legendre polynomials. Several Turán-type inequalities for various special functions can be found in [3, 4, 11, 12] and references cited therein. It will be interesting to see from (1.6) that stronger results on monotonicity of ratios of functions of the form (1.6) with upper or lower constants as unity can be obtained using these types of inequalities. These inequalities have applications in the theory of transmutation operators for estimating transmulation kernels and norms [18] and also in the problems of function expansions by system of integer shifts of Gaussians [10].

This paper is organized as follows. In the next Section, we present some useful lemmas which are required to complete the proofs of the main results. Using the integral representation (1.5) of the Barnes Mittag-Leffler function, complete monotonicity property and several interesting inequalities and their consequences are established in Section 3. Turán type inequalities for $E_{\kappa,\nu}^{(a)}(s;z)$ are obtained in Section 4. Moreover, monotonicity, convexity and log-convexity properties for the ratios of Barnes Mittag-Leffler function are also studied in this section. Section 5 is devoted to derive further interesting inequalities for $E_{\kappa,\nu}^{(a)}(s;z)$.

2. Useful lemmas

Before proving the main results in this paper, we need the following preliminary lemmas.

Lemma 1 ([12]). *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers. If $b_n > 0$ for $n \geq 0$ and if the sequence (a_n/b_n) is increasing (decreasing), then $\left(\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n}\right)_{n\geq 0}$ is increasing (decreasing).*

Lemma 2 ([17]). *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers and let the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ be convergent for all $|x| < r$. If $b_n > 0$ for $n \geq 0$ and if the sequence (a_n/b_n) is increasing (decreasing), then the function $x \mapsto f(x)/g(x)$ is increasing (decreasing) on $(0,r)$.*

Lemma 3 ([1]). *Suppose that $f,g : [a,b] \to \mathbb{R}$ are continuous functions, which are differentiable on (a,b). Moreover, assume that $g'(x) \neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the following ratios*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}$$

are also increasing (or decreasing) on (a,b).

3. Complete monotonicity property and related inequalities

In the next theorem, we will study complete monotonicity property of the function $E_{\kappa,\nu}^{(a)}(s;-z)$. A C^∞ function $f(x)$ is said to be completely monotonic on an interval I if

$$(-1)^{n-1} f^{n-1}(x) \geq 0,$$

for any $n \in \mathbb{N}$ and $x \in I$.

Theorem 1. *Assume that* $\min(\Re(a),\Re(s)) > 0, \kappa > 0$. *Under the following conditions*

$$(\kappa,\nu \in (0,1) \ \ \nu \geq \kappa) \ \text{ or } \ (\kappa \in (0,1] \text{ and } \nu \geq \kappa),$$

the function $z \mapsto E_{\kappa,\nu}^{(a)}(s;-z)$ is completely monotonic on $(0,\infty)$. Moreover, the following inequalities

$$E_{\kappa,\nu}^{(a)}(s;-z_1)E_{\kappa,\nu}^{(a)}(s;-z_2) \leq E_{\kappa,\nu}^{(a)}(s;-z_1-z_2), \ (\min(z_1,z_2) > 0), \tag{3.1}$$

$$E_{\kappa,\nu}^{(a)}(s;-(z_1+z_2)/2) \leq \sqrt{E_{\kappa,\nu}^{(a)}(s;-z_1)E_{\kappa,\nu}^{(a)}(s;-z_2)},\ (\min(z_1,z_2)>0) \tag{3.2}$$

$$\frac{1}{a^s\Gamma(\nu)}\exp\left(-\frac{a^s\Gamma\nu)}{(a+1)^s\Gamma(\nu+\kappa)}z\right) \leq E_{\kappa,\nu}^{(a)}(s;-z),\ z>0, \tag{3.3}$$

hold true under the given hypothesis.

Proof. By using (1.5) with [20, Corollary 3] and [8, Theorem 2.1], we establish that $E_{\kappa,\nu}^{(a)}(s;-z)$ is completely monotonic on $(0,\infty)$, under the given hypothesis. However, we see that the function $z \mapsto a^s\Gamma(\nu)E_{\kappa,\nu}^{(a)}(s;-z)$ is completely monotonic on $(0,\infty)$ and maps $(0,\infty)$ to $(0,1)$. According to [9, Theorem 3], we get the result (3.1). Let us now focus on the inequality (3.2). By using the fact that every completely monotonic function is log-convex [22, p. 167], we deduce that the function $E_{\kappa,\nu}^{(a)}(s;-z)$ is log-convex and consequently the inequality (3.2) holds true. It remains to prove the inequality (3.3). Let

$$F(z) = \log\left(a^s\Gamma(\nu)E_{\kappa,\nu}^{(a)}(s;-z)\right) =: \log\tilde{E}_{\kappa,\nu}^{(a)}(s;z) \ \text{ and } G(z)=z.$$

Observe that the function $z \mapsto (\tilde{E}_{\kappa,\nu}^{(a)}(s;z))'/\tilde{E}_{\kappa,\nu}^{(a)}(s;z)$ is increasing on $(0,\infty)$ and consequently the function

$$z \mapsto \frac{F(z)-F(0)}{G(z)-G(0)} =: \Delta(z)$$

is also increasing on $(0,\infty)$ by Lemma 3. Furthermore,

$$\lim_{z\to 0}\Delta(z) = \lim_{z\to 0}\frac{(\tilde{E}_{\kappa,\nu}^{(a)}(s;z))'}{\tilde{E}_{\kappa,\nu}^{(a)}(s;z)} = -\frac{a^s\Gamma(\nu)}{(a+1)^s\Gamma(\nu+\kappa)},$$

which completes the proof of Theorem 1. □

Theorem 2. *Under the assumptions of Theorem 1, the following inequality holds true:*

$$E_{\kappa,\nu}(e^{-\frac{s}{a}}z) \leq E_{\kappa,\nu}^{(a)}(s;z) \leq \frac{(a+1)^s-\Gamma(s)}{(a+1)^s\Gamma(\kappa+\nu)} + \frac{\Gamma(s)}{(a+1)^s}E_{\kappa,\nu}(z). \tag{3.4}$$

Proof. We recall the Jensen's integral inequality [14, Chap. I, Eq. (7.15)],

$$\varphi\left(\int_a^b f(s)d\sigma(s)\Big/\int_a^b d\sigma(s)\right) \leq \int_a^b \varphi(f(s))d\sigma(s)\Big/\int_a^b d\sigma(s), \tag{3.5}$$

where, φ is convex and f is integrable with respect to a probability measure σ. Let

$$\varphi_z(t) = E_{\kappa,\nu}(ze^{-t}),\ f(t)=t \ \text{ and } \ d\sigma(t) = \frac{1}{\Gamma(s)}t^{s-1}e^{-at}dt.$$

Using an elementary property of Gamma-function:

$$\frac{\Gamma(s)}{a^s} = \int_0^\infty t^{s-1}e^{-at}dt,\ (\Re(a)>0,\ \Re(a)>0), \tag{3.6}$$

and the representation (1.5), we obtain

$$\int_a^b d\sigma(t) = \frac{1}{a^s}, \quad \int_a^b f(t)d\sigma(t) = \frac{s}{a^{s+1}} \text{ and } \int_a^b \varphi(f(t))d\sigma(t) = E^{(a)}_{\kappa,\nu}(s;z).$$

Again, from (3.5), we have

$$E_{\kappa,\nu}(e^{-\frac{s}{a}}z) \leq E^{(a)}_{\kappa,\nu}(s;z),$$

which proves the lower bound of (3.4). In order to demonstrate the upper bound, we will apply the converse Jensen inequality, due to Lah and Ribarić, which reads as follows. Set

$$A(f) = \int_m^M f(s)d\sigma(s) \Big/ \int_m^M d\sigma(s),$$

where σ is a non-negative measure and f is a continuous function. If $-\infty < m < M < \infty$ and φ is convex on $[m,M]$, then according to [16, Theorem 3.37], we have

$$(M-m)A(\varphi(f)) \leq (M-A(f))\varphi(m) + (A(f)-m)\varphi(M). \tag{3.7}$$

Rewriting the representation (1.5) as follows:

$$E^{(a)}_{\kappa,\nu}(s;z) = \frac{1}{\Gamma(s)} \int_0^1 \log^{s-1}(1/u)u^{a-1}E_{\kappa,\nu}(zu)du, \tag{3.8}$$

and taking

$$[m,M] = [0,1], \varphi(u) = E_{\kappa,\nu}(zu), f(u) = u \text{ and } d\sigma(u) = \frac{\log^{s-1}(1/u)u^{a-1}du}{\Gamma(s)},$$

in (3.7), in view of (3.6) and (3.8), we obtain

$$\frac{E^{(a)}_{\kappa,\nu}(s;z)}{a^s} \leq \frac{(a(a+1))^s - \Gamma(s)}{(a(a+1))^s\Gamma(\kappa+\nu)} + \frac{\Gamma(s)}{(a(a+1))^s}E_{\kappa,\nu}(z).$$

□

Theorem 3. *The following inequalities hold true:*

(a) *Let a,ν,s,z be positive real numbers such that $s \leq 1$ and $a \geq \nu$. Then the following inequality*

$$\frac{E_{1,\nu+1}(z)}{(a-\nu+1)^s} \leq E^{(a)}_{1,\nu}(s,z) \tag{3.9}$$

holds true. Furthermore, the above inequality is reversed if $s \geq 1$ and $\nu - 1 < a \leq \nu$.

(b) *Let $a,\nu,s,z > 0$ be such that $a > \nu$ and $0 < s < 1$. Then the inequality*

$$E^{(a)}_{1,\nu}(s,z)E^{(a)}_{1,\nu}(s+2,z) \leq \frac{2^{2s}s\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s+2)} E_{1,\nu+1}(z)E^{(2a-\nu)}_{1,\nu}(2s+1,z) \tag{3.10}$$

is valid.

Proof. (**a**) First of all, we recall the Chebyshev integral inequality [13, p. 40]. If $f,g:[a,b]\longrightarrow\mathbb{R}$ are integrable functions, both increasing or both decreasing, and $p:[a,b]\longrightarrow\mathbb{R}$ is a positive integrable function, then

$$\int_a^b p(t)f(t)dt\int_a^b p(t)g(t)dt\leq\int_a^b p(t)dt\int_a^b p(t)f(t)g(t)dt. \tag{3.11}$$

It can be noted that if one of the functions f or g is decreasing and the other is increasing, then (3.11) is reversed. We will use (3.8) and (3.11) to prove (3.9). We set

$$p(t)=1,\ f(t)=\frac{t^{a-\nu}}{\Gamma(s)}\log^{s-1}(1/t)\ \text{ and }\ g(t)=t^{\nu-1}E_{1,\nu}(zt).$$

From the differentiation formula [7, Eq. (4.3.1), p. 51]

$$\left(\frac{d}{dz}\right)^m\left[z^{\nu-1}E_{\kappa,\nu}(z^\kappa)\right]=z^{\nu-m-1}E_{\kappa,\nu-m}(z^\kappa),\ (m\geq 1),$$

we see that the function g is increasing on $(0,1]$. Moreover, the function f is decreasing (increasing) if $s\geq 1$ and $\nu-1<a\geq\nu$ $(0<s\leq 1$ and $a\geq\nu)$ on $(0,1]$. Furthermore, we have

$$\begin{aligned}\int_0^1 p(t)f(t)dt&=\frac{t^{a-\nu}}{\Gamma(s)}\int_0^1\log^{s-1}(1/t)dt\\&=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(a-\nu+1)t}dt=\frac{1}{(a-\nu+1)^s},\quad (\text{by (3.6)}).\end{aligned}$$

Taking into account [7, Eq. (4.4.4), p. 61]

$$\int_0^z t^{\nu-1}E_{\kappa,\nu}(\lambda t^\kappa)dt=z^\kappa E_{\kappa,\nu+1}(\lambda z^\kappa)\quad(\nu>0),$$

we obtain

$$\int_0^1 p(t)g(t)dt=E_{1,\nu+1}(z).$$

By applying (3.11), we obtain the desired bound (3.9).

(**b**) We set

$$p(t)=t^{\nu-1}E_{1,\nu}(zt),\ \ f(t)=\frac{t^{a-\nu}}{\Gamma(s)}\log^{s-1}(1/t)\ \text{ and } g(t)=\frac{t^{a-\nu}}{\Gamma(s+2)}\log^{s+1}(1/t).$$

Moreover, by (3.8) we obtain

$$\int_0^1 p(t)f(t)dt=E_{1,\nu}^{(a)}(s;z),\quad \int_0^1 p(t)g(t)dt=E_{1,\nu}^{(a)}(s+2;z),$$

and

$$\int_0^1 p(t)g(t)f(t)dt=\frac{\Gamma(2s+1)}{\Gamma(s)\Gamma(s+2)}E_{1,\nu}^{(2a-\nu)}(s+2;z).$$

Observe that the function f and g are decreasing if $0<s<1$ and $\nu-1<a<\nu$. Then, using (3.11) and the Legendre duplication formula, we obtain (3.10). $\square$

On setting $\nu = 1$ in the above theorem, in view of the fact that

$$E_{1,2}(z) = \frac{e^z - 1}{z},$$

we obtain the following result.

Corollary 1.

(a) *Let a, s, z be positive real numbers such that $s \leq 1$ and $a \geq 1$. Then*

$$\frac{e^z - 1}{z a^s} \leq E_{1,1}^{(a)}(z). \tag{3.12}$$

Moreover, the inequality (3.12) is reversed if $a \leq 1$ and $s \geq 1$.

(b) *Let $a, s, z > 0$ be such that $a > 1$ and $0 < s < 1$. Then the following inequality holds:*

$$E_{1,1}^{(a)}(s,z) E_{1,1}^{(a)}(s+2,z) \leq \frac{2^{2s} s \Gamma(s+1/2)(e^z - 1)}{\sqrt{\pi}\Gamma(s+2) z} E_{1,1}^{(2a-1)}(2s+1,z). \tag{3.13}$$

Theorem 4. *Suppose that p, q, r, s, κ and ν are positive real numbers such that $\frac{1}{r} + \frac{1}{s} = 1$ with $r > 1$. Then the following inequality holds true:*

$$E_{\kappa,\nu}^{(a)}\left(\frac{p}{r} + \frac{q}{s} + 1; z\right) \leq \frac{\Gamma^{1/r}(p+1)\Gamma^{1/s}(q+1)}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)} \left[E_{\kappa,\nu}^{(a)}(p+1;z)\right]^{1/r} \left[E_{\kappa,\nu}^{(a)}(q+1;z)\right]^{1/s}. \tag{3.14}$$

Proof. Using Hölder's inequality and (1.5), we have

$$\begin{aligned}
E_{\kappa,\nu}^{(a)}\left(\frac{p}{r} + \frac{q}{s} + 1; z\right) &= \frac{1}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)} \int_0^\infty t^{\left(\frac{p}{r} + \frac{q}{s}\right)} e^{-at} E_{\kappa,\nu}(ze^{-t}) dt \\
&= \frac{1}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)} \int_0^\infty \{t^p e^{-at} E_{\kappa,\nu}(ze^{-t})\}^{1/r} \{t^q e^{-at} E_{\kappa,\nu}(ze^{-t})\}^{1/s} dt \\
&\leq \frac{1}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)} \left(\int_0^\infty t^p e^{-at} E_{\kappa,\nu}(ze^{-t}) dt\right)^{1/r} \left(\int_0^\infty t^q e^{-at} E_{\kappa,\nu}(ze^{-t}) dt\right)^{1/s} \\
&= \frac{\Gamma^{1/r}(p+1)\Gamma^{1/s}(q+1)}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)} \left[E_{\kappa,\nu}^{(a)}(p+1;z)\right]^{1/r} \left[E_{\kappa,\nu}^{(a)}(q+1;z)\right]^{1/s},
\end{aligned}$$

which completes the proof of the theorem. □

Remark 1. Setting $r = s = 2$ in (3.14) of Theorem 4, we obtain the following inequality:

$$\left[\Gamma\left(\frac{p+q+2}{2}\right) E_{\kappa,\nu}^{(a)}\left(\frac{p+q+2}{2}; z\right)\right]^2 \leq \Gamma(p+1)\Gamma(q+1) E_{\kappa,\nu}^{(a)}(p+1;z) E_{\kappa,\nu}^{(a)}(q+1;z).$$

Theorem 5. *Let a,b,p,q,κ and ν be positive real numbers satisfying the conditions $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds true:*

$$E_{\kappa,\nu}^{\left(\frac{a}{p}+\frac{b}{q}\right)}(s;z) \le \left[E_{\kappa,\nu}^{(a)}(s;z)\right]^{1/p}\left[E_{\kappa,\nu}^{(b)}(s;z)\right]^{1/q}. \tag{3.15}$$

Proof. Using Hölder's inequality and (1.5), obtain

$$\begin{aligned}
E_{\kappa,\nu}^{\left(\frac{a}{p}+\frac{b}{q}\right)}(s;z) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\left(\frac{a}{p}+\frac{b}{q}\right)t}E_{\kappa,\nu}(ze^{-t})dt\\
&= \int_0^\infty \left(\frac{t^{(s-1)}e^{-at}E_{\kappa,\nu}(ze^{-t})}{\Gamma(s)}\right)^{1/p}\left(\frac{t^{(s-1)}e^{-bt}E_{\kappa,\nu}(ze^{-t})}{\Gamma(s)}\right)^{1/q}dt\\
&\le \left(\frac{1}{\Gamma(s)}\int_0^\infty t^{(s-1)}e^{-at}E_{\kappa,\nu}(ze^{-t})dt\right)^{1/p}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{(s-1)}e^{-bt}E_{\kappa,\nu}(ze^{-t})dt\right)^{1/q}\\
&= \left[E_{\kappa,\nu}^{(a)}(s;z)\right]^{1/p}\left[E_{\kappa,\nu}^{(b)}(s;z)\right]^{1/q},
\end{aligned}$$

which proves the theorem. □

Remark 2. Setting $p=q=2$ in (3.15) of Theorem 5, we obtain the following inequality:

$$\left[E_{\kappa,\nu}^{\left(\frac{a+b}{2}\right)}(s;z)\right]^2 \le E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\nu}^{(b)}(s;z).$$

4. Turán type inequalities for Barnes Mittag-Leffler function

In this section, we discuss the monotonicity property of the following Turán ratio, which will be useful to establish the sharp Turán type inequality for $E_{\kappa,\nu}^{(a)}(s;z)$:

$$\mathcal{E}_{\kappa,\nu,\mu}^{(a)}(s;z) = \frac{E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\mu}^{(a)}(s;z)}{\left[E_{\kappa,(\nu+\mu)/2}^{(a)}(s;z)\right]^2}. \tag{4.1}$$

Theorem 6. *Let ν,μ,κ be three distinct positive real numbers and $s\ge 0$. If $\max(\kappa,\nu)<\mu$, then the function $z\mapsto \mathcal{E}_{\kappa,\nu,\mu}^{(a)}(s;z)$ defined by (4.1) is increasing on $(0,\infty)$. Furthermore, the following sharp Turán type inequality*

$$\left[E_{\kappa,\nu+1}^{(a)}(s;z)\right]^2 - E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\nu+2}^{(a)}(s;z) \le \left(\frac{1}{\nu+1}\right)\left[E_{\kappa,\nu+1}^{(a)}(s;z)\right]^2, \tag{4.2}$$

holds for all $\nu>0$.

Proof. By means of the Cauchy product, we have

$$E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\mu}^{(a)}(s;z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}A_{j,k}z^k,$$

$$[E^{(a)}_{\kappa,(\nu+\mu)/2}(s;z)]^2 = \sum_{k=0}^{\infty}\sum_{j=0}^{k} B_{j,k} z^k,$$

where

$$A_{j,k} = \frac{1}{\Gamma(\kappa j+\nu)\Gamma(\kappa(k-j)+\mu)(j+a)^s(k-j+a)^s},$$
$$B_{j,k} = \frac{1}{\Gamma(\kappa j+(\nu+\mu)/2)\Gamma(\kappa(k-j)+(\nu+\mu)/2)(j+a)^s(k-j+a)^s}.$$

Now, we define the sequence $(C_{j,k})$ by $C_{j,k} = A_{j,k}/B_{j,k}$. Therefore,

$$\frac{C_{j+1,k}}{C_{j,k}} = \left[\frac{\Gamma(\kappa j+(\nu+\mu)/2+\kappa)\Gamma(\kappa j+\nu)}{\Gamma(\kappa j+\nu+\kappa)\Gamma(\kappa j+(\nu+\mu)/2)}\right]\left[\frac{\Gamma(\kappa(k-j)+\mu)\Gamma(\kappa(k-j)+(\nu+\mu)/2-\kappa)}{\Gamma(\kappa(k-j)+\mu-\kappa)\Gamma(\kappa(k-j)+(\nu+\mu)/2)}\right]. \tag{4.3}$$

In virtue of the following inequality for the gamma function

$$\Gamma(x+a)\Gamma(x+b) \le \Gamma(x)\Gamma(x+a+b),\ a>0, b>0, x>0, \tag{4.4}$$

and choosing $x = \kappa(k-j)+(\nu+\mu)/2-\kappa, a = (\mu-\nu)/2$ and $b = \kappa$, we obtain

$$\frac{\Gamma(\kappa(k-j)+\mu)\Gamma(\kappa(k-j)+(\nu+\mu)/2-\kappa)}{\Gamma(\kappa(k-j)+\mu-\kappa)\Gamma(\kappa(k-j)+(\nu+\mu)/2)} \ge 1. \tag{4.5}$$

On the other hand, uppon setting $x = \kappa j+\nu, a = (\mu-\nu)/2$ and $b = \kappa$ in (4.4), we get

$$\frac{\Gamma(\kappa j+(\nu+\mu)/2+\kappa)\Gamma(\kappa j+\nu)}{\Gamma(\kappa j+\nu+\kappa)\Gamma(\kappa j+(\nu+\mu)/2)} \ge 1. \tag{4.6}$$

Combining (4.3), (4.5) and (4.6), we deduce that the sequence $(C_{j,k})_{j\ge0}$ is increasing and consequently the sequence $(S_k)_{k\ge0}$ defined by $S_k = \left(\sum_{j=0}^{k} A_{j,k}/\sum_{j=0}^{k} B_{j,k}\right)$ is increasing too, by means of Lemma 1. Further, using Lemma 2, we deduce that the function $z \mapsto \mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;z)$ is increasing on $(0,\infty)$. Moreover,

$$\mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;z) \ge \mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;0) = \frac{\Gamma^2((\nu+\mu)/2)}{\Gamma(\nu)\Gamma(\mu)}.$$

Finally, putting $\mu = \nu+2$ in the above inequality, we get (4.2) for all $\nu > 0$, which completes the proof of the theorem. □

Corollary 2. *The following assertions are true.*

(i) *Let* $\kappa > 0,\ s \ge 0,\ a > 0$ *and* $z > 0$, *then the function* $\nu \mapsto \frac{E^{(a)}_{\kappa,\nu}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)}$, *is convex on* $(0,\infty)$.

(ii) *Under the hypothesis of Theorem 6, the function* $\nu \mapsto \frac{E_{\kappa,\nu}^{(a)}(s;z_1)}{E_{\kappa,\nu}^{(a)}(s;z_2)}$ *is log-convex on* $(0,\infty)$, *for all* $0 < z_2 < z_1$.

Proof. (i) From Theorem 6, we deduce that the function $z \mapsto \log \mathcal{E}_{\kappa,\nu,\mu}^{(a)}(s;z)$ is increasing on $(0,\infty)$. We have,

$$\frac{\partial}{\partial z}\log \mathcal{E}_{\kappa,\nu,\mu}^{(a)}(s;z) = \frac{\frac{\partial}{\partial z}E_{\kappa,\nu}^{(a)}(s;z)}{E_{\kappa,\nu}^{(a)}(s;z)} + \frac{\frac{\partial}{\partial z}E_{\kappa,\mu}^{(a)}(s;z)}{E_{\kappa,\mu}^{(a)}(s;z)} - 2\frac{\frac{\partial}{\partial z}E_{\kappa,(\nu+\mu)/2}^{(a)}(s;z)}{E_{\kappa,(\nu+\mu)/2}^{(a)}(s;z)} \geq 0,$$

and consequently, the function $\nu \mapsto \frac{\frac{\partial}{\partial z}E_{\kappa,\nu}^{(a)}(s;z)}{E_{\kappa,\nu}^{(a)}(s;z)}$ is convex on $(0,\infty)$. In view of (4.9), we get

$$\frac{\frac{\partial}{\partial z}E_{\kappa,\nu+1}^{(a)}(s;z)}{E_{\kappa,\nu+1}^{(a)}(s;z)} = \frac{E_{\kappa,\nu}^{(a)}(s;z)}{\kappa z E_{\kappa,\nu+1}^{(a)}(s;z)} - \frac{\nu}{\kappa z},$$

which implies that the function $\nu \mapsto \frac{E_{\kappa,\nu}^{(a)}(s;z)}{E_{\kappa,\nu+1}^{(a)}(s;z)}$ is convex on $(0,\infty)$.

(ii) Again, by means of Theorem 6, we have $\mathcal{E}_{\kappa,\nu,\mu}^{(a)}(s;z_1) > \mathcal{E}_{\kappa,\nu,\mu}^{(a)}(s;z_2)$ for $0 < z_2 < z_1$, which yields

$$\frac{E_{\kappa,\nu}^{(a)}(s;z_1)}{E_{\kappa,\nu}^{(a)}(s;z_2)}\frac{E_{\kappa,\mu}^{(a)}(s;z_1)}{E_{\kappa,\mu}^{(a)}(s;z_2)} \geq \frac{[E_{\kappa,(\nu+\mu)/2}^{(a)}(s;z_1)]^2}{[E_{\kappa,(\nu+\mu)/2}^{(a)}(s;z_2)]^2},$$

which implies that the function $\nu \mapsto \frac{E_{\kappa,\nu}^{(a)}(s;z_1)}{E_{\kappa,\nu}^{(a)}(s;z_2)}$ is log-convex on $(0,\infty)$. □

Theorem 7. *Suppose that* μ, ν_1, ν_2 *and* a *are positive real numbers and* $s \geq 0$. *If* $\nu_1 < \nu_2$, *then the function* $z \mapsto \frac{E_{\kappa,\nu_2}^{(a)}(s;z)}{E_{\kappa,\nu_1}^{(a)}(s;z)}$ *is decreasing on* $(0,\infty)$. *Moreover, the following Turán type inequality*

$$[E_{\kappa,\nu+1}^{(a)}(s;z)]^2 - E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\nu+2}^{(a)}(s;z) \leq (\kappa+1)[E_{\kappa,\nu+1}^{(a)}(s;z)]^2, \tag{4.7}$$

holds true for all $z > 0$.

Proof. From (1.3), we have

$$\frac{E_{\kappa,\nu_1}^{(a)}(s;z)}{E_{\kappa,\nu_2}^{(a)}(s;z)} = \frac{\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\kappa n+\nu_1)(n+a)^s}}{\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\kappa n+\nu_2)(n+a)^s}} = \frac{\sum_{n=0}^{\infty}a_n z^n}{\sum_{n=0}^{\infty}b_n z^n}, \tag{4.8}$$

where

$$a_n = \frac{1}{\Gamma(\kappa n+\nu_1)(n+a)^s}, \quad b_n = \frac{1}{\Gamma(\kappa n+\nu_2)(n+a)^s}.$$

Now, we define the sequence $(c_n)_{n\geq 0}$ by $c_n = a_n/b_n$. Therefore,

$$c_{n+1}-c_n = \frac{\Gamma(\kappa n+\nu_2+\kappa)\Gamma(\kappa n+\nu_1)-\Gamma(\kappa n+\nu_1+\kappa)\Gamma(\kappa n+\nu_2)}{\Gamma(\kappa n+\nu_1+\kappa)\Gamma(\kappa n+\nu_1)}.$$

Setting $z=\kappa n+\nu_1, a=\kappa$ and $b=\nu_2-\nu_1$ in (4.4), we get

$$\Gamma(\kappa n+\nu_2+\kappa)\Gamma(\kappa n+\nu_1)\geq \Gamma(\kappa n+\nu_1+\kappa)\Gamma(\kappa n+\nu_2).$$

This yields that the sequence $(c_n)_{n\geq 0}$ is increasing, and consequently the function $z\mapsto \frac{E^{(a)}_{\kappa,\nu_2}(s;z)}{E^{(a)}_{\kappa,\nu_1}(s;z)}$ is decreasing on $(0,\infty)$ if $\nu_2>\nu_1$. Now, setting $\nu_2=\nu+2$ and $\nu_1=\nu+1$, we deduce that the function $z\mapsto \frac{zE^{(a)}_{\kappa,\nu+1}(s;z)}{E^{(a)}_{\kappa,\nu+2}(s;z)}$ is increasing on $(0,\infty)$. This implies that

$$\frac{\partial}{\partial z}\frac{zE^{(a)}_{\kappa,\nu+1}(s;z)}{E^{(a)}_{\kappa,\nu+2}(s;z)}\geq 0.$$

Using the recurrence formula

$$\frac{\partial}{\partial z}E^{(a)}_{\kappa,\nu+1}(s;z) = \frac{E^{(a)}_{\kappa,\nu}(s;z)-\nu E^{(a)}_{\kappa,\nu+1}(s;z)}{\kappa z}, \tag{4.9}$$

we obtain

$$\begin{aligned}[E^{(a)}_{\kappa,\nu+2}(s;z)]^2\frac{\partial}{\partial z}\frac{zE^{(a)}_{\kappa,\nu+1}(s;z)}{E^{(a)}_{\kappa,\nu+2}(s;z)} &= \left(1+\frac{1}{\kappa}\right)E^{(a)}_{\kappa,\nu+1}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z)\\ &\quad+\frac{E^{(a)}_{\kappa,\nu}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z)-[E^{(a)}_{\kappa,\nu+1}(s;z)]^2}{\kappa}\geq 0,\end{aligned} \tag{4.10}$$

which yields

$$[E^{(a)}_{\kappa,\nu+1}(s;z)]^2-E^{(a)}_{\kappa,\nu}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z)\leq (\kappa+1)E^{(a)}_{\kappa,\nu+1}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z). \tag{4.11}$$

Moreover, we observe that

$$E^{(a)}_{\kappa,\nu+2}(s;z)\leq E^{(a)}_{\kappa,\nu+1}(s;z) \text{ for all } z>0.$$

Combining the above inequality with (4.11), we obtain (4.7). □

5. Further inequalities for Barnes Mittag-Leffler function

This section is devoted to study further new inequalities for Barnes Mittag-Leffler function.

Theorem 8. *Let $\nu>0$, $\mu>0$ and $s\geq 0$. If $\mu<\nu$, then the following inequality*

$$\left[E^{(a)}_{\kappa,\nu+1}(s;z)\right]^{\frac{1}{\nu}}\leq\frac{a^{\frac{s(\nu-\mu)}{\nu\mu}}[\Gamma(\mu+1)]^{\frac{1}{\mu}}}{[\Gamma(\nu+1)]^{\frac{1}{\nu}}}\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{1}{\mu}},\tag{5.1}$$

holds true for all $z>0$.

Proof. Suppose that $0<\mu<\nu$ and $s\geq 0$ and the function $F^{(a)}_{\kappa,\nu,\mu}:[0,\infty)\to\mathbb{R}$ is defined by

$$F^{(a)}_{\kappa,\nu,\mu}(s;z)=\frac{\mu}{\nu}\log\tilde{E}^{(a)}_{\kappa,\nu+1}(s;z)-\log\tilde{E}^{(a)}_{\kappa,\mu+1}(s;z),$$

where

$$\tilde{E}^{(a)}_{\kappa,\nu}(s;z)=a^s\Gamma(\nu)E^{(a)}_{\kappa,\nu}(s;z).$$

In view of the recurrence relation (4.9), we have

$$\begin{aligned}\frac{\partial}{\partial z}F^{(a)}_{\kappa,\nu,\mu}(s;z)&=\frac{1}{\kappa z}\left[\frac{\mu}{\nu}\frac{E^{(a)}_{\kappa,\nu}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)}-\frac{E^{(a)}_{\kappa,\mu}(s;z)}{E^{(a)}_{\kappa,\mu+1}(s;z)}\right]\\&=\frac{\mu}{\kappa z}\left[\frac{\tilde{E}^{(a)}_{\kappa,\nu}(s;z)}{\tilde{E}^{(a)}_{\kappa,\nu+1}(s;z)}-\frac{\tilde{E}^{(a)}_{\kappa,\mu}(s;z)}{\tilde{E}^{(a)}_{\kappa,\mu+1}(s;z)}\right].\end{aligned}\tag{5.2}$$

Moreover, the function $\nu\mapsto\tilde{E}^{(a)}_{\kappa,\nu}(s;z)$ is log-convex on $(0,\infty)$. Indeed, for convenience, let us write

$$\tilde{E}^{(a)}_{\kappa,\nu}(s;z)=\sum_{n=0}^{\infty}a_n(\nu)z^n,\ \text{ where }\ a_n(\nu)=\frac{a^s\Gamma(\nu)}{\Gamma(\kappa n+\nu)(n+a)^s},\ n\geq 0.$$

Using the fact that the sum of log-convex functions is log-convex too, we just need to prove the log-convexity of $a_n(\nu)$. Hence, for all $n\geq 0$, we have

$$\frac{\partial^2}{\partial\nu^2}\log(a_n(\nu))=\psi'(\nu)-\psi'(\kappa n+\nu),$$

where $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. Since ψ is concave, we deduce that $\frac{\partial^2}{\partial\nu^2}\log(a_n(\nu))\geq 0$, and consequently $\nu\to a_n(\nu)$ is log-convex on $(0,\infty)$, which implies that the function $\nu\mapsto\tilde{E}^{(a)}_{\kappa,\nu}(s;z)$ is log-convex on $(0,\infty)$. Therefore, the function $\nu\mapsto\log\tilde{E}^{(a)}_{\kappa,\nu+1}(s;z)-\log\tilde{E}^{(a)}_{\kappa,\nu}(s;z)$ is increasing on $(0,\infty)$ and consequently the function $\nu\mapsto\frac{\tilde{E}^{(a)}_{\kappa,\nu+1}(s;z)}{\tilde{E}^{(a)}_{\kappa,\nu}(s;z)}$ is increasing on $(0,\infty)$. Combining this fact with (5.2), we obtain $\frac{\partial}{\partial z}F^{(a)}_{\kappa,\nu,\mu}(s;z)\leq 0$. Therefore, $F^{(a)}_{\kappa,\nu,\mu}(s;z)\leq F^{(a)}_{\kappa,\nu,\mu}(s;0)=0$, which proves the desired result. □

Corollary 3. *Under the assumption of Theorem 8, the following inequality*

$$\frac{a^{\frac{s(\nu-\mu)}{\mu}}[\Gamma(\mu+1)]^{\frac{\nu}{\mu}}}{[\Gamma(\nu+1)]}\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{\nu-\mu}{\mu}}+\frac{E^{(a)}_{\kappa,\mu+1}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)}\geq 2, \tag{5.3}$$

holds true for all $z>0$.

Proof. Using (5.1), we obtain

$$\frac{a^{\frac{s(\nu-\mu)}{\mu}}[\Gamma(\mu+1)]^{\frac{\nu}{\mu}}}{[\Gamma(\nu+1)]}\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{\nu}{\mu}}\left[E^{(a)}_{\kappa,\nu+1}(s;z)\right]^{-1}\geq 1.$$

With the help of the above inequality and the arithmetic-geometric mean inequality, we get

$$\frac{1}{2}\left(\frac{a^{\frac{s(\nu-\mu)}{\mu}}[\Gamma(\mu+1)]^{\frac{\nu}{\mu}}}{\Gamma(\nu+1)}\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{\nu-\mu}{\mu}}+\frac{E^{(a)}_{\kappa,\mu+1}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)}\right)$$
$$\geq\sqrt{\frac{a^{\frac{s(\nu-\mu)}{\mu}}[\Gamma(\mu+1)]^{\frac{\nu}{\mu}}}{\Gamma(\nu+1)}\frac{\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{\nu}{\mu}}}{E^{(a)}_{\kappa,\nu+1}(s;z)}}\geq 1,$$

which proves (5.3). □

ACKNOWLEDGEMENTS

The authors wish to thank the anonymous referee for the careful reading of the manuscript and for constructive comments, observations, and suggestions that has helped in significantly improving the manuscript.

REFERENCES

[1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, "Inequalities for quasiconformal mappings in space," *Pacific J. Math.*, vol. 160, no. 1, pp. 1–18, 1993. [Online]. Available: http://projecteuclid.org/euclid.pjm/1102624560

[2] Á. Baricz, "Functional inequalities involving Bessel and modified Bessel functions of the first kind," *Expo. Math.*, vol. 26, no. 3, pp. 279–293, 2008, doi: 10.1016/j.exmath.2008.01.001.

[3] Á. Baricz, "Turán type inequalities for modified Bessel functions," *Bull. Aust. Math. Soc.*, vol. 82, no. 2, pp. 254–264, 2010, doi: 10.1017/S000497271000002X.

[4] Á. Baricz, K. Raghavendar, and A. Swaminathan, "Turán type inequalities for q-hypergeometric functions," *J. Approx. Theory*, vol. 168, pp. 69–79, 2013, doi: 10.1016/j.jat.2013.01.002.

[5] Á. Baricz and J. Sándor, "Extensions of the generalized Wilker inequality to Bessel functions," *J. Math. Inequal.*, vol. 2, no. 3, pp. 397–406, 2008, doi: 10.7153/jmi-02-35.

[6] E. W. Barnes, "The asymptotic expansion of integral functions defined by Taylor's series," *Phil. Trans. Roy. Soc. London (A)*, vol. 206, pp. 249–297, 1906, doi: 10.1098/rsta.1906.0019.

[7] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, ser. Springer Monographs in Mathematics. Springer, Heidelberg, 2014. doi: 10.1007/978-3-662-43930-2.

[8] K. Górska, A. Horzela, A. Lattanzi, and T. K. Pogány, "On complete monotonicity of three parameter Mittag-Leffler function," *Appl. Anal. Discrete Math.*, vol. 15, no. 1, pp. 118–128, 2021, doi: 10.2298/aadm190226025g.

[9] C. H. Kimberling, "A probabilistic interpretation of complete monotonicity," *Aequationes Math.*, vol. 10, pp. 152–164, 1974, doi: 10.1007/BF01832852.

[10] E. A. Kiselev, L. A. Minin, I. Y. Novikov, and S. M. Sitnik, "On the Riesz constants for systems of integer translates," *Math. Notes*, vol. 96, no. 1-2, pp. 228–238, 2014, translation of Mat. Zametki **96** (2014), no. 2, 239–250, doi: 10.1134/S0001434614070244.

[11] K. Mehrez and S. M. Sitnik, "Turán type inequalities for classical and generalized Mittag-Leffler functions," *Anal. Math.*, vol. 44, no. 4, pp. 521–541, 2018, doi: 10.1007/s10476-018-0404-9.

[12] K. Mehrez and S. M. Sitnik, "Functional inequalities for the Fox-Wright functions," *Ramanujan J.*, vol. 50, no. 2, pp. 263–287, 2019, doi: 10.1007/s11139-018-0071-2.

[13] D. S. Mitrinović, *Analytic inequalities*, ser. Die Grundlehren der mathematischen Wissenschaften, Band 165. Springer-Verlag, New York-Berlin, 1970, in cooperation with P. M. Vasić.

[14] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and new inequalities in analysis*, ser. Mathematics and its Applications (East European Series). Kluwer Academic Publishers Group, Dordrecht, 1993, vol. 61, doi: 10.1007/978-94-017-1043-5.

[15] G. M. Mittag-Leffler, "Sur la nouvelle fonction $e_\alpha(x)$," *C. R. Acad. Sci. Paris*, vol. 137, pp. 554–558, 1903.

[16] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, ser. Mathematics in Science and Engineering. Academic Press, Inc., Boston, MA, 1992, vol. 187.

[17] S. Ponnusamy and M. Vuorinen, "Asymptotic expansions and inequalities for hypergeometric functions," *Mathematika*, vol. 44, no. 2, pp. 278–301, 1997, doi: 10.1112/S0025579300012602.

[18] S. M. Sitnik, "Factorization and estimates of norms in weighted Lebesgue spaces of Buschman-Erdelyi operators," *Dokl. Akad. Nauk SSSR*, vol. 320, no. 6, pp. 1326–1330, 1991.

[19] H. M. Srivastava, R. K. Saxena, T. K. Pogány, and R. Saxena, "Integral and computational representations of the extended Hurwitz-Lerch zeta function," *Integral Transforms Spec. Funct.*, vol. 22, no. 7, pp. 487–506, 2011, doi: 10.1080/10652469.2010.530128.

[20] Ž. Tomovski, T. K. Pogány, and H. M. Srivastava, "Laplace type integral expressions for a certain three-parameter family of generalized Mittag-Leffler functions with applications involving complete monotonicity," *J. Franklin Inst.*, vol. 351, no. 12, pp. 5437–5454, 2014, doi: 10.1016/j.jfranklin.2014.09.007.

[21] P. Turán, "On the zeros of the polynomials of Legendre," *Časopis Pěst. Mat. Fys.*, vol. 75, pp. 113–122, 1950.

[22] D. V. Widder, *The Laplace Transform*, ser. Princeton Mathematical Series, vol. 6. Princeton University Press, Princeton, N. J., 1941.

[23] J. B. Wilker, "Problem E3306," *Amer. Math. Monthly*, vol. 96, no. 1, p. 55, 1989, doi: 10.2307/2323259.

[24] A. Wiman, "Über den Fundamentalsatz in der Teorie der Funktionen $E^a(x)$," *Acta Math.*, vol. 29, no. 1, pp. 191–201, 1905, doi: 10.1007/BF02403202.

[25] L. Zhu, "On Wilker-type inequalities," *Math. Inequal. Appl.*, vol. 10, no. 4, pp. 727–731, 2007, doi: 10.7153/mia-10-67.

Authors' addresses

Khaled Mehrez
Department of Mathematics, Kairouan Preparatory Institute for Engineering Studies, University of Kairouan, 3100 Kairouan, Tunisia
E-mail address: k.mehrez@yahoo.fr

Sourav Das
(**Corresponding author**) Department of Mathematics, National Institute of Technology Jamshedpur, Jamshedpur-831014, Jharkhand, India
E-mail address: souravdas.math@nitjsr.ac.in

Anish Kumar
Department of Mathematics, National Institute of Technology Jamshedpur, Jamshedpur-831014, Jharkhand, India
E-mail address: ak8107690@gmail.com