Miskolc Mathematical Notes
Vol. 14 (2013), No 1, pp. 19-30

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2013.399

On common fixed point theorems for (ψ, ϕ)-generalized f-weakly contractive mappings

H. Aydi

Miskolc Mathematical Notes
Vol. 14 (2013), No. 1, pp. 19–30

HU e-ISSN 1787-2413

ON COMMON FIXED POINT THEOREMS FOR (ψ,φ)-GENERALIZED f-WEAKLY CONTRACTIVE MAPPINGS

H. AYDI

Received September 15, 2011

Abstract. In this paper, we present some common fixed point theorems for (ψ,φ)-generalized f-weakly contractive mappings in metric and ordered metric spaces. Our results extend, generalize and improve some well-known results in the literature. Also, we give an example to illustrate our results.

2000 *Mathematics Subject Classification:* 54H25; 47H10; 54E50

Keywords: common fixed point, commuting maps, f-weakly contractive maps, generalized f-weakly contractive maps, ordered metric space

1. Introduction and preliminaries

The first important result on fixed points for contractive type mapping was the much celebrated Banach's contraction principle by Banach [2] in 1922. After this, Kannan [9, 10] proved the following result:

Theorem 1. *Let (X,d) be a complete metric space. If $T : X \to X$ satisfies*

$$d(Tx,Ty) \le k[d(x,Tx)+d(y,Ty)],$$

where $0<k<\frac{1}{2}$ and $x,y \in X$, then T has a unique fixed point.

A similar type of contractive condition has been studied by Chatterjee [5] and he proved the following result:

Theorem 2. *Let (X,d) be a complete metric space. If $T : X \to X$ satisfies a C-contraction given as follows:*

$$d(Tx,Ty) \le k[d(x,Ty)+d(y,Tx)],$$

where $0<k<\frac{1}{2}$ and $x,y \in X$, then T has a unique fixed point.

Alber and Guerre-Delabriere [1] introduced the definition of weak Φ-contraction.

Definition 1. A self mapping T on a metric space X is called weak Φ-contraction if there exists a function $\Phi : [0,+\infty) \to [0,+\infty)$ such that for each $x,y \in X$,

$$d(Tx,Ty) \le d(x,y) - \Phi(d(x,y))).$$

The notion of Φ-contraction and weak Φ-contraction has been studied by many authors, see [3, 12, 15, 17, 19]. In recent years, many results related to fixed point theorems in partially ordered metric spaces are given, for more details see [8, 12–16].

Choudhury in [6] introduced a generalization of C-contraction given by the following definition.

Definition 2 ([6])**.** Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be weakly C-contractive (or a weak C-contraction) if for all $x, y \in X$,

$$d(Tx,Ty) \le \frac{1}{2}(d(x,Ty)+d(y,Tx)) - \varphi(d(x,Ty),d(y,Tx)),$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is a continuous function such that $\varphi(x,y) = 0$ if and only $x = y = 0$.

In [6] the author proves that if X is complete then every weak C-contraction has a unique fixed point. Recently, Harjani et al, [8] presented this last result in the context of ordered metric spaces.

Chandok [4] introduced the following definition : A map $T : X \to X$ is generalized f-weakly contractive if for each $x, y \in X$,

$$d(Tx,Ty) \le \frac{1}{2}(d(fx,Ty)+d(fy,Tx)) - \varphi(d(fx,Ty),d(fy,Tx)),$$

where $\varphi : [0,+\infty) \to [0,+\infty) \to [0,+\infty)$ is a continuous function such that $\varphi(x,y) = 0$ if and only $x = y = 0$.

If $f = I_X$, the identity mapping, then generalized f-weakly contractive mapping is weakly C-contractive.

Khan et al. [11] introduced the concept of altering distance function as follows:

Definition 3 (altering distance function, [11])**.** The function $\psi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is continuous and non-decreasing.
(2) $\psi(t) = 0$ if and only if $t = 0$.

Following the above definitions, we introduce the following:

Definition 4. A map $T : X \to X$ is called (ψ,φ)-generalized f-weakly contractive if for each $x, y \in X$,

$$\psi(d(Tx,Ty)) \le \psi(\frac{1}{2}[d(fx,Ty)+d(fy,Tx)]) - \varphi(d(fx,Ty),d(fy,Tx)),$$

where

(1) $\psi : [0,+\infty) \to [0,+\infty)$ is an altering distance function.
(2) $\varphi : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function with $\varphi(t,s) = 0$ if and only if $t = s = 0$.

If $\psi(t) = t$, then (ψ,φ)-generalized f-weakly contractive mapping is generalized f-weakly contractive.

The aim of this paper is to study some common fixed point theorems for (ψ,φ)-generalized f-weakly contractive in metric and ordered metric spaces.

2. Main results

First, we state the following known definition:

Definition 5. Let X a non-empty set. A point $x \in X$ is a coincidence point (common fixed point) of $f : X \to X$ and $T : X \to X$ if $fx = Tx$ ($x = fx = Tx$). The pair $\{f,T\}$ is called commuting if $Tfx = fTx$ for all $x \in X$.

We start with a common fixed point theorem for (ψ,φ)-generalized f-weakly contractive mappings in complete metric spaces.

Theorem 3. *Let (X,d) be a metric space. Let $f,T : X \to X$ satisfy $T(X) \subset f(X)$, $(f(X),d)$ is complete and*

$$\psi(d(Tx,Ty)) \le \psi(\frac{1}{2}[d(fx,Ty)+d(fy,Tx)]) - \varphi(d(fx,Ty),d(fy,Tx)), \tag{2.1}$$

for all $x,y \in X$, where

(1) *$\psi : [0,+\infty) \to [0,+\infty)$ is an altering distance function,*
(2) *$\varphi : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is a continuous function with $\varphi(t,s) = 0$ if and only if $t = s = 0$, then T and f have a coincidence point in X. Further, if T and f commute at their coincidence points, then T and f have a common fixed point.*

Proof. Let $x_0 \in X$. Since $T(X) \subset f(X)$, we can choose $x_1 \in X$, so that $fx_1 = Tx_0$. Since $Tx_1 \in f(X)$, there exists $x_2 \in X$ such that $fx_2 = Tx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$, for every $n \in \mathbb{N}$. By inequality (2.1), we have

$$\begin{aligned}
\psi(d(Tx_{n+1},Tx_n)) &\le \psi(\frac{1}{2}[d(fx_{n+1},Tx_n)+d(fx_n,Tx_{n+1})]) \\
&\quad - \varphi(d(fx_{n+1},Tx_n),d(fx_n,Tx_{n+1})) \\
&= \psi(\frac{1}{2}d(Tx_{n-1},Tx_{n+1})) - \varphi(0,d(Tx_{n-1},Tx_{n+1})) \\
&\le \psi(\frac{1}{2}d(Tx_{n-1},Tx_{n+1})) \\
&\le \psi(\frac{1}{2}[d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})]).
\end{aligned} \tag{2.2}$$

Since ψ is a non-decreasing function, we get that

$$d(Tx_n,Tx_{n+1}) \le d(Tx_{n-1},Tx_n) \quad \text{for any } n \in \mathbb{N}^*. \tag{2.3}$$

Thus, $\{d(Tx_n, Tx_{n+1})\}$ is a monotone non-increasing sequence of non-negative real numbers and hence is convergent. Hence there is $r \geq 0$ such that

$$\lim_{n\to+\infty} d(Tx_n, Tx_{n+1}) = r.$$

Using a triangular inequality, we have

$$d(Tx_{n+1}, Tx_n) \leq \frac{1}{2}d(Tx_{n-1}, Tx_{n+1}) \leq \frac{1}{2}[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})].$$

Letting $n \to +\infty$, we get

$$r \leq \frac{1}{2}\lim_{n\to+\infty} d(Tx_{n-1}, Tx_{n+1}) \leq \frac{1}{2}r + \frac{1}{2}r,$$

that is $\lim_{n\to+\infty} d(Tx_{n-1}, Tx_{n+1}) = 2r$. Using the continuity of ψ and φ, and inequality (2.2), we have, letting $n \to +\infty$

$$\psi(r) \leq \psi(r) - \varphi(0, 2r),$$

and consequently, $\varphi(0, 2r) \leq 0$. Thus, by a property of φ, $r = 0$, so

$$\lim_{n\to+\infty} d(Tx_{n+1}, Tx_n) = 0. \tag{2.4}$$

Now, we show that $\{Tx_n\}$ is a Cauchy sequence. If otherwise, then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k) > m(k) > k$ such that for every k,

$$d(Tx_{m(k)}, Tx_{n(k)}) \geq \varepsilon, \quad d(Tx_{m(k)}, Tx_{n(k)-1}) < \varepsilon. \tag{2.5}$$

By triangular inequality, we have from (2.5)

$$\begin{aligned}\varepsilon &\leq d(Tx_{m(k)}, Tx_{n(k)})\\ &\leq d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)})\\ &< \varepsilon + d(Tx_{n(k)-1}, Tx_{n(k)}).\end{aligned}$$

Using (2.4), we get

$$\lim_{k\to+\infty} d(Tx_{m(k)}, Tx_{n(k)}) = \lim_{k\to+\infty} d(Tx_{m(k)}, Tx_{n(k)-1}) = \varepsilon. \tag{2.6}$$

On the other hand,

$$\begin{aligned}d(Tx_{m(k)}, Tx_{n(k)-1}) \leq{}& d(Tx_{m(k)}, Tx_{m(k)-1}) + d(Tx_{m(k)-1}, Tx_{n(k)})\\ &+ d(Tx_{n(k)}, Tx_{n(k)-1}),\end{aligned}$$

and

$$d(Tx_{m(k)-1}, Tx_{n(k)}) \leq d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{m(k)}, Tx_{n(k)}).$$

Letting $k \to +\infty$ in the two above inequalities, we have thanks to (2.4) and (2.6),

$$\lim_{k\to+\infty} d(Tx_{m(k)-1}, Tx_{n(k)}) = \varepsilon. \tag{2.7}$$

From (2.1), we have

$$\begin{aligned}\psi(\varepsilon) &\le \psi(d(Tx_{m(k)},Tx_{n(k)}))\\ &\le \psi(\frac{1}{2}[d(fx_{m(k)},Tx_{n(k)})+d(fx_{n(k)},Tx_{m(k)})])\\ &\quad -\varphi(d(fx_{m(k)},Tx_{n(k)}),d(fx_{n(k)},Tx_{m(k)}))\\ &= \psi(\frac{1}{2}[d(Tx_{m(k)-1},Tx_{n(k)})+d(Tx_{n(k)-1},Tx_{m(k)})])\\ &\quad -\varphi(d(Tx_{m(k)-1},Tx_{n(k)}),d(Tx_{n(k)-1},Tx_{m(k)})).\end{aligned}$$

Taking $k\to+\infty$, using the continuity of ψ and φ, we obtain from (2.6), (2.7)

$$\psi(\varepsilon)\le\psi(\varepsilon)-\varphi(\varepsilon,\varepsilon),$$

hence $\varphi(\varepsilon,\varepsilon)=0$, so $\varepsilon=0$, it is a contradiction. Thus $\{Tx_n\}$ is a Cauchy sequence. Since $fx_n=Tx_{n-1}$, hence $\{fx_n\}$ is a Cauchy sequence in $(f(X),d)$, which is complete. Thus there is $z\in X$ such that

$$\lim_{n\to+\infty} fx_n = fz. \tag{2.8}$$

Moreover, (2.4) reads

$$\lim_{n\to+\infty} d(fx_n,fx_{n+1})=0. \tag{2.9}$$

By (2.1), we have

$$\begin{aligned}&\psi(d(Tz,fx_{n+1}))=\psi(d(Tz,Tx_n))\\ &\le\psi(\frac{1}{2}[d(fz,Tx_n)+d(fx_n,Tz)])-\varphi(d(fz,Tx_n),d(fx_n,Tz))\\ &=\psi(\frac{1}{2}[d(fz,fx_{n+1})+d(fx_n,Tz)])-\varphi(d(fz,fx_{n+1}),d(fx_n,Tz)),\end{aligned}$$

and letting $n\to+\infty$, using the continuity of ψ and φ and by (2.8), (2.9), we find

$$\psi(d(Tz,fz))\le\psi(\frac{1}{2}d(Tz,fz))-\varphi(0,d(fz,Tz))\le\psi(\frac{1}{2}d(Tz,fz)).$$

Consequently, $d(Tz,fz)\le\frac{1}{2}d(Tz,fz)$, that is, $d(Tz,fz)=0$, i,e. $Tz=fz$, hence z is a coincidence point of T and f. Now suppose that T and f commute at z. Let $w=Tz=fz$. Then $Tw=T(fz)=f(Tz)=fw$. By inequality (2.1)

$$\begin{aligned}\psi(d(Tz,Tw) &\le \psi(\frac{1}{2}[d(fz,Tw)+d(fw,Tz)])-\varphi(d(fz,Tw),d(fw,Tz))\\ &=\psi(\frac{1}{2}[d(Tz,Tw)+d(Tw,Tz)])-\varphi(d(Tz,Tw),d(Tw,Tz))\\ &=\psi(\frac{1}{2}[d(Tz,Tw)+d(Tw,Tz)])-\varphi(d(Tz,Tw),d(Tw,Tz))\\ &=\psi(d(Tz,Tw))-\varphi(d(Tz,Tw),d(Tw,Tz)).\end{aligned}$$

This implies that $d(Tz,Tw)=0$, by the property of φ. Therefore, $Tw=fw=w$. This completes the proof of Theorem 3. $\square$

Example 1. Let $X=[0,+\infty)$. Let d be defined by $d(x,y)=|x-y|$ for all $x,y\in X$. We set $fx=\frac{x}{2}$ and $Tx=\frac{x}{4}$ for all $x\in X$. It is clear that $T(X)\subset f(X)$ and $(f(X),d)$ is a complete metric space. Define $\psi:[0,+\infty)\to[0,+\infty)$ and $\varphi:[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ by

$$\psi(t)=\frac{t}{2}\quad\text{and}\quad\varphi(t,s)=\frac{1}{16}(t+s).$$

It is obvious that ψ and φ satisfy the hypotheses of Theorem 3. We need to show that the inequality (2.1) holds for any $x,y\in X$. First, the left-hand side of (2.1) is

$$\psi(d(Tx,Ty))=\frac{1}{8}|x-y|. \tag{2.10}$$

While, the right-hand side of (2.1) is

$$\begin{aligned}&\psi(\frac{1}{2}(d(fx,Ty)+d(fy,Tx))-\varphi(d(fx,Ty),d(fy,Tx)\\ &=\frac{1}{4}[|\frac{x}{2}-\frac{y}{4}|+|\frac{y}{2}-\frac{x}{4}|]-\frac{1}{16}[|\frac{x}{2}-\frac{y}{4}|+|\frac{y}{2}-\frac{x}{4}|]\\ &=\frac{3}{16}[|\frac{x}{2}-\frac{y}{4}|+|\frac{y}{2}-\frac{x}{4}|].\end{aligned} \tag{2.11}$$

By symmetry of (2.10) and (2.11), and without loss of generality, we suppose that $x\geq y$. In particular, (2.10) reads

$$\psi(d(Tx,Ty))=\frac{1}{8}(x-y).$$

We distinguish two cases:

• If $2y\geq x$. Here, we have from (2.11)

$$\begin{aligned}&\psi(\frac{1}{2}(d(fx,Ty)+d(fy,Tx))-\varphi(d(fx,Ty),d(fy,Tx)\\ &=\frac{3}{16}[|\frac{x}{2}-\frac{y}{4}|+|\frac{y}{2}-\frac{x}{4}|]=\frac{3}{16}[(\frac{x}{2}-\frac{y}{4})+(\frac{y}{2}-\frac{x}{4})]\\ &=\frac{3}{64}(x+y)\geq\frac{1}{8}(x-y)=\psi(d(Tx,Ty)).\end{aligned} \tag{2.12}$$

• If $2y<x$. Here, we have from (2.11)

$$\begin{aligned}&\psi(\frac{1}{2}(d(fx,Ty)+d(fy,Tx))-\varphi(d(fx,Ty),d(fy,Tx)\\ &=\frac{3}{16}[|\frac{x}{2}-\frac{y}{4}|+|\frac{y}{2}-\frac{x}{4}|]=\frac{3}{16}[(\frac{x}{2}-\frac{y}{4})+(-\frac{y}{2}+\frac{x}{4})]\\ &=\frac{9}{64}(x-y)\geq\frac{1}{8}(x-y)=\psi(d(Tx,Ty)).\end{aligned} \tag{2.13}$$

By (2.12) and (2.13), the inequality (2.1) is satisfied. Then by Theorem 3, T and f have a common fixed point, which is $z = 0$.

Corollary 1. *Let (X,d) be a complete metric space. If $T : X \to X$ satisfies*

$$\psi(d(Tx,Ty)) \leq \psi(\frac{1}{2}[d(x,Ty)+d(y,Tx)]) - \varphi(d(x,Ty),d(y,Tx)), \quad (2.14)$$

for all $x, y \in X$, where

(1) *$\psi : [0,+\infty) \to [0,+\infty)$ is an altering distance function,*
(2) *$\varphi : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is a continuous function with $\varphi(t,s) = 0$ if and only if $t = s = 0$, then T has a unique fixed point.*

Proof. It follows by taking $f = I_X$ in Theorem 3. The uniqueness of the fixed point follows by the following: suppose u and v are fixed points of T. By (2.14), we have

$$\begin{aligned}\psi(d(u,v)) &= \psi(d(Tu,Tv)) \\ &\leq \psi(\frac{1}{2}[d(u,Tv)+d(v,Tu)]) - \varphi(d(u,Tv),d(v,Tu)) \\ &= \psi(\frac{1}{2}[d(u,v)+d(v,u)]) - \varphi(d(u,v),d(v,u)) \\ &= \psi(d(u,v)) - \varphi(d(u,v),d(v,u)),\end{aligned}$$

which implies that $\varphi(d(u,v),d(v,u)) = 0$, and by a property of φ, we get $u = v$. □

Corollary 2. *Let (X,d) be a metric space. If $T, f : X \to X$ are such that $T(X) \subset f(X)$, $(f(X),d)$ is complete and*

$$d(Tx,Ty) \leq \frac{1}{2}[d(fx,Ty)+d(fy,Tx)] - \varphi(d(fx,Ty),d(fy,Tx)), \quad (2.15)$$

for all $x, y \in X$, where $\varphi : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is a continuous function with $\varphi(t,s) = 0$ if and only if $t = s = 0$, then T and f have a coincidence point in X. Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. The proof follows by taking $\psi(t) = t$ in Theorem 3. □

Corollary 3. *Let (X,d) be a complete metric space. If $T : X \to X$ satisfies for all $x, y \in X$*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \varphi(d(x,Ty),d(y,Tx)), \quad (2.16)$$

where $\varphi : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is a continuous function with $\varphi(t,s) = 0$ if and only if $t = s = 0$, then T has a unique fixed point.

Proof. It follows by taking $f = Id_X$ in Corollary 2. The uniqueness of the fixed point follows from Corollary 1. □

Remark 1.
- Corollary 1 corresponds to Corollary 2.1 of Shatanawi [18].
- Corollary 2 corresponds to Theorem 1 of Chandok [4].
- Corollary 3 corresponds to Theorem 2.1 of Choudhury [6].

Now, we extend Theorem 3 and we prove a common fixed point theorem for f-non-decreasing generalized nonlinear contraction mappings in the context of ordered metric spaces.

Definition 6 ([7]). Suppose $(X,\leq)$ is a partially ordered set and $T,f:X\to X$. T is said to be monotone f-nondecreasing if for all $x,y\in X$,

$$fx\leq fy \quad \text{implies} \quad Tx\leq Ty. \tag{2.17}$$

If $f=I_X$ in Definition 6, then T is monotone non-decreasing.

Theorem 4. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let f and T are self-mappings of X such that $T(X)\subset f(X)$, $f(X)$ is closed and T is f-non-decreasing mapping. Suppose that f and T satisfy for all $x,y\in X$, for which $f(x)\leq f(y)$*

$$\psi(d(Tx,Ty))\leq\psi(\frac{1}{2}[d(fx,Ty)+d(fy,Tx)])-\varphi(d(fx,Ty),d(fy,Tx)) \tag{2.18}$$

where

1. *$\psi:[0,+\infty)\to[0,+\infty)$ is an altering distance function,*
2. *$\varphi:[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is a continuous function with $\varphi(x,y)=0$ if and only if $x=y=0$.*

Also, suppose that if $\{f(x_n)\}\subset X$ is a non-decreasing sequence with $f(x_n)\to f(z)$ in $f(X)$, then $f(x_n)\leq f(z)$ and $f(z)\leq f(f(z))$ for every n.
If there exists $x_0\in X$ with $fx_0\leq Tx_0$, then T and f have a coincidence point. Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. Let $x_0\in X$ such that $fx_0\leq Tx_0$. Since $T(X)\subset f(X)$, we can choose $x_1\in X$, so that $fx_1=Tx_0$. Since $Tx_1\in f(X)$, there exists $x_2\in X$ such that $fx_2=Tx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that

$$fx_{n+1}=Tx_n.$$

Since $fx_0\leq Tx_0$, $Tx_0=fx_1$, so $fx_0\leq fx_1$. T is f-non-decreasing mapping, we get $Tx_0\leq Tx_1$. Similarly $fx_1\leq fx_2$, $Tx_1\leq Tx_2$, hence $fx_2\leq fx_3$. Continuing, we obtain

$$fx_0\leq fx_1\leq fx_2\leq\ldots\leq fx_n\leq fx_{n+1}\leq\ldots$$

If for some n, $Tx_{n+1}=Tx_n$, then $Tx_{n+1}=fx_{n+1}$, i.e. T and f have a coincidence point x_{n+1}, and so we have the result. For the rest, assume that $d(Tx_n,Tx_{n+1})>0$

for all $n \in \mathbb{N}$. By (2.18), we have

$$\begin{aligned}\psi(d(Tx_n, Tx_{n+1})) &\leq \psi(\frac{1}{2}[d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})]) \\ &\quad - \varphi(d(fx_{n+1}, Tx_n), d(fx_n, Tx_{n+1})) \\ &= \psi(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1})) - \varphi(0, d(Tx_{n-1}, Tx_{n+1})) \\ &\leq \psi(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1})) \\ &\leq \psi(\frac{1}{2}d(Tx_{n-1}, Tx_n) + \frac{1}{2}d(Tx_n, Tx_{n+1})).\end{aligned}$$

It follows that, for any $n \in \mathbb{N}^*$

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n).$$

Thus $\{d(Tx_n, Tx_{n+1})\}$ is a monotone non-increasing sequence, hence it is convergent. Now, proceeding as in Theorem 3, we can prove that

$$\lim_{n\to+\infty} d(Tx_n, Tx_{n+1}) = 0. \tag{2.19}$$

Moreover, $\{Tx_n\}$ is a Cauchy sequence. Since $Tx_n = fx_{n+1}$ and $f(X)$ is closed, so there exists $z \in X$ such that

$$\lim_{n\to+\infty} fx_n = fz. \tag{2.20}$$

Having in mind $\{fx_n\}$ is a non-decreasing sequence, so by (2.20) we have for every $n \in \mathbb{N}$

$$fx_n \leq fz, \quad f(z) \leq f(fz). \tag{2.21}$$

Having $fx_n \leq fz$, so from inequality (2.18), we have

$$\begin{aligned}&\psi(d(fx_{n+1}, Tz)) = \psi(d(Tx_n, Tz)) \\ &\leq \psi(\frac{1}{2}[d(fz, Tx_n) + d(fx_n, Tz)]) - \varphi(d(fz, Tx_n), d(fx_n, Tz)) \\ &= \psi(\frac{1}{2}[d(fz, fx_{n+1}) + d(fx_n, Tz)] - \varphi(d(fz, fx_{n+1}), d(fx_n, Tz)).\end{aligned}$$

Taking $n \to +\infty$, using the continuity of ψ and φ, we get from (2.19), (2.20)

$$\psi(d(Tz, fz)) \leq \psi(\frac{1}{2}d(fz, fz)) - \varphi(0, d(fz, Tz)),$$

that is, $d(Tz, fz) = 0$, hence $Tz = fz$, so z is a coincidence point of T and f.

Now suppose that T and f commute at z. Let $w = Tz = fz$. Then $Tw = T(fz) = f(Tz) = fw$. From (2.21), we have $fz \leq f(fz) = fw$, so the inequality (2.18) gives us

$$\psi(d(Tz, Tw) \leq \psi(\frac{1}{2}[d(fz, Tw) + d(fw, Tz)]) - \varphi(d(fz, Tw), d(fw, Tz))$$

$$= \psi(\frac{1}{2}[d(Tz,Tw)+d(Tw,Tz)]) - \varphi(d(Tz,Tw),d(Tw,Tz))$$
$$= \psi(\frac{1}{2}[d(Tz,Tw)+d(Tw,Tz)]) - \varphi(d(Tz,Tw),d(Tw,Tz))$$
$$= \psi(d(Tz,Tw)) - \varphi(d(Tz,Tw),d(Tw,Tz)).$$

This implies that $d(Tz,Tw) = 0$, by the property of φ. Therefore, $Tw = fw = w$. This completes the proof of Theorem 4. □

Corollary 4. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let f and T are self-mappings of X such that $T(X) \subset f(X)$, $f(X)$ is closed and T is f-non-decreasing mapping. Assume that f and T satisfy for all $x,y \in X$, for which $f(x) \leq f(y)$*

$$d(Tx,Ty) \leq \frac{1}{2}[d(fx,Ty)+d(fy,Tx)] - \varphi(d(fx,Ty),d(fy,Tx)), \tag{2.22}$$

where $\varphi : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is a continuous function with $\varphi(x,y) = 0$ if and only if $x = y = 0$.
Also, suppose that if $\{f(x_n)\} \subset X$ is a non-decreasing sequence with $f(x_n) \to f(z)$ in $f(X)$, then $f(x_n) \leq f(z)$ and $f(z) \leq f(f(z))$ for every n.
If there exists $x_0 \in X$ with $fx_0 \leq Tx_0$, then T and f have a coincidence point. Further, if T and f commute at their coincidence points, then T and f have a common fixed point.

Proof. It follows by taking $\psi(t) = t$ in Theorem 4. □

Corollary 5. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T : X \to X$ be a monotone non-decreasing mapping. Suppose that T satisfies for all $x,y \in X$, for which $x \leq y$,*

$$\psi(d(Tx,Ty)) \leq \psi(\frac{1}{2}[d(x,Ty)+d(y,Tx)]) - \varphi(d(x,Ty),d(y,Tx)), \tag{2.23}$$

where

(1) *$\psi : [0,+\infty) \to [0,+\infty)$ is an altering distance function,*
(2) *$\varphi : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is a continuous function with $\varphi(x,y) = 0$ if and only if $x = y = 0$.*

Also suppose either

(i) *$\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$, then $x_n \leq z$ for every n, or*
(ii) *T is continuous.*

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. If (i) holds, then taking $f = I_X$ in Theorem 4, we get the result.
If (ii) holds, then proceeding as in Theorem 4 with $f = I_X$, we can prove that $\{Tx_n\}$ is a Cauchy sequence and

$$z = \lim_{n\to+\infty} x_{n+1} = \lim Tx_n = T(\lim_{n\to+\infty} x_n) = Tz.$$

Hence the proof is completed. □

Corollary 6. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T: X \to X$ be a monotone non-decreasing mapping. Suppose that T satisfies for all $x,y \in X$, for which $x \leq y$,*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \varphi(d(x,Ty),d(y,Tx)), \tag{2.24}$$

where $\varphi : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is a continuous function with $\varphi(x,y) = 0$ if and only if $x = y = 0$.
Also, suppose either

(i) *If $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$, then $x_n \leq z$ for every n, or*
(ii) *T is continuous.*

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. It follows by taking $\psi(t) = t$ in Corollary 5. □

Remark 2. Corollary 6 corresponds to Theorem 2.1 and Theorem 2.2 of Harjani et al. [8].

Corollary 7. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T: X \to X$ be a monotone non-decreasing mapping. Suppose that T satisfies for all $x,y \in X$, for which $x \leq y$,*

$$d(Tx,Ty) \leq k[d(x,Ty)+d(y,Tx)], \tag{2.25}$$

where $0 < k < \frac{1}{2}$.
Also, suppose either

(i) *If $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$, then $x_n \leq z$ for every n, or*
(ii) *T is continuous.*

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. It follows by taking $\varphi(t) = (\frac{1}{2} - k)t$ in Corollary 6. □

REFERENCES

[1] Y. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," in *New results in operator theory and its applications: the Israel M. Glazman memorial volume*, ser. Oper. Theory, Adv. Appl., I. Gohberg, Ed. Basel: Birkhäuser, 1997, vol. 98, pp. 7–22.

[2] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta math.*, vol. 3, pp. 133–181, 1922.

[3] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proc. Am. Math. Soc.*, vol. 20, pp. 458–464, 1969.

[4] S. Chandok, "Some common fixed point theorems for generalized f-weakly contractive mappings," *J. Appl. Math. Inform.*, vol. 29, no. 1-2, pp. 257–265, 2011.

[5] S. K. Chatterjee, "Fixed point theorem," *C. R. Acad. Bulgare Sci.*, vol. 25, pp. 727–730, 1972.

[6] B. S. Choudhury, "Unique fixed point theorem for weakly C-contractive mappings," *Kathmandu University J. Sci. Engg. Tech.*, vol. 5, pp. 6–13, 2009.

[7] L. Ćirić, N. Cakić, M. Rajović, and J. S. Ume, "Monotone generalized nonlinear contractions in partially ordered metric spaces," *Fixed Point Theory Appl.*, vol. 2008, p. 11, 2008.

[8] J. Harjani, B. López, and K. Sadarangani, "Fixed point theorems for weakly c-contractive mappings in ordered metric spaces," *Comput. Math. Appl.*, vol. 61, no. 4, pp. 790–796, 2011.

[9] R. Kannan, "Some results on fixed points. ii." *Am. Math. Mon.*, vol. 76, pp. 405–408, 1969.

[10] R. Kannan, "Some results on fixed points," *Bull. Calcutta Math. Soc.*, vol. 60, pp. 71–76, 1986.

[11] M. S. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points," *Bull. Aust. Math. Soc.*, vol. 30, pp. 1–9, 1984.

[12] H. K. Nashine and B. Samet, "Fixed point results for mappings satisfying (ψ,φ)-weakly contractive condition in partially ordered metric spaces," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 74, no. 6, pp. 2201–2209, 2011.

[13] J. J. Nieto, R. L. Pouso, and R. Rodríguez-López, "Fixed point theorems in ordered abstract spaces," *Proc. Am. Math. Soc.*, vol. 135, no. 8, pp. 2505–2517, 2007.

[14] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[15] O. Popescu, "Fixed points for (ψ,φ)-weak contractions," *Appl. Math. Lett.*, vol. 24, no. 1, pp. 1–4, 2011.

[16] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proc. Am. Math. Soc.*, vol. 132, no. 5, pp. 1435–1443, 2004.

[17] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Anal., Theory Methods Appl.*, vol. 47, no. 4, pp. 2683–2693, 2001.

[18] W. Shatanawi, "Fixed point theorems for nonlinear weakly-contractive mappings in metric spaces," *Math. Comput. Modelling*, vol. 54, no. 11-12, pp. 2816–2826, 2011.

[19] Q. Zhang and Y. Song, "Fixed point theory for generalized φ-weak contractions," *Appl. Math. Lett.*, vol. 22, no. 1, pp. 75–78, 2009.

Author's address

H. Aydi

Université de Sousse, Institut Supérieur d'Informatique et des Technologies de Communication de Hammam Sousse, Route GP1-4011, H. Sousse, Tunisia

E-mail address: hassen.aydi@isima.rnu.tn