On common fixed point theorems for
\((\psi, \phi)\)-generalized \(f\)-weakly contractive mappings

\(H. \ Aydi\)
ON COMMON FIXED POINT THEOREMS FOR
($\psi, \varphi$)-GENERALIZED $f$-WEAKLY CONTRACTIVE MAPPINGS

H. AYDI

Received September 15, 2011

Abstract. In this paper, we present some common fixed point theorems for ($\psi, \varphi$)-generalized $f$-weakly contractive mappings in metric and ordered metric spaces. Our results extend, generalize and improve some well-known results in the literature. Also, we give an example to illustrate our results.

2000 Mathematics Subject Classification: 54H25; 47H10; 54E50

Keywords: common fixed point, commuting maps, $f$-weakly contractive maps, generalized $f$-weakly contractive maps, ordered metric space

1. INTRODUCTION AND PRELIMINARIES

The first important result on fixed points for contractive type mapping was the much celebrated Banach’s contraction principle by Banach [2] in 1922. After this, Kannan [9, 10] proved the following result:

**Theorem 1.** Let $(X, d)$ be a complete metric space. If $T : X \to X$ satisfies

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)],$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then $T$ has a unique fixed point.

A similar type of contractive condition has been studied by Chatterjee [5] and he proved the following result:

**Theorem 2.** Let $(X, d)$ be a complete metric space. If $T : X \to X$ satisfies a $C$-contraction given as follows:

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)],$$

where $0 < k < \frac{1}{2}$ and $x, y \in X$, then $T$ has a unique fixed point.

Alber and Guerre-Delabriere [1] introduced the definition of weak $\Phi$-contraction.

**Definition 1.** A self mapping $T$ on a metric space $X$ is called weak $\Phi$-contraction if there exists a function $\Phi : [0, +\infty) \to [0, +\infty)$ such that for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \Phi(d(x, y)).$$

© 2013 Miskolc University Press
The notion of $\Phi$-contraction and weak $\Phi$-contraction has been studied by many authors, see [3, 12, 15, 17, 19]. In recent years, many results related to fixed point theorems in partially ordered metric spaces are given, for more details see [8,12–16].

Choudhury in [6] introduced a generalization of $C$-contraction given by the following definition.

**Definition 2 (6).** Let $(X,d)$ be a metric space. A mapping $T : X \to X$ is said to be weakly $C$-contractive (or a weak $C$-contraction) if for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)), $$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only $x = y = 0$.

In [6] the author proves that if $X$ is complete then every weak $C$-contraction has a unique fixed point. Recently, Harjani et al, [8] presented this last result in the context of ordered metric spaces.

Chandok [4] introduced the following definition: A map $T : X \to X$ is generalized $f$-weakly contractive if for each $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}(d(fx, Ty) + d(fy, Tx)) - \varphi(d(fx, Ty), d(fy, Tx)),$$

where $\varphi : [0, +\infty) \to [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only $x = y = 0$.

If $f = I_X$, the identity mapping, then generalized $f$-weakly contractive mapping is weakly $C$-contractive.

Khan et al. [11] introduced the concept of altering distance function as follows:

**Definition 3 (altering distance function, [11]).** The function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

1. $\psi$ is continuous and non-decreasing.
2. $\psi(t) = 0$ if and only if $t = 0$.

Following the above definitions, we introduce the following:

**Definition 4.** A map $T : X \to X$ is called $(\psi, \varphi)$-generalized $f$-weakly contractive if for each $x, y \in X$,

$$\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right) - \varphi(d(fx, Ty), d(fy, Tx)),$$

where

1. $\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function.
2. $\varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function with $\varphi(t, s) = 0$ if and only if $t = s = 0$. 
If \( \psi(t) = t \), then \((\psi, \varphi)\)-generalized \( f \)-weakly contractive mapping is generalized \( f \)-weakly contractive.

The aim of this paper is to study some common fixed point theorems for \((\psi, \varphi)\)-generalized \( f \)-weakly contractive in metric and ordered metric spaces.

2. MAIN RESULTS

First, we state the following known definition:

**Definition 5.** Let \( X \) a non-empty set. A point \( x \in X \) is a coincidence point (common fixed point) of \( f : X \to X \) and \( T : X \to X \) if \( fx = Tx \) \((x = fx = Tx)\). The pair \( \{f, T\} \) is called commuting if \( Tf x = f Tx \) for all \( x \in X \).

We start with a common fixed point theorem for \((\psi, \varphi)\)-generalized \( f \)-weakly contractive mappings in complete metric spaces.

**Theorem 3.** Let \((X, d)\) be a metric space. Let \( f, T : X \to X \) satisfy \( T(X) \subset f(X) \), \((f(X), d)\) is complete and

\[
\psi(d(Tx, Ty)) \leq \psi\left( \frac{1}{2} [d(fx, Tx) + d(fy, Ty)] \right) - \psi(d(fx, Ty), d(fy, Tx),)
\]

for all \( x, y \in X \), where

1. \( \psi : [0, +\infty) \to [0, +\infty) \) is an altering distance function,
2. \( \varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is a continuous function with \( \varphi(t, s) = 0 \) if and only if \( t = s = 0 \), then \( T \) and \( f \) have a coincidence point in \( X \). Further, if \( T \) and \( f \) commute at their coincidence points, then \( T \) and \( f \) have a common fixed point.

**Proof.** Let \( x_0 \in X \). Since \( T(X) \subset f(X) \), we can choose \( x_1 \in X \), so that \( fx_1 = Tx_0 \). Since \( Tx_1 \in f(X) \), there exists \( x_2 \in X \) such that \( fx_2 = Tx_1 \). By induction, we construct a sequence \( \{x_n\} \) in \( X \) such that \( fx_{n+1} = Tx_n \), for every \( n \in \mathbb{N} \). By inequality (2.1), we have

\[
\psi(d(Tx_{n+1}, Tx_n)) \leq \psi\left( \frac{1}{2} [d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})] \right) - \psi(d(fx_{n+1}, Tx_n), d(fx_n, Tx_{n+1}))
\]

\[
= \psi\left( \frac{1}{2} d(Tx_{n+1},Tx_{n+1}) \right) - \psi(0,d(Tx_{n-1},Tx_{n+1}))
\]

\[
\leq \psi\left( \frac{1}{2} d(Tx_{n-1},Tx_{n+1}) \right) - \psi(0,d(Tx_{n-1},Tx_{n+1}))
\]

\[
\leq \psi\left( \frac{1}{2} [d(Tx_{n-1},Tx_n) + d(Tx_n, Tx_{n+1})] \right).
\]

Since \( \psi \) is a non-decreasing function, we get that

\[
d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n) \text{ for any } n \in \mathbb{N}^*.
\]
Thus, \( \{d(Tx_n, Tx_{n+1})\} \) is a monotone non-increasing sequence of non-negative real numbers and hence is convergent. Hence there is \( r \geq 0 \) such that
\[
\lim_{n \to +\infty} d(Tx_n, Tx_{n+1}) = r.
\]
Using a triangular inequality, we have
\[
d(Tx_{n+1}, Tx_n) \leq \frac{1}{2} d(Tx_{n-1}, Tx_n) \leq \frac{1}{2} [d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})].
\]
Letting \( n \to +\infty \), we get
\[
r \leq \frac{1}{2} \lim_{n \to +\infty} d(Tx_{n-1}, Tx_n) \leq \frac{1}{2} r + \frac{1}{2} r,
\]
that is
\[
\lim_{n \to +\infty} d(Tx_{n-1}, Tx_{n+1}) = 2r.
\]
Using the continuity of \( \psi \) and \( \varphi \), and inequality (2.2), we have, letting \( n \to +\infty \)
\[
\psi(r) \leq \psi(r) - \varphi(0, 2r),
\]
and consequently, \( \varphi(0, 2r) \leq 0 \). Thus, by a property of \( \varphi, r = 0 \), so
\[
\lim_{n \to +\infty} d(Tx_{n+1}, Tx_n) = 0. \tag{2.4}
\]
Now, we show that \( \{Tx_n\} \) is a Cauchy sequence. If otherwise, then there exists \( \varepsilon > 0 \) for which we can find subsequences \( \{Tx_{m(k)}\} \) and \( \{Tx_{n(k)}\} \) of \( \{Tx_n\} \) with \( n(k) > m(k) > k \) such that for every \( k \),
\[
d(Tx_{m(k)}, Tx_{n(k)}) \geq \varepsilon, \quad d(Tx_{m(k)}, Tx_{n(k)-1}) < \varepsilon. \tag{2.5}
\]
By triangular inequality, we have from (2.5)
\[
\varepsilon \leq d(Tx_{m(k)}, Tx_{n(k)}) \\
\leq d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)}) \\
< \varepsilon + d(Tx_{n(k)-1}, Tx_{n(k)}).
\]
Using (2.4), we get
\[
\lim_{k \to +\infty} d(Tx_{m(k)}, Tx_{n(k)}) = \lim_{k \to +\infty} d(Tx_{m(k)}, Tx_{n(k)-1}) = \varepsilon. \tag{2.6}
\]
On the other hand,
\[
d(Tx_{m(k)}, Tx_{n(k)-1}) \leq d(Tx_{m(k)}, Tx_{m(k)-1}) + d(Tx_{m(k)-1}, Tx_{n(k)}) \\
+ d(Tx_{n(k)}, Tx_{n(k)-1}),
\]
and
\[
d(Tx_{m(k)-1}, Tx_{n(k)}) \leq d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{m(k)}, Tx_{n(k)}).
\]
Letting \( k \to +\infty \) in the two above inequalities, we have thanks to (2.4) and (2.6),
\[
\lim_{k \to +\infty} d(Tx_{m(k)-1}, Tx_{n(k)}) = \varepsilon. \tag{2.7}
\]
From (2.1), we have
\[
\psi(\varepsilon) \leq \psi\left(\frac{1}{2}[d(fx_{m(k)} \cdot Tx_{n(k)}) + d(fx_{n(k)} \cdot Tx_{m(k)})]\right)
\]
\[
= \psi\left(\frac{1}{2}(d(Tx_{m(k)} \cdot Tx_{n(k)}) + d(Tx_{n(k)} \cdot Tx_{m(k)})\right)
\]
\[
= \psi(\varepsilon) - \psi(\varepsilon) \leq \psi(\varepsilon) - \psi(\varepsilon),
\]
\[
\psi(\varepsilon) \leq \psi(\varepsilon) - \psi(\varepsilon),
\]
hence \(\psi(\varepsilon, \varepsilon) = 0\), so \(\varepsilon = 0\), it is a contradiction. Thus \(\{Tx_n\}\) is a Cauchy sequence. Since \(fx_n = Tx_{n-1}\), hence \(\{fx_n\}\) is a Cauchy sequence in \((f(X), d)\), which is complete. Thus there is \(z \in X\) such that
\[
\lim_{n \to \infty} fx_n = f z.
\]
Moreover, (2.4) reads
\[
\lim_{n \to \infty} d(fx_n, fx_{n+1}) = 0.
\]
By (2.1), we have
\[
\psi(d(Tz, fx_{n+1})) = \psi(d(Tz, Tx_n))
\]
\[
\leq \psi(\frac{1}{2}[d(fz, Tx_n) + d(fx_n, Tz)]) - \psi(d(fz, Tx_n), d(fx_n, Tz))
\]
\[
= \psi(\frac{1}{2}[d(fz, fx_{n+1}) + d(fx_n, Tz)]) - \psi(d(fz, fx_{n+1}), d(fx_n, Tz)),
\]
and letting \(n \to +\infty\), using the continuity of \(\psi\) and \(\varphi\) and by (2.8), (2.9), we find
\[
\psi(d(Tz, fz)) \leq \psi(\frac{1}{2}d(Tz, fz)) - \psi(0, d(fz, Tz)) \leq \psi(\frac{1}{2}d(Tz, fz)).
\]
Consequently, \(d(Tz, fz) \leq \frac{1}{2}d(Tz, fz)\), that is, \(d(Tz, fz) = 0\), i.e. \(Tz = fz\), hence \(z\) is a coincidence point of \(T\) and \(f\). Now suppose that \(T\) and \(f\) commute at \(z\). Let \(w = Tz = fz\). Then \(Tw = T(fz) = f(Tz) = fw\). By inequality (2.1)
\[
\psi(d(Tz, Tw)) \leq \psi(\frac{1}{2}[d(fz, Tw) + d(fw, Tz)]) - \psi(d(fz, Tw), d(fw, Tz))
\]
\[
= \psi(\frac{1}{2}[d(Tz, Tw) + d(Tw, Tz)]) - \psi(d(Tz, Tw), d(Tw, Tz))
\]
\[
= \psi(\frac{1}{2}[d(Tz, Tw) + d(Tw, Tz)]) - \psi(d(Tz, Tw), d(Tw, Tz))
\]
\[
= \psi(d(Tz, Tw)) - \psi(d(Tz, Tw), d(Tw, Tz)).
\]
This implies that $d(Tz, Tw) = 0$, by the property of $\varphi$. Therefore, $Tw = fw = w$. This completes the proof of Theorem 3. □

Example 1. Let $X = [0, +\infty)$. Let $d$ be defined by $d(x, y) = |x - y|$ for all $x, y \in X$. We set $fx = \frac{x}{2}$ and $Tx = \frac{x}{4}$ for all $x \in X$. It is clear that $T(X) \subset f(X)$ and $(f(X), d)$ is a complete metric space. Define $\psi: [0, +\infty) \to [0, +\infty)$ and $\varphi: [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \frac{t}{2} \quad \text{and} \quad \varphi(t, s) = \frac{1}{16}(t + s).$$

It is obvious that $\psi$ and $\varphi$ satisfy the hypotheses of Theorem 3. We need to show that the inequality (2.1) holds for any $x, y \in X$. First, the left-hand side of (2.1) is

$$\psi(d(Tx, Ty)) = \frac{1}{8}|x - y|. \quad (2.10)$$

While, the right-hand side of (2.1) is

$$\psi\left(\frac{1}{2}(d(fx, Ty) + d(fy, Tx)) - \varphi(d(fx, Ty), d(fy, Tx))\right) \quad (2.11)$$

By symmetry of (2.10) and (2.11), and without loss of generality, we suppose that $x \geq y$. In particular, (2.10) reads

$$\psi(d(Tx, Ty)) = \frac{1}{8}(x - y).$$

We distinguish two cases:

- If $2y \geq x$. Here, we have from (2.11)

$$\psi\left(\frac{1}{2}(d(fx, Ty) + d(fy, Tx)) - \varphi(d(fx, Ty), d(fy, Tx))\right) = \frac{3}{64}(x + y) \geq \frac{1}{8}(x - y) = \psi(d(Tx, Ty)). \quad (2.12)$$

- If $2y < x$. Here, we have from (2.11)

$$\psi\left(\frac{1}{2}(d(fx, Ty) + d(fy, Tx)) - \varphi(d(fx, Ty), d(fy, Tx))\right) = \frac{9}{64}(x - y) \geq \frac{1}{8}(x - y) = \psi(d(Tx, Ty)). \quad (2.13)$$
By (2.12) and (2.13), the inequality (2.1) is satisfied. Then by Theorem 3, \( T \) and \( f \) have a common fixed point, which is \( z = 0 \).

**Corollary 1.** Let \((X, d)\) be a complete metric space. If \( T : X \to X \) satisfies
\[
\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2}[d(x, Ty) + d(y, Tx)]\right) - \varphi(d(x, Ty), d(y, Tx)),
\]
for all \( x, y \in X \), where
\begin{enumerate}
  \item \( \psi : [0, +\infty) \to [0, +\infty) \) is an altering distance function,
  \item \( \varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is a continuous function with \( \varphi(t, s) = 0 \) if and only if \( t = s = 0 \), then \( T \) has a unique fixed point.
\end{enumerate}

**Proof.** It follows by taking \( f = I_X \) in Theorem 3. The uniqueness of the fixed point follows by the following: suppose \( u \) and \( v \) are fixed points of \( T \). By (2.14), we have
\[
\psi(d(u, v)) = \psi(d(Tu, Tv)) \\
\leq \psi\left(\frac{1}{2}[d(u, Tv) + d(v, Tu)]\right) - \varphi(d(u, Tv), d(v, Tu)) \\
\leq \psi\left(\frac{1}{2}[d(u, v) + d(v, u)]\right) - \varphi(d(u, v), d(v, u)) \\
= \psi(d(u, v)) - \varphi(d(u, v), d(v, u)),
\]
which implies that \( \varphi(d(u, v), d(v, u)) = 0 \), and by a property of \( \varphi \), we get \( u = v \). \( \square \)

**Corollary 2.** Let \((X, d)\) be a metric space. If \( T, f : X \to X \) are such that \( T(X) \subset f(X) \), \((f(X), d)\) is complete and
\[
d(Tx, Ty) \leq \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \varphi(d(fx, Ty), d(fy, Tx)),
\]
for all \( x, y \in X \), where \( \varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is a continuous function with \( \varphi(t, s) = 0 \) if and only if \( t = s = 0 \), then \( T \) and \( f \) have a coincidence point in \( X \). Further, if \( T \) and \( f \) commute at their coincidence points, then \( T \) and \( f \) have a common fixed point.

**Proof.** The proof follows by taking \( \psi(t) = t \) in Theorem 3. \( \square \)

**Corollary 3.** Let \((X, d)\) be a complete metric space. If \( T : X \to X \) satisfies for all \( x, y \in X \)
\[
d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \varphi(d(x, Ty), d(y, Tx)),
\]
where \( \varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is a continuous function with \( \varphi(t, s) = 0 \) if and only if \( t = s = 0 \), then \( T \) has a unique fixed point.

**Proof.** It follows by taking \( f = I_X \) in Corollary 2. The uniqueness of the fixed point follows from Corollary 1. \( \square \)
Remark 1. • Corollary 1 corresponds to Corollary 2.1 of Shatanawi [18].
• Corollary 2 corresponds to Theorem 1 of Chandok [4].
• Corollary 3 corresponds to Theorem 2.1 of Choudhury [6].

Now, we extend Theorem 3 and we prove a common fixed point theorem for \( f \)-non-decreasing generalized nonlinear contraction mappings in the context of ordered metric spaces.

**Definition 6** ([7]). Suppose \((X, \leq)\) is a partially ordered set and \(T, f : X \to X\). \(T\) is said to be monotone \(f\)-non-decreasing if for all \(x, y \in X\),

\[
fx \leq fy \quad \text{implies} \quad Tx \leq Ty.
\]

(2.17)

If \(f = I_X\) in Definition 6, then \(T\) is monotone non-decreasing.

**Theorem 4.** Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(f\) and \(T\) are self-mappings of \(X\) such that \(T(X) \subseteq f(X)\), \(f(X)\) is closed and \(T\) is \(f\)-non-decreasing mapping. Suppose that \(f\) and \(T\) satisfy for all \(x, y \in X\), for which \(f(x) \leq f(y)\)

\[
\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2} [d(fx, Ty) + d(fy, Tx)]\right) - \varphi(d(fx, Ty), d(fy, Tx))
\]

where

1. \(\psi : [0, +\infty) \to [0, +\infty)\) is an altering distance function,
2. \(\varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)\) is a continuous function with \(\varphi(x, y) = 0\) if and only if \(x = y = 0\).

Also, suppose that if \(\{f(x_n)\} \subseteq X\) is a non-decreasing sequence with \(f(x_n) \to f(z)\) in \(f(X)\), then \(f(x_n) \leq f(z)\) and \(f(z) \leq f(f(z))\) for every \(n\). If there exists \(x_0 \in X\) with \(fx_0 \leq Tx_0\), then \(T\) and \(f\) have a coincidence point. Further, if \(T\) and \(f\) commute at their coincidence points, then \(T\) and \(f\) have a common fixed point.

**Proof.** Let \(x_0 \in X\) such that \(fx_0 \leq Tx_0\). Since \(T(X) \subseteq f(X)\), we can choose \(x_1 \in X\) so that \(fx_1 = Tx_0\). Since \(Tx_1 \in f(X)\), there exists \(x_2 \in X\) such that \(fx_2 = Tx_1\). By induction, we construct a sequence \(\{x_n\}\) in \(X\) such that

\[
fx_{n+1} = Tx_n.
\]

Since \(fx_0 \leq Tx_0\), \(Tx_0 = fx_1\), so \(fx_0 \leq fx_1\). \(T\) is \(f\)-non-decreasing mapping, we get \(Tx_0 \leq Tx_1\). Similarly \(fx_1 \leq fx_2\), \(Tx_1 \leq Tx_2\), hence \(fx_2 \leq fx_3\). Continuing, we obtain

\[
fx_0 \leq fx_1 \leq fx_2 \leq \ldots \leq fx_n \leq fx_{n+1} \leq \ldots
\]

If for some \(n\), \(Tx_{n+1} = Tx_n\), then \(Tx_{n+1} = fx_{n+1}\), i.e. \(T\) and \(f\) have a coincidence point \(x_{n+1}\), and so we have the result. For the rest, assume that \(d(Tx_n, Tx_{n+1}) > 0\)
for all \( n \in \mathbb{N} \). By (2.18), we have
\[
\psi(d(Tx_n, Tx_{n+1})) \leq \frac{1}{2} [d(fx_{n+1},Tx_n) + d(fx_n,Tx_{n+1})]
- \varphi(d(fx_{n+1},Tx_n), d(fx_n,Tx_{n+1}))
= \psi\left(\frac{1}{2} d(Tx_{n-1},Tx_{n+1})\right) - \varphi(0,d(Tx_{n-1},Tx_{n+1}))
\leq \psi\left(\frac{1}{2} d(Tx_{n-1},Tx_{n+1})\right)
\leq \psi\left(\frac{1}{2} d(Tx_{n-1},Tx_n) + \frac{1}{2} d(Tx_n,Tx_{n+1})\right).
\]

It follows that, for any \( n \in \mathbb{N}^* \)
\[
d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1},Tx_n).
\]
Thus \( \{d(Tx_n, Tx_{n+1})\} \) is a monotone non-increasing sequence, hence it is convergent. Now, proceeding as in Theorem 3, we can prove that
\[
\lim_{n \to +\infty} d(Tx_n, Tx_{n+1}) = 0. \quad (2.19)
\]
Moreover, \( \{Tx_n\} \) is a Cauchy sequence. Since \( Tx_n = fx_{n+1} \) and \( f(X) \) is closed, so there exists \( z \in X \) such that
\[
\lim_{n \to +\infty} fx_n = f(z). \quad (2.20)
\]
Having in mind \( \{fx_n\} \) is a non-decreasing sequence, so by (2.20) we have for every \( n \in \mathbb{N} \)
\[
fx_n \leq f(z), \quad f(z) \leq f(f(z)). \quad (2.21)
\]
Having \( fx_n \leq f(z) \), so from inequality (2.18), we have
\[
\psi(d(fx_{n+1},Tx)) = \psi(d(Tx_n,Tz))
\leq \psi\left(\frac{1}{2} [d(fz,Tx_n) + d(fx_n,Tz)]\right) - \varphi(d(fz,Tx_n), d(fx_n,Tz))
= \psi\left(\frac{1}{2} [d(fz,fx_{n+1}) + d(fx_n,Tz)] - \varphi(d(fz,fx_{n+1}), d(fx_n,Tz)).
\]
Taking \( n \to +\infty \), using the continuity of \( \psi \) and \( \varphi \), we get from (2.19), (2.20)
\[
\psi(d(Tz,fz)) \leq \psi\left(\frac{1}{2} d(fz,fz)\right) - \varphi(0,d(fz,Tz)),
\]
that is, \( d(Tz,fz) = 0 \), hence \( Tz = fz \), so \( z \) is a coincidence point of \( T \) and \( f \).

Now suppose that \( T \) and \( f \) commute at \( z \). Let \( w = Tz = fz \). Then \( Tw = T(fz) = f(Tz) = fw \). From (2.21), we have \( fz \leq f(fz) = fw \), so the inequality (2.18) gives us
\[
\psi(d(Tz,Tw)) \leq \psi\left(\frac{1}{2} [d(fz,Tw) + d(fw,Tz)]\right) - \varphi(d(fz,Tw), d(fw,Tz))
\]
This implies that \( d(Tz, Tw) = 0 \), by the property of \( \varphi \). Therefore, \( Tw = fw = w \).

This completes the proof of Theorem 4.

**Corollary 4.** Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Let \( f \) and \( T \) are self-mappings of \( X \) such that \( T(X) \subset f(X) \), \( f(X) \) is closed and \( T \) is \( f \)-non-decreasing mapping. Assume that \( f \) and \( T \) satisfy for all \( x, y \in X \), for which \( f(x) \leq f(y) \)

\[
d(Tx, Ty) \leq \frac{1}{2} [d(fx, Ty) + d(fy, Tx)] - \varphi(d(fx, Ty), d(fy, Tx)),
\]

where \( \varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is a continuous function with \( \varphi(x, y) = 0 \) if and only if \( x = y = 0 \).

Also, suppose that if \( \{ f(x_n) \} \subset X \) is a non-decreasing sequence with \( f(x_n) \to f(z) \) in \( f(X) \), then \( f(x_n) \leq f(z) \) and \( f(z) \leq f(f(z)) \) for every \( n \).

If there exists \( x_0 \in X \) with \( f(x_0) \leq Tx_0 \), then \( T \) and \( f \) have a coincidence point. Further, if \( T \) and \( f \) commute at their coincidence points, then \( T \) and \( f \) have a common fixed point.

**Proof.** It follows by taking \( \psi(t) = t \) in Theorem 4.

**Corollary 5.** Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Let \( T : X \to X \) be a monotone non-decreasing mapping. Suppose that \( T \) satisfies for all \( x, y \in X \), for which \( x \leq y \),

\[
\psi(d(Tx, Ty)) \leq \psi\left(\frac{1}{2} [d(x, Ty) + d(y, Tx)]\right) - \varphi(d(x, Ty), d(y, Tx)),
\]

where

(1) \( \psi : [0, +\infty) \to [0, +\infty) \) is an altering distance function,
(2) \( \varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is a continuous function with \( \varphi(x, y) = 0 \) if and only if \( x = y = 0 \).

Also suppose either

(i) \( \{ x_n \} \subset X \) is a non-decreasing sequence with \( x_n \to z \), then \( x_n \leq z \) for every \( n \), or
(ii) \( T \) is continuous.

If there exists \( x_0 \in X \) with \( x_0 \leq Tx_0 \), then \( T \) has a fixed point.
Proof. If (i) holds, then taking $f = I_X$ in Theorem 4, we get the result. If (ii) holds, then proceeding as in Theorem 4 with $f = I_X$, we can prove that \( \{Tx_n\} \) is a Cauchy sequence and

\[
x_n = \lim_{n \to +\infty} x_{n+1} = \lim_{n \to +\infty} T x_n = T(\lim_{n \to +\infty} x_n) = Tz.
\]

Hence the proof is completed. \( \square \)

**Corollary 6.** Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a monotone non-decreasing mapping. Suppose that $T$ satisfies for all $x, y \in X$, for which $x \leq y$,

\[
d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \varphi(d(x, Ty), d(y, Tx)), \tag{2.24}
\]

where $\varphi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function with $\varphi(x, y) = 0$ if and only if $x = y = 0$. Also, suppose either

(i) If \( \{x_n\} \subset X \) is a non-decreasing sequence with $x_n \to z$, then $x_n \leq z$ for every $n$, or

(ii) $T$ is continuous.

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then $T$ has a fixed point.

Proof. It follows by taking $\psi(t) = t$ in Corollary 5. \( \square \)

**Remark 2.** Corollary 6 corresponds to Theorem 2.1 and Theorem 2.2 of Harjani et al. \[8\].

**Corollary 7.** Let \((X, \leq)\) be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a monotone non-decreasing mapping. Suppose that $T$ satisfies for all $x, y \in X$, for which $x \leq y$,

\[
d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \tag{2.25}
\]

where $0 < k < \frac{1}{2}$. Also, suppose either

(i) If \( \{x_n\} \subset X \) is a non-decreasing sequence with $x_n \to z$, then $x_n \leq z$ for every $n$, or

(ii) $T$ is continuous.

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then $T$ has a fixed point.

Proof. It follows by taking $\varphi(t) = (\frac{1}{2} - k)t$ in Corollary 6. \( \square \)
REFERENCES


Author’s address

H. Aydi
Université de Sousse, Institut Supérieur d’Informatique et des Technologies de Communication de Hammam Sousse, Route GP1-4011, H. Sousse, Tunisia
E-mail address: hassen.aydi@isima.rnu.tn