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ON THE VON KARMAN'S EQUATION IN THE NONLINEAR THEORY OF GAS DYNAMICS

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Abstract. In this paper, by using invariants of Riemann and general solutions of Euler–Poisson–Darboux–Riemann equations, a new class of exact solutions of von Karman's equation in the nonlinear theory of gas dynamics is given.

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In the plane of independent variables x and y, consider quasilinear von Karman's equation arising in a variety of physical problems such as nonlinear vibrations and irrotational transonic flows of baritropic gas (see [1–5, 7, 13])

$$(u_x)^\alpha u_{xx} - u_{yy} = 0. \tag{1}$$

Equation (1) is considered in the class of hyperbolic solutions, which in this case is determined by the condition

$$u_x > 0. \tag{2}$$

Let

$$m := \frac{\alpha}{2(\alpha+2)}, \quad -2 \neq \alpha \in \mathbb{R} := (-\infty, +\infty). \tag{3}$$

Theorem 1. *If the condition $m \in \mathbb{N} := \{1,2,3,\dots\}$ is fulfilled, then the general classical solution $u \in C^2$ of equation (1) is given by the formulas*

$$\begin{cases} x = (X-Y)^{2m+1} \dfrac{\partial^{2m}}{\partial X^m \partial Y^m} \dfrac{F(X)-G(Y)}{X-Y}, \\ y = m[2(1-2m)]^{2m} \dfrac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}} \dfrac{F'(X)-G'(Y)}{X-Y}, \\ u = m[2(1-2m)]^{2m} \Bigg[\left(\dfrac{m-1}{2m-1} X + \dfrac{m}{2m-1} Y \right) \dfrac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}} \dfrac{F'(X)-G'(Y)}{X-Y} \\ \qquad - \dfrac{m-1}{2m-1} \dfrac{\partial^{2m-3}}{\partial X^{m-2}\partial Y^{m-1}} \dfrac{F'(X)-G'(Y)}{X-Y} \Bigg] \quad \textit{for} \quad m = 2,3,\dots \end{cases} \tag{4}$$

and

$$\begin{cases} x = -2[F(X) - G(Y)] + [F'(X) + G'(Y)](X - Y), \\ y = \dfrac{4[F'(X) - G'(Y)]}{X - Y}, \\ u = \dfrac{4[YF'(X) - XG'(Y)]}{X - Y} \quad \textit{for} \quad m = 1. \end{cases} \tag{5}$$

Here $F, G \in C^{m+1}$ are arbitrary functions with respect to the variables X and Y, respectively.

Proof. Let us find the Riemann invariants, that is, what is conserved along the corresponding characteristic curves of equation (1) (see [8,9]). For any $u \in C^2$, classical solution of equation (1), introduce the designation of Monge-Ampere:

$$p := u_x, \;\; q := u_y, \;\; r := u_{xx}, \;\; s := u_{xy}, \;\; t := u_{yy}.$$

In this notation equation (1) has the form

$$t = p^{\alpha} r. \tag{6}$$

Due to (2), equation (1) along each regular solution is hyperbolic and therefore two families of characteristic curves pass through any fixed point of the plane of independent variables x and y. Along one of these curves, for example, the one whose equation has the form

$$dx = -p^{\frac{\alpha}{2}} dy,$$

by virtue of equality (6), we have

$$dp = rdx + sdy = (-p^{\frac{\alpha}{2}} r + s)dy$$

and

$$dq = sdx + tdy = (-p^{\frac{\alpha}{2}} s + p^{\alpha} r)dy.$$

Whence it follows that

$$dq + p^{\frac{\alpha}{2}} dp = 0$$

and hence

$$q + \frac{2}{\alpha + 2} p^{\frac{\alpha+2}{2}} = C_1.$$

Similarly, along the family of other characteristic curves, whose equation has the form

$$dx = p^{\frac{\alpha}{2}} dy,$$

we obtain

$$q - \frac{2}{\alpha + 2} p^{\frac{\alpha+2}{2}} = C_2.$$

Here C_i, $i = 1, 2$, are arbitrary constants.

Let us introduce the Riemann invariants of equation (1) as independent variables

$$\begin{cases} X = q + \frac{2}{\alpha+2} p^{\frac{\alpha+2}{2}}, \\ Y = q - \frac{2}{\alpha+2} p^{\frac{\alpha+2}{2}}, \end{cases} \tag{7}$$

in terms of which equation (1) can be rewritten in the form of a system of equations of the first order [5, 8, 9]

$$\begin{cases} X_y + X_x \frac{dx}{dy} = X_y - p^{\frac{\alpha}{2}} X_x = 0, \\ Y_y + Y_x \frac{dx}{dy} = Y_y + p^{\frac{\alpha}{2}} Y_x = 0. \end{cases} \tag{8}$$

In system (8), we choose X and Y as independent variables, while $x(X,Y)$ and $y(X,Y)$ as unknown functions. Applying the formulas of differentiation of implicit functions of two variables

$$x_X = DY_y, \quad x_Y = -DX_y, \quad y_X = -DY_x, \quad y_Y = DX_x,$$

where $D := \frac{D(x,y)}{D(X,Y)}$ is the Jacobian of transformation, from system (8) we obtain

$$\begin{cases} x_Y + p^{\frac{\alpha}{2}} y_Y = 0, \\ x_X - p^{\frac{\alpha}{2}} y_X = 0. \end{cases} \tag{9}$$

Here

$$p^{\frac{\alpha}{2}} = \left[\frac{X-Y}{2(1-2m)} \right]^{2m}, \tag{10}$$

due to (2), (3) and (7).

Eliminating the function $y(X,Y)$ from system (9), we receive that the function $x(X,Y)$ satisfies the Euler–Poisson–Darboux–Riemann equation (see [4, 11])

$$x_{XY} + \frac{m}{X-Y} x_X - \frac{m}{X-Y} x_Y = 0. \tag{11}$$

By a similar way, for the function $y(X,Y)$ we get

$$y_{XY} - \frac{m}{X-Y} y_X + \frac{m}{X-Y} y_Y = 0. \tag{12}$$

First, let us get the general solution of equation

$$w_{XY} + \frac{n}{X-Y} w_X - \frac{m}{X-Y} w_Y = 0 \tag{13}$$

for all $n, m \in \mathbb{N}$.

Denote by $z(\alpha, \beta)$ a solution of equation

$$z_{XY} - \frac{\beta}{X-Y} z_X + \frac{\alpha}{X-Y} z_Y = 0,$$

then we have

$$z(\alpha, \beta) = (X-Y)^{1-\alpha-\beta} z(1-\beta, 1-\alpha). \tag{14}$$

(see, for example, [10, 11]).

Putting $\alpha = \alpha' + m$, $\beta = \beta' + n$ in (14) and taking into account the equality

$$z(\alpha' + m, \beta' + n) = \frac{\partial^{m+n} z(\alpha', \beta')}{\partial X^m \partial Y^n},$$

we get

$$(X-Y)^{1-m-n-\alpha'-\beta'} z(1-\beta'-n, 1-\alpha'-m) = \frac{\partial^{m+n} z(\alpha', \beta')}{\partial X^m \partial Y^n}.$$

Let us use again formula (14), then we have

$$(X-Y)^{1-m-n-\alpha'-\beta'} z(1-\beta'-n, 1-\alpha'-m) = \frac{\partial^{m+n}}{\partial X^m \partial Y^n}\left[\frac{z(1-\beta', 1-\alpha')}{(X-Y)^{\alpha'+\beta'-1}}\right].$$

Replacing α', β', m, n, respectively, by $1-\beta'$, $1-\alpha'$, n, m, we obtain

$$z(\alpha'-m, \beta'-n) = (X-Y)^{m+n+1-\alpha'-\beta'} \frac{\partial^{m+n}}{\partial X^m \partial Y^n}\left[\frac{z(\alpha', \beta')}{(X-Y)^{1-\alpha'-\beta'}}\right].$$

Putting $\alpha' = \beta' = 0$, we get the solution of equation (13) by the formula

$$w(X,Y) = z(-m,-n) = (X-Y)^{m+n+1} \frac{\partial^{m+n}}{\partial X^m \partial Y^n}\left[\frac{F_1(X) - G_1(Y)}{X-Y}\right], \tag{15}$$

where $F_1(X)$ and $G_1(Y)$ are the arbitrary functions of its arguments.

Let us now get a general solution of equation

$$w_{XY} - \frac{n}{X-Y} w_X + \frac{m}{X-Y} w_Y = 0. \tag{16}$$

First, we obtain a general solution to the equation

$$(X-Y)v_{XY} - v_X + v_Y = 0. \tag{17}$$

Proposition 1. *The general solution of equation* (17) *can be represented by the formula*

$$v(X,Y) = \frac{F_2(X) - G_2(Y)}{X-Y}, \tag{18}$$

where $F_2(X)$ and $G_2(Y)$ are the arbitrary functions of its arguments.

Proof. Indeed, let us introduce the function $\chi := (X-Y)v$. Then by (17), the function χ satisfies the equation $\chi_{XY} = 0$, from which (18) immediately follows. □

Proposition 2. *The general solution of equation* (16) *has the form*

$$w(X,Y) = \frac{\partial^{m+n-2}}{\partial X^{m-1} \partial Y^{m-1}} \frac{F_2(X) - G_2(Y)}{X-Y}. \tag{19}$$

Proof. It is easy to show that the following equalities are valid:

$$\frac{\partial^m}{\partial X^m}[(X-Y)v] = m\frac{\partial^{m-1} v}{\partial X^{m-1}} + (X-Y)\frac{\partial^m v}{\partial X^m} \tag{20}$$

and

$$\frac{\partial^n}{\partial Y^n}[(X-Y)v] = -n\frac{\partial^{n-1}v}{\partial Y^{n-1}} + (X-Y)\frac{\partial^n v}{\partial Y^n}. \tag{21}$$

Indeed, let us prove equality (20) by induction. For $m=1$, we have that

$$\frac{\partial}{\partial X}[(X-Y)v] = v + (X-Y)\frac{\partial v}{\partial X}$$

is true.

Suppose that (20) is true for m, and show for $m+1$. We have

$$\begin{aligned}\frac{\partial^{m+1}}{\partial X^{m+1}}[(X-Y)v] &= \frac{\partial}{\partial X}\left[m\frac{\partial^{m-1}v}{\partial X^{m-1}} + (X-Y)\frac{\partial^m v}{\partial X^m}\right] \\ &= m\frac{\partial^m v}{\partial X^m} + \frac{\partial^m v}{\partial X^m} + (X-Y)\frac{\partial^{m+1}v}{\partial X^{m+1}} \\ &= (m+1)\frac{\partial^m v}{\partial X^m} + (X-Y)\frac{\partial^{m+1}v}{\partial X^{m+1}}\end{aligned}$$

and therefore (20) is true. The validity of (21) can be shown analogously.

Differentiating equation (17) $(m-1)$ times with respect to the variable x and $(n-1)$ times with respect to the variable y, respectively, by virtue of (20) and (21), we have

$$\begin{aligned}0 &= \frac{\partial^{m+n-2}}{\partial X^{m-1}\partial Y^{n-1}}\left[(X-Y)v_{XY} - v_X + v_Y\right] \\ &= \frac{\partial^{n-1}}{\partial Y^{n-1}}\left[(m-1)\frac{\partial^{m-2}v_{XY}}{\partial X^{m-2}} + (X-Y)\frac{\partial^{m-1}v_{XY}}{\partial X^{m-1}}\right] - \frac{\partial^{m+n-2}v_X}{\partial X^{m-1}\partial Y^{n-1}} + \frac{\partial^{m+n-2}v_Y}{\partial X^{m-1}\partial Y^{n-1}} \\ &= (m-1)\frac{\partial^{m+n-3}v_{XY}}{\partial X^{m-2}\partial Y^{n-1}} - (n-1)\frac{\partial^{m+n-3}v_{XY}}{\partial X^{m-1}\partial Y^{n-2}} + (X-Y)\frac{\partial^{m+n-2}v_{XY}}{\partial X^{m-1}\partial Y^{n-1}} \\ &\quad - \frac{\partial^{m+n-2}v_X}{\partial X^{m-1}\partial Y^{n-1}} + \frac{\partial^{m+n-2}v_Y}{\partial X^{m-1}\partial Y^{n-1}} \\ &= m\frac{\partial^{m+n-2}v_Y}{\partial X^{m-1}\partial Y^{n-1}} - n\frac{\partial^{m+n-2}v_X}{\partial X^{m-1}\partial Y^{n-1}} + (X-Y)\frac{\partial^{m+n-2}v_{XY}}{\partial X^{m-1}\partial Y^{n-1}}.\end{aligned}$$

Introduce the notation

$$w := \frac{\partial^{m+n-2}v}{\partial X^{m-1}\partial Y^{n-1}}. \tag{22}$$

Due the last equality, it is easy to see that the function $w(X,Y)$, taking into account (18), can be represented by formula (19) and satisfies equation (16).

Finally, considering (15), (18) and (22) for $n=m$, we get that the general solutions of equations (11) and (12) under the conditions of Theorem 1 have the following form (see [10, 12])

$$x = (X-Y)^{2m+1}\frac{\partial^{2m}}{\partial X^m \partial Y^m}\frac{F_1(X)-G_1(Y)}{X-Y}, \tag{23}$$

$$y=\frac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}}\frac{F_2(X)-G_2(Y)}{X-Y}, \tag{24}$$

respectively. Here $F_1, G_1 \in C^{m+2}$ and $F_2, G_2 \in C^{m+1}$ are arbitrary functions. □

The following statement is true.

Proposition 3. *For equalities (9) to hold, it is necessary and sufficient that the equalities*

$$F_2(X)=m[2(1-2m)]^{2m}F_1'(X),\quad G_2(Y)=m[2(1-2m)]^{2m}G_1'(Y) \tag{25}$$

were satisfied.

Proof. Necessity. Obviously, the equalities

$$\frac{\partial^{2m}}{\partial X^m\partial Y^m}\frac{F_1(X)-G_1(Y)}{X-Y}=\frac{\partial^{2m}}{\partial X^m\partial Y^m}\frac{F_1(X)}{X-Y}+\frac{\partial^{2m}}{\partial Y^m\partial X^m}\frac{G_1(Y)}{Y-X} \tag{26}$$

and

$$\frac{\partial^j}{\partial X^j}\frac{1}{(X-Y)^k}=\frac{(-1)^j k(k+1)\cdots(k+j-1)}{(X-Y)^{k+j}}\quad \forall j,k\in\mathbb{N} \tag{27}$$

are valid.

In particular, from (27) it follows that for $j=m$, $k=1$,

$$\frac{\partial^m}{\partial X^m}\frac{1}{X-Y}=\frac{(-1)^m m!}{(X-Y)^{m+1}} \tag{28}$$

and for $j=m-i$, $k=m+1$,

$$\frac{\partial^{m-i}}{\partial X^{m-i}}\frac{1}{(X-Y)^{m+1}}=\frac{(-1)^{m-i}(m+1)(m+2)\cdots(2m-i)}{(X-Y)^{2m-i+1}}. \tag{29}$$

Further, due to (28) and (29), we obtain

$$\begin{aligned}\frac{\partial^{2m}}{\partial X^m\partial Y^m}\frac{F_1(X)}{X-Y}&=-\frac{\partial^m}{\partial X^m}\left[F_1(X)\frac{\partial^m}{\partial Y^m}\frac{1}{Y-X}\right]=m!\frac{\partial^m}{\partial X^m}\left[F_1(X)\frac{1}{(X-Y)^{m+1}}\right]\\&=m!\sum_{i=0}^m C_m^i F_1^{(i)}(X)\frac{\partial^{m-i}}{\partial X^{m-i}}\frac{1}{(X-Y)^{m+1}}\\&=\sum_{i=0}^m C_m^i(-1)^{m-i}(2m-i)!\frac{F_1^{(i)}(X)}{(X-Y)^{2m-i+1}}.\end{aligned} \tag{30}$$

Analogously,

$$\frac{\partial^{2m}}{\partial Y^m\partial X^m}\frac{G_1(Y)}{Y-X}=(-1)^{m+1}\sum_{i=0}^m C_m^i(2m-i)!\frac{G_1^{(i)}(Y)}{(X-Y)^{2m-i+1}}. \tag{31}$$

Therefore, due to (23), (24), (26), (30) and (31), we have

$$\begin{aligned} x(X,Y) = &\sum_{i=0}^{m} C_m^i (-1)^{m-i}(2m-i)!F_1^{(i)}(X)(X-Y)^i \\ &+(-1)^{m+1}\sum_{i=0}^{m} C_m^i (2m-i)!G_1^{(i)}(Y)(X-Y)^i \end{aligned} \tag{32}$$

and

$$\begin{aligned} y(X,Y) = &\sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i-1}(2m-i-2)!\frac{F_2^{(i)}(X)}{(X-Y)^{2m-i-1}} \\ &+(-1)^{m}\sum_{i=0}^{m-1} C_{m-1}^i (2m-i-2)!\frac{G_2^{(i)}(Y)}{(X-Y)^{2m-i-1}}. \end{aligned} \tag{33}$$

Differentiating equalities (32) and (33) with respect to the variable X, we get

$$\begin{aligned} x_X = &\sum_{i=0}^{m} C_m^i (-1)^{m-i}(2m-i)!F_1^{(i+1)}(X)(X-Y)^i \\ &+\sum_{i=0}^{m-1} C_m^{i+1} (-1)^{m-i-1}(i+1)(2m-i-1)!F_1^{(i+1)}(X)(X-Y)^i \\ &+(-1)^{m+1}\sum_{i=0}^{m-1} C_m^{i+1}(i+1)(2m-i-1)!G_1^{(i+1)}(Y)(X-Y)^i \end{aligned} \tag{34}$$

and

$$\begin{aligned} y_X = &\sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i-1}(2m-i-2)!\frac{F_2^{(i+1)}(X)}{(X-Y)^{2m-i-1}} \\ &+\sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i}(2m-i-1)!\frac{F_2^{(i)}(X)}{(X-Y)^{2m-i}} \\ &+(-1)^{m+1}\sum_{i=0}^{m-1} C_{m-1}^i (2m-i-1)!\frac{G_2^{(i)}(Y)}{(X-Y)^{2m-i}}, \end{aligned} \tag{35}$$

respectively.

Substituting (34) and (35) in the second equality of (9), we obtain

$$
\begin{aligned}
&\sum_{i=0}^{m} C_m^i (-1)^{m-i} (2m-i)! F_1^{(i+1)}(X)(X-Y)^i \\
&\quad + \sum_{i=0}^{m-1} C_m^{i+1} (-1)^{m-i-1} (i+1)(2m-i-1)! F_1^{(i+1)}(X)(X-Y)^i \\
&\quad + (-1)^{m+1} \sum_{i=0}^{m-1} C_m^{i+1} (i+1)(2m-i-1)! G_1^{(i+1)}(Y)(X-Y)^i \\
&= [2(1-2m)]^{-2m} \Bigg[\sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i-1} (2m-i-2)! F_2^{(i+1)}(X)(X-Y)^{i+1} \\
&\quad + \sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X)(X-Y)^i \\
&\quad + (-1)^{m+1} \sum_{i=0}^{m-1} C_{m-1}^i (2m-i-1)! G_2^{(i)}(Y)(X-Y)^i \Bigg]. \qquad (36)
\end{aligned}
$$

First, equating in (36) the expressions before the functions F_1 and F_2 and their derivatives, we obtain

$$
\begin{aligned}
&\sum_{i=0}^{m} C_m^i (-1)^{m-i} (2m-i)! F_1^{(i+1)}(X)(X-Y)^i \\
&\quad + \sum_{i=0}^{m-1} C_m^{i+1} (-1)^{m-i-1} (i+1)(2m-i-1)! F_1^{(i+1)}(X)(X-Y)^i \\
&= [2(1-2m)]^{-2m} \Bigg[\sum_{i=1}^{m} C_{m-1}^{i-1} (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X)(X-Y)^i \\
&\quad + \sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X)(X-Y)^i \Bigg].
\end{aligned}
$$

From here

$$
\begin{aligned}
&C_m^i (-1)^{m-i} (2m-i)! F_1^{(i+1)}(X) + C_m^{i+1} (-1)^{m-i-1} (i+1)(2m-i-1)! F_1^{(i+1)}(X) \\
&= [2(1-2m)]^{-2m} \Big[C_{m-1}^{i-1} (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X) \\
&\quad + C_{m-1}^i (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X) \Big],
\end{aligned}
$$

for $i = 1, 2, \ldots, m-1$.

Consequently,

$$
F_2^{(i)}(X) = m[2(1-2m)]^{2m} F_1^{(i+1)}(X), \quad i = 1, 2, \ldots, m-1. \qquad (37)
$$

Let us now equate together the present members for $i=0$ and $i=m$, we get

$$C_m^0(-1)^m(2m)!F_1'(X)+C_m^m m!F_1^{(m+1)}(X)(X-Y)^m+C_m^1(-1)^{m-1}(2m-1)!F_1'(X)$$
$$=[2(1-2m)]^{-2m}\Big[C_{m-1}^{m-1}(m-1)!F_2^{(m)}(X)(X-Y)^m+C_{m-1}^0(-1)^m(2m-1)!F_2(X)\Big].$$

Hence

$$F_2(X)=m[2(1-2m)]^{2m}F_1'(X) \quad \text{and} \quad F_2^{(m)}(X)=m[2(1-2m)]^{2m}F_1^{(m+1)}(X).$$

Let us equate the expressions before the functions G_1 and G_2 and their derivatives, we obtain

$$\sum_{i=0}^{m-1}C_m^{i+1}(i+1)(2m-i-1)!G_1^{(i+1)}(Y)(X-Y)^i$$
$$=[2(1-2m)]^{-2m}\sum_{i=0}^{m-1}C_{m-1}^i(2m-i-1)!G_2^{(i)}(Y)(X-Y)^i,$$

this immediately implies that

$$G_2^{(i)}(Y)=m[2(1-2m)]^{2m}G_1^{(i+1)}(Y), \quad i=0,1,\dots,m-1.$$

Consequently, taking into account (37) and the last equality, (25) is shown.

Remark 1. We obtain the same equality if we differentiate equalities (32) and (33) with respect to the variable Y and substitute them in the first equality of (9), and equate the expressions before the functions F_1, F_2 and G_1, G_2 and their derivatives, respectively.

Sufficiency. Let conditions (25) be fulfilled and show that equalities (9) hold. Indeed, by virtue of the obvious equality

$$\frac{F_1'(X)-G_1'(Y)}{X-Y}=\frac{\partial}{\partial X}\frac{F_1(X)-G_1(Y)}{X-Y}+\frac{\partial}{\partial Y}\frac{F_1(X)-G_1(Y)}{X-Y},$$

(10) and conditions (23)–(25), we have

$$x_X-p^{\frac{\alpha}{2}}y_X=(X-Y)^{2m}\Bigg[(2m+1)\frac{\partial^{2m}}{\partial X^m\partial Y^m}\frac{F_1(X)-G_1(Y)}{X-Y}$$
$$+(X-Y)\frac{\partial^{2m+1}}{\partial X^{m+1}\partial Y^m}\frac{F_1(X)-G_1(Y)}{X-Y}-m\frac{\partial^{2m-1}}{\partial X^m\partial Y^{m-1}}\frac{F_1'(X)-G_1'(Y)}{X-Y}\Bigg]$$

$$
\begin{aligned}
&= (X-Y)^{2m}\Bigg[(m+1)\frac{\partial^{2m}}{\partial X^m \partial Y^m}\frac{F_1(X)-G_1(Y)}{X-Y}\\
&\quad - m\frac{\partial^{2m}}{\partial X^{m+1}\partial Y^{m-1}}\frac{F_1(X)-G_1(Y)}{X-Y} + (X-Y)\frac{\partial^{2m+1}}{\partial X^{m+1}\partial Y^m}\frac{F_1(X)-G_1(Y)}{X-Y}\Bigg]\\
&= (X-Y)^{2m}\frac{\partial}{\partial X}\Bigg[(X-Y)\frac{\partial^2}{\partial X\partial Y}\frac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}}\frac{F_1(X)-G_1(Y)}{X-Y}\\
&\quad - m\frac{\partial}{\partial X}\frac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}}\frac{F_1(X)-G_1(Y)}{X-Y}\\
&\quad + m\frac{\partial}{\partial Y}\frac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}}\frac{F_1(X)-G_1(Y)}{X-Y}\Bigg]\\
&= 0,
\end{aligned}
$$

by virtue of equality (24), if we replace there formally the functions F_2 and G_2 with the functions F_1 and G_1, respectively.

The first equality in (9) can be proved in a similar way. Thus Proposition 3 is proved. □

Further, to obtain the final form of the function u, due (3), (7), (9) and (10), we have

$$
\begin{aligned}
du &= pdx + qdy = (px_X + qy_X)dX + (px_Y + qy_Y)dY\\
&= \left(q + p^{\frac{\alpha+2}{2}}\right)y_X dX + \left(q - p^{\frac{\alpha+2}{2}}\right)y_Y dY\\
&= \left(\frac{m-1}{2m-1}X + \frac{m}{2m-1}Y\right)y_X dX + \left(\frac{m}{2m-1}X + \frac{m-1}{2m-1}Y\right)y_Y dY,
\end{aligned}
$$

whence

$$
U_X = \left(\frac{m-1}{2m-1}X + \frac{m}{2m-1}Y\right)y_X,\quad U_Y = \left(\frac{m}{2m-1}X + \frac{m-1}{2m-1}Y\right)y_Y. \tag{38}
$$

By virtue of the first equality of (38), we obtain

$$
\begin{aligned}
U(X,Y) &= \frac{m-1}{2m-1}\int Xy_X dX + \frac{m}{2m-1}Yy + \varphi(Y)\\
&= \frac{m-1}{2m-1}\left(Xy - \int y dX\right) + \frac{m}{2m-1}Yy + \varphi(Y),
\end{aligned} \tag{39}
$$

where φ is an arbitrary function.

According to the second equality from (38), for definition of the function φ, we get

$$
\frac{m-1}{2m-1}\left(Xy_Y - \int y_Y dX\right) + \frac{m}{2m-1}(y + Yy_Y) + \varphi'(Y) = \left(\frac{m}{2m-1}X + \frac{m-1}{2m-1}Y\right)y_Y. \tag{40}
$$

By virtue of (12), we receive

$$m\int y_Y dX = \int \Big[(Y-X)y_{XY} + my_X\Big]dX = (Y-X)y_Y + \int y_Y dX + my.$$

Thus

$$\int y_Y dX = \frac{Y-X}{m-1}y_Y + \frac{m}{m-1}y, \quad \text{for} \quad m = 2,3,\dots.$$

Taking into account the latter equality, from (40) we obtain

$$\varphi'(Y) \equiv 0, \Rightarrow \varphi = const \quad \text{for} \quad m = 2,3,\dots. \tag{41}$$

Analogously, from (39) for $m = 1$, we get

$$U(X,Y) = Yy + \varphi(Y). \tag{42}$$

According to the second equality from (38), in this case, for definition of the function φ, we receive

$$\varphi'(Y) = (X-Y)y_Y - y. \tag{43}$$

Since

$$\Big\{(X-Y)y_Y - y\Big\}_X = y_Y + (X-Y)y_{XY} - y_X = 0,$$

by virtue of equation (12) for $m = 1$, from (43) implies that

$$\varphi'(Y) = -G_2'(Y),$$

therefore

$$\varphi(Y) = -G_2(Y). \tag{44}$$

Now, introducing the notation $F := F_1, \quad G := G_1$ and taking into account (23)–(25), (41), (42), (44), we obtain (4) and (5), respectively. □

Remark 2. In the case $m = 1$, i.e., for $\alpha = -4$, solution (5) of equation (1) has been obtained in [6] by the method of Lee's group.

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