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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR FRACTIONAL *Q*-DIFFERENCE EQUATIONS

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Abstract. In this work, two different boundary value problems of fractional q-difference equation are discussed. The first one is fractional q-difference equation with integral boundary conditions and the other one is three point boundary value problem of fractional q-difference equation with p-Laplacian. By using fixed point theorems, some sufficient conditions that guarantee the existence and uniqueness of solutions are given.

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1. INTRODUCTION

The quantum calculus deals with quantum derivatives and integrals, and has proven to be relevant for quantum mechanics. It allows to deal with continuous functions, which do not need to be smooth. There are several types of quantum calculus: h-calculus (also known as the calculus of finite differences), the q-calculus and Hahn's calculus. In this paper we are concerned with the q-calculus. The q-derivative and the q-integral (also known as Jackson derivative and Jackson integral, respectively) were first defined by Jackson [12, 13] and had proven to have important applications on many subjects, like in hypergeometric series, complex analysis, particle physics and quantum mechanics. For a general introduction to the q-calculus we refer the reader to the books [10, 14]. Fractional calculus, on the other hand, generalizes integer-order analysis, by considering derivatives and integrals of non-integer order, and found many applications in several fields of applied sciences and engineering such as physics, chemistry, biology, economics, signal and image processing, calculus of variations, control theory, electrochemistry, electrical networks and many other fields for example [3, 7, 9, 21]. Several definitions of fractional derivatives are available in the literature, including the Riemann-Liouville, Grunwald-Letnikov, Caputo, Riesz, Riesz-Caputo, Weyl, Hadamard, and Chen derivatives. In this paper we are concerned with the Riemann-Liouville type fractional derivative. For an excellent book on the theory of fractional calculus we refer the reader to [17].

The natural extension, which we investigate here, is to consider a quantum fractional calculus, which unifies these two theories by considering quantum derivatives of non-integer order. Nowadays, fractional q-difference calculus has been given in wide applications of different science areas, which include basic hyper-geometric functions, mechanics, the theory of relativity, combinatorics and discrete mathematics. Therefore, fractional q-difference calculus has been of great interest and many good results can be found in [2,4–6] and references therein. For different problems of fractional q-difference equations, the existence and the uniqueness of solutions have been always considered in literature. To solve these boundary value problems, some techniques have been applied, such as the monotone iterative technique, the lower-upper solution method, the Schauder fixed point theorem and the Krasnoselskii fixed point theorem. For details, one can see [19, 20, 23, 24].

By using Schauder fixed point theorem and the Banach fixed point theorem, Yang [22] discussed a fractional q-difference equation with three-point boundary conditions:

$$D_q^\alpha x(t)+f(t,x(t),x(t))+g(t,x(t))=0, \qquad 0<t<1,\ \ 2<\alpha<3,$$
$$x(0)=D_q x(0)=0,\ \ D_q x(1)=\beta D_q x(\eta),$$

where $0<\beta\eta^{\alpha-2}<1,\ 0<q<1$ and D_q^α is the Riemann-Liouville fractional q-derivative of order α.

In [25], Zhao studied the existence of positive solutions for the folowing q-fractional boundary value problems with p-Laplacian

$$D_q^\beta\left(\varphi_p\left(D_q^\alpha u(t)\right)\right)=f(t,u(t)),\ \ t\in(0,1),$$
$$u(0)=0,\quad u(1)=\int_0^1 h(t)u(t)\,d_qt,\quad D_q^\alpha u(0)=0,\quad D_q^\alpha u(1)=bD_q^\alpha u(\eta),$$

where $D_q^\alpha,\ D_q^\beta$ are the fractional q-derivative of Riemann-Liouville type with $1<\alpha,\ \beta\le 2,\ 0\le b<1,\ 0<\eta<1,\ \varphi_p(s)=|s|^{p-2}s,\ \varphi_p^{-1}=\varphi_r,\ p^{-1}+r^{-1}=1,$ $p>1,\ r>1$ and $f\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+),\ h\in C([0,1],\mathbb{R}^+).$

In the light of the studies mentioned, in this paper firstly we investigate the existence and uniqueness of positive solutions for the following q-fractional boundary value problem

$$D_q^\alpha x(t)+f(t,x(t))=0, \qquad t\in(0,1) \tag{1.1}$$
$$x(0)=a\int_0^1 f_1(t)x(t)\,d_qt, \qquad x(1)=b\int_0^1 f_2(t)x(t)\,d_qt, \tag{1.2}$$

where D_q^α is the fractional q-derivative of Riemann-Liouville type with $1<\alpha\le 2,\ a,b\in\mathbb{R}^+$ and $f\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+),\ f_1,f_2\in C([0,1],\mathbb{R}^+).$

Secondly, we will give the existence result for the following q-fractional boundary value problems with p-Laplacian

$$D_q^\beta\left(\varphi_p\left(D_q^\alpha x(t)\right)\right)=f(t,x(t)) \tag{1.3}$$

$$x(0) = a\int_0^1 f_1(t)x(t)d_qt, \ \ x(1) = b\int_0^1 f_2(t)x(t)d_qt,$$
$$D_q^{\alpha}x(0) = 0, \quad D_q^{\alpha}x(1) = cD_q^{\alpha}x(\eta), \tag{1.4}$$

where D_q^{α}, D_q^{β} are the fractional q-derivative of Riemann-Liouville type with $1 < \alpha,\ \beta \le 2,\ a,b \in \mathbb{R}^+\ 0 \le c < 1,\ 0 < \eta < 1,\ \varphi_p(s) = |s|^{p-2}s$ such that $\varphi_p^{-1} = \varphi_r$ with $p^{-1}+r^{-1} = 1,\ p > 1,\ r > 1$ and $f \in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+),\ f_1, f_2 \in C([0,1],\mathbb{R}^+)$.

2. PRELIMINARIES

In this section, we list some useful definitions and preliminaries, which will be used in the proofs of the main results.

Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^k$, $k \in N_0 = \{0,1,2,\dots\}$ is

$$(a-b)^0 = 1\ , \quad (a-b)^{(k)} = \prod_{i=0}^{k-1}\left(a-bq^i\right), \quad k \in N,\ a,b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^{\alpha}\prod_{n=0}^{\infty}\frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^{\alpha}$.

The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$$

then

$$\Gamma_q(x+1) = [x]\Gamma_q(x).$$

The q-derivative of a function f is here defined by

$$(D_qf)(x) = \frac{f(x)-f(qx)}{(1-q)^x}, (D_qf)(0) = \lim_{x\to 0}(D_qf)(x) \quad \text{for } x \ne 0$$

and q-derivatives of higher order by

$$\left(D_q^0f\right)(x) = f(x) \quad \text{and} \quad \left(D_q^nf\right)(x) = D_q\left(D_q^{n-1}f\right)(x), \quad n \in N.$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$(I_qf)(x) = \int_0^x f(t)d_qt = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)d_qt.$$

Similarly as done for derivatives, an operator I_q^n can be defined, i.e.,

$$\left(I_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x),\ n \in N.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x)$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

We now point out two formulas that will be used later (${}_tD_q$ denotes the derivative with respect to variable t)

$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$

$$\left({}_xD_q \int_0^x f(x,t)\,d_qt\right)(x) = \int_0^x {}_xD_q f(x,t)d_qt + f(qx,x).$$

Remark 1. If $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$.

Definition 1 ([1])**.** Let $\alpha \ge 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is

$$(I_q^0 f)(x) = f(x)$$

and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)d_qt, \qquad x \in [0,1].$$

Definition 2 ([16])**.** The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by

$$(D_q^\alpha f)(x) = f(x)$$

and

$$(D_q^\alpha f)(x) = \left(D_q^p I_q^{p-\alpha} f\right)(x), \quad \alpha > 0,$$

where p is the smallest integer greater than or equal to α.

Next, we list some properties about q-derivative and q-integral that are already known in the literature, which are helpful in proofs of our main results.

Lemma 1 ([15])**.** *(1) If f and g are q-integral on the interval* $[a,b]$*,* $\alpha \in \mathbb{R}$*,* $c \in [a,b]$*, then*

(1) $\int_a^b (f(t)+g(t))\,d_qt = \int_a^b f(t)\,d_qt + \int_a^b g(t)d_qt$
(2) $\int_a^b \alpha f(t)\,d_qt = \alpha \int_a^b f(t)d_qt$

(3) $\int_a^b f(t) d_q t = \int_a^c f(t) d_q t + \int_c^b f(t) d_q t$

(4) $\int x^\alpha d_q s = \frac{x^{\alpha+1}}{[\alpha+1]}$, $(\alpha \neq -1)$;

(2) If $|f|$ *is q-integral on the interval* $[0,x]$, *then*

$$\left| \int_0^x f(t) d_q t \right| \leq \int_0^x |f(t)| \, d_q t;$$

(3) If f and g are q-integral on the interval $[0,x]$, $f(t) \leq g(t)$, $\forall t \in [0,x]$, *then*

$$\int_0^x f(t) d_q t \leq \int_0^x g(t) d_q t.$$

Lemma 2 ([8]). *Let* $\alpha > 0$ *and p be a positive integer. Then, the following equality holds:*

$$\left(I_q^\alpha D_q^p f\right)(x) = \left(D_q^p I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Now, we will give the existence theorems used in our main results.

Theorem 1 ([11]). *Let* $T : X \to X$ *be a map on a complete non-empty metric space. If some iterate* T^n *of T is a contraction, then T has a unique fixed point.*

Theorem 2 ([18]). *(Schauder Fixed Point Theorem) Let* $(X, \|.\|)$ *be a Banach space and* $S \subset X$ *is compact, convex and nonempty. Then any continuous operator* $A : S \to S$ *has at least one fixed point.*

Theorem 3 ([16]). *(Nonlinear alternative for single valued maps) Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and* $0 \in U$. *Suppose that* $F : \overline{U} \to C$ *is a continuous, compact (that is,* $F(\overline{U})$ *is a relatively compact subset of C) map. Then either*

(1) *F has a fixed point in* $\overline{U}$, *or*

(2) *There is a* $u \in \partial U$ *(the boundary of U in C) and* $\lambda \in (0,1)$ *with* $u = \lambda F(u)$.

Suppose that the following condition is satisfied.

$$(H_1): 1 - b\int_{t=0}^1 f_2(t) t^{\alpha-2} d_q t - a \int_{t=0}^1 f_1(t) t^{\alpha-2} d_q t > 0.$$

Lemma 3. *Let* $h(t) \in C[0,1]$ *and* $1 < \alpha \leq 2$, *then the boundary value problem*

$$\left(D_q^\alpha x\right)(t) + h(t) = 0 , \qquad 0 < t < 1, \tag{2.1}$$

$$x(0) = a \int_0^1 f_1(t) \; x(t) d_q t, \quad x(1) = b \int_0^1 f_2(t) \; x(t) d_q t \tag{2.2}$$

has a unique solution

$$x(t) = \int_{s=0}^1 G(t, qs) h(s) d_q s$$

such that

$$\begin{aligned}G(t,s) = g(t,s) + \frac{at^{\alpha-2}(1-t)}{\Delta}\Bigg\{\left[1-b\int_0^1 t^{\alpha-1}\, f_2(t)\, d_qt\right]\int_0^1 g(t,s) f_1(t)\, d_qt \\ +b\int_0^1 t^{\alpha-1} f_1(t)\, d_qt \int_0^1 g(t,s) f_2(t)\, d_qt\Bigg\} \\ +\frac{bt^{\alpha-1}}{\Delta}\Bigg\{\left[1-a\int_0^1 t^{\alpha-2}(1-t)\, f_1(t)\, d_qt\right]\int_0^1 g(t,s) f_2(t)\, d_qt \\ +a\int_0^1 t^{\alpha-2}(1-t) f_2(t)\, d_qt \int_0^1 g(t,s) f_1(t)\, d_qt\Bigg\}\end{aligned}$$

and

$$g(t,s) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} (t(1-s))^{(\alpha-1)} - (t-s)^{(\alpha-1)} , & s \le t \\ (t(1-s))^{(\alpha-1)} , & s \ge t. \end{cases}$$

Proof. From Lemma 2, since $D_q^\alpha x(t) = -h(t)$ we have

$$\begin{aligned}x(t) &= -I_q^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \qquad c_1, c_2 \in \mathbb{R} \\ &= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_qs + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.\end{aligned}$$

Using the first boundary condition

$$x(0) = a\int_0^1 f_1(t)\, x(t)\, d_qt$$

we get

$$c_2 = a\int_0^1 f_1(t)\, x(t)\, d_qt.$$

So we have

$$x(t) = -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} h(s)\, d_qs + c_1 t^{\alpha-1} + at^{\alpha-2}\int_0^1 f_1(t)\, x(t)\, d_qt.$$

Then using the second boundary condition

$$\begin{aligned}x(1) &= b\int_0^1 f_2(t)\ x(t) d_qt \\ &= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (1-qs)^{(\alpha-1)} h(s)\, d_qs + c_1 + a\int_0^1 f_1(t)\, x(t)\, d_qt\end{aligned}$$

we get

$$c_1 = \frac{1}{\Gamma_q(\alpha)}\int_0^t (1-qs)^{(\alpha-1)} h(s)\, d_qs - a\int_0^1 f_1(t)\, x(t)\, d_qt + b\int_0^1 f_2(t)\ x(t) d_qt.$$

Therefore, we have

$$\begin{aligned}
x(t) &= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}h(s)\,d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (1-qs)^{(\alpha-1)}h(s)\,d_qs \\
&\quad - at^{\alpha-1}\int_0^1 f_1(t)x(t)\,d_qt + bt^{\alpha-1}\int_0^1 f_2(t)\ x(t)\,d_qt + at^{\alpha-2}\int_0^1 f_1(t)x(t)\,d_qt \\
&= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}h(s)\,d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (1-qs)^{(\alpha-1)}h(s)\,d_qs \\
&\quad - at^{\alpha-2}(t-1)\int_0^1 f_1(t)x(t)\,d_qt + bt^{\alpha-1}\int_0^1 f_2(t)\ x(t)\,d_qt \\
&= \int_0^1 g(t,qs)\ h(s)\,d_qs + at^{\alpha-2}(1-t)\int_0^1 f_1(t)x(t)\,d_qt + bt^{\alpha-1}\int_0^1 f_2(t)\ x(t)\,d_qt.
\end{aligned} \tag{2.3}$$

At first, multiply both sides of (2.3) by $f_1(t)$, then take the integral from 0 to 1, we get

$$\begin{aligned}
\int_0^1 f_1(t)x(t)\,d_qt &= \int_{t=0}^1 f_1(t)\int_{s=0}^1 g(t,qs)\ h(s)\,d_qs\ d_qt \\
&\quad + a\int_0^1 \left(f_1(t)t^{\alpha-2}(1-t)\int_0^1 f_1(t)x(t)\,d_qt\right)d_qt \\
&\quad + b\int_0^1 \left(f_1(t)t^{\alpha-1}\int_0^1 f_2(t)\ x(t)\,d_qt\right)d_qt
\end{aligned}$$

and so

$$\begin{aligned}
&\int_0^1 f_1(t)x(t)\,d_qt\ \left[1-a\int_0^1 f_1(t)t^{\alpha-2}(1-t)d_qt\right] \\
&\qquad - b\int_0^1 f_2(t)\ x(t)\,d_qt\int_0^1 f_1(t)t^{\alpha-1}d_qt = \int_{t=0}^1 f_1(t)\int_{s=0}^1 g(t,qs)\ h(s)\,d_qs\ d_qt.
\end{aligned}$$

Similarly, multiply both sides of (2.3) by $f_2(t)$, then take the integral from 0 to 1, we have

$$\begin{aligned}
\int_0^1 f_2(t)x(t)\,d_qt &= \int_{t=0}^1 f_2(t)\int_{s=0}^1 g(t,qs)\ h(s)\,d_qs\ d_qt \\
&\quad + a\int_0^1 \left(f_2(t)t^{\alpha-2}(1-t)\int_0^1 f_1(t)x(t)\,d_qt\right)d_qt \\
&\quad + b\int_0^1 \left(f_2(t)t^{\alpha-1}\int_0^1 f_2(t)\ x(t)\,d_qt\right)d_qt
\end{aligned}$$

and so

$$\int_0^1 f_1(t)x(t)\,d_qt\ \left[-a\int_0^1 f_2(t)t^{\alpha-2}(1-t)d_qt\right]$$

$$+\int_0^1 f_2(t)\ x(t)\,d_qt\left[1-b\int_0^1 f_2(t)\,t^{\alpha-1}d_qt\right]=\int_{t=0}^1 f_2(t)\int_{s=0}^1 g(t,qs)\ h(s)\,d_qs\,d_qt.$$

If we define

$$\begin{aligned}\Delta &= \begin{vmatrix} 1-a\int_0^1 f_1(t)\,t^{\alpha-2}(1-t)d_qt & -b\int_0^1 f_1(t)\,t^{\alpha-1}d_qt \\ -a\int_0^1 f_2(t)\,t^{\alpha-2}(1-t)\,d_qt & 1-b\int_0^1 f_2(t)\,t^{\alpha-1}d_qt \end{vmatrix} \\ &= (1-a\int_0^1 f_1(t)\,t^{\alpha-2}(1-t)d_qt\ )(1-b\int_0^1 f_2(t)\,t^{\alpha-1}d_qt) \\ &\quad -ab\int_0^1 f_2(t)\,t^{\alpha-2}(1-t)\,d_qt\ \int_0^1 f_1(t)\,t^{\alpha-1}d_qt,\end{aligned}$$

we have

$$\begin{aligned}&\int_0^1 f_1(t)\,x(t)\,d_qt \\ &\quad= \frac{1}{\Delta}\begin{vmatrix} \int_{t=0}^1 f_1(t)\int_{s=0}^1 g(t,qs)\ h(s)\,d_qs\,d_qt & -b\int_0^1 f_1(t)\,t^{\alpha-1}d_qt \\ \int_{t=0}^1 f_2(t)\int_{s=0}^1 g(t,qs)\ h(s)\,d_qs\,d_qt & 1-b\int_0^1 f_2(t)\,t^{\alpha-1}d_qt \end{vmatrix} \\ &\quad= \frac{1}{\Delta}\left\{\left(1-b\int_0^1 f_2(t)\,t^{\alpha-1}d_qt\right)\int_{s=0}^1\int_{t=0}^1 g(t,qs)\ h(s)\ f_1(t)\ d_qt\ d_qs\right. \\ &\qquad \left.+b\int_0^1 f_1(t)\,t^{\alpha-1}d_qt\int_{s=0}^1\int_{t=0}^1 g(t,qs)\ h(s)\,f_2(t)\ d_qt\ d_qs\right\}\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 f_2(t)\,x(t)\,d_qt \\ &\quad= \frac{1}{\Delta}\begin{vmatrix} 1-a\int_0^1 f_1(t)\,t^{\alpha-2}(1-t)d_qt & \int_{t=0}^1 f_1(t)\int_{s=0}^1 g(t,qs)\ h(s)\,d_qs\ d_qt \\ -a\int_0^1 f_2(t)\,t^{\alpha-2}(1-t)\,d_qt & \int_{t=0}^1 f_2(t)\int_{s=0}^1 g(t,qs)\ h(s)\,d_qs\ d_qt \end{vmatrix} \\ &\quad= \frac{1}{\Delta}\left\{\left(1-a\int_0^1 f_1(t)\,t^{\alpha-2}(1-t)d_qt\right)\int_{s=0}^1\int_{t=0}^1 g(t,qs)\ h(s)\ f_2(t)\ d_qt\ d_qs\right. \\ &\qquad \left.+a\int_0^1 f_2(t)\,t^{\alpha-2}(1-t)d_qt\int_{s=0}^1\int_{t=0}^1 g(t,qs)\ h(s)\,f_1(t)\ d_qt\ d_qs\right\}.\end{aligned}$$

Because of the condition (H_1), we can easily see that $\Delta>0$. Therefore, we have

$$\begin{aligned}x(t) &= \int_0^1 g(t,qs)\ h(s)\ d_qs \\ &\quad + at^{\alpha-2}(1-t)\frac{1}{\Delta}\left\{\left(1-b\int_0^1 f_2(t)\,t^{\alpha-1}d_qt\right)\int_{s=0}^1\int_{t=0}^1 g(t,qs)\,h(s)\ f_1(t)\ d_qt\ d_qs\right. \\ &\quad \left.+b\int_0^1 f_1(t)\,t^{\alpha-1}d_qt\int_{s=0}^1\int_{t=0}^1 g(t,qs)\,h(s)\ f_2(t)\ d_qt\ d_qs\right\}\end{aligned}$$

$$+bt^{\alpha-1}\frac{1}{\Delta}\left\{\left(1-a\int_0^1 f_1(t)\,t^{\alpha-2}(1-t)d_qt\right)\int_{s=0}^1\int_{t=0}^1 g(t,qs)\,h(s)\;f_2(t)\;d_qt\;d_qs\right.$$
$$\left.+a\int_0^1 f_2(t)\,t^{\alpha-2}(1-t)d_qt\int_{s=0}^1\int_{t=0}^1 g(t,qs)\,h(s)\;f_1(t)\;d_qt\;d_qs\right\}$$

or

$$\begin{aligned}
x(t) &= \int_0^1 g(t,qs)\;h(s)\;d_qs \\
&\quad + at^{\alpha-2}(1-t)\frac{1}{\Delta}\int_{s=0}^1\left[\int_{t=0}^1 g(t,qs)\left\{f_1(t)\left(1-b\int_0^1 f_2(t)\,t^{\alpha-1}d_qt\right)\right.\right. \\
&\quad \left.\left.+bf_2(t)\int_0^1 f_1(t)\,t^{\alpha-1}d_qt\right\}d_qt\right]h(s)d_qs \\
&\quad + bt^{\alpha-1}\frac{1}{\Delta}\int_{s=0}^1\left[\int_{t=0}^1 g(t,qs)\left\{f_2(t)\left(1-a\int_0^1 f_1(t)\,t^{\alpha-2}(1-t)d_qt\right)\right.\right. \\
&\quad \left.\left.+af_1(t)\int_0^1 f_2(t)\,t^{\alpha-2}(1-t)d_qt\right\}d_qt\right]h(s)d_qs \\
&= \int_0^1 G(t,qs)\,h(s)d_qs.
\end{aligned}$$

Consequently, we can write

$$\begin{aligned}
G(t,s) &= g(t,s)+\frac{at^{\alpha-2}(1-t)}{\Delta}\left\{\left[1-b\int_0^1 f_2(t)\,t^{\alpha-1}d_qt\right]\int_0^1 g(t,s)f_1(t)d_qt\right. \\
&\quad \left.+b\int_0^1 f_1(t)\,t^{\alpha-1}d_qt\int_0^1 g(t,s)f_2(t)d_qt\right\} \\
&\quad +\frac{bt^{\alpha-1}}{\Delta}\left\{\left[1-a\int_0^1 f_1(t)\,t^{\alpha-2}(1-t)d_qt\right]\int_0^1 g(t,s)f_2(t)d_qt\right. \\
&\quad \left.+a\int_0^1 f_2(t)\,t^{\alpha-2}(1-t)d_qt\int_0^1 g(t,s)f_1(t)d_qt\right\}.
\end{aligned}$$

$\square$

Using the definition of the Green's function for our problem, we get the following inequality

$$\begin{aligned}
G(t,s) \le \frac{1}{\Gamma_q(\alpha)}&\left\{1+\frac{as^{\alpha-2}}{\Delta}\left[B_1\int_0^1 f_1(t)d_qt+B_2\int_0^1 f_2(t)d_qt\right]\right. \\
&\left.+\frac{bs^{\alpha-1}}{\Delta}\left[A_1\int_0^1 f_2(t)d_qt+A_2\int_0^1 f_1(t)d_qt\right]\right\}\left(\gamma(1-s)^{\alpha-1}\right).
\end{aligned}$$

So, if we define

$$\frac{1}{\Gamma_q(\alpha)}\left\{1+\frac{a}{\Delta}\left[B_1\int_0^1 f_1(t)d_qt+B_2\int_0^1 f_2(t)d_qt\right]\right.$$
$$\left.+\frac{b}{\Delta}\left[A_1\int_0^1 f_2(t)d_qt+A_2\int_0^1 f_1(t)d_qt\right]\right\}:=M, \quad (2.4)$$

we get $G(t,s)\leq M$.

3. Main Results

In this section, primarily, we will give the existence and uniqueness results for the problem (1.1)-(1.2). Let $B=C([0,1],\mathbb{R})$ is the Banach space with the norm $\|x\|=\max_{t\in[0,1]}|x(t)|$.

We define the operator $T: B\to B$ by

$$Tx(t):=\int_0^1 G(t,qs)f(s,x(s))d_qs.$$

Let $x\in B$, in view of the continuity of $G(t,s)$ and $f(t,x(t))$, we have $T: B\to B$ is continuous. Let $\Omega\subset B$ be bounded, it is easy to see that $T(\Omega)$ is uniformly bounded. On the other hand, since $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$ we can get that $T(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, we have $T: B\to B$ is completely continuous.

Theorem 4. *Assume that $f(t,0)\neq 0$ and there exist $k,h\in L^1[0,1]$ such that*

$$|f(t,x)|\leq k(t)|x|+h(t), \qquad (t,x)\in[0,1]\times\mathbb{R}$$

and

$$M\int_0^1 k(s)d_qs<1,$$

where M is defined by (2.4). Then the boundary value problem (1.1)-(1.2) has at least one solution $x^*\in C[0,1]$ which is different from 0.

Proof. Let we define the numbers

$$\int_0^1 k(s)\,d_qs=K$$

and

$$\int_0^1 h(s)\,d_qs=L.$$

Thus we get $MK<1$, since $f(t,0)\neq 0$ there exist such $[a,b]\subseteq[0,1]$ that $\min_{a\leq t\leq b}|f(t,0)|>0$. Moreover $h(t)>|f(t,0)|$ then $L>0$.

Assume that $A=ML(1-MK)^{-1}$ and $\Omega=\{x\in E: \ \|x\|<A\}$. Let $x\in\partial\Omega$ and $\lambda>1$ with $T(x)=\lambda x$.

In that case, we have

$$\lambda A=\lambda\|x\|=\|Tx\|=\max_{t\in[0,1]}|Tx(t)|$$

$$\begin{aligned}
&\leq \max_{t\in[0,1]} \left| \int_0^1 G(t,qs) f(s,x(s)) d_q s \right| \\
&\leq \max_{t\in[0,1]} \int_0^1 |G(t,qs)| \, |f(s,x(s))| \, d_q s \\
&\leq \max_{t\in[0,1]} \int_0^1 M \cdot \{k(s)\,|x(s)| + h(s)\} \, d_q s \\
&= MK \cdot \|x\| + ML.
\end{aligned}$$

Therefore, we get

$$\lambda \leq MK + \frac{ML}{A} = MK + \frac{ML}{ML(1-MK)^{-1}} = 1.$$

This conflicts with $\lambda > 1$.

Consequently, by the nonlinear alternative of Leray Schauder type, we deduce that T has a fixed point $x^* \in \overline{\Omega}$ which is a solution of ((1.1)-(1.2). □

Now, we will give the uniqueness of solution for our problem, using the following well-known contraction mapping theorem named also as the Banach fixed point theorem. Let B be a Banach space and S a nonempty closed subset of B. Assume $A: S \to S$ is a contraction, i.e., there is a λ $(0 < \lambda < 1)$ such that $\|Ax - Ay\| \leq \lambda \|x - y\|$ for all x, y in S. Then A has a unique fixed point in S.

Theorem 5. *Suppose that (H_1) holds. Also, we assume that*
(H_2): Let the function $f(t,x)$ satisfy the following Lipschitz condition: there is a constant $L > 0$ such that $|f(t,x) - f(t,y)| \leq L|x-y|$, for $t \in [0,1]$ for all $x, y \in C[0,1]$. Moreover, $LM < 1$ where M is defined in (2.4). Then the problem (1.1)-(1.2) has a unique solution in $C[0,1]$.

Proof. For $x, y \in C[0,1]$ and $t \in [0,1]$, we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \left| \int_0^1 G(t,qs)\,(f(s,x(s)) - f(s,y(s)))\, d_q s \right| \\
&\leq \int_0^1 |G(t,qs)|\, |f(s,x(s)) - f(s,y(s))|\, d_q s \\
&\leq M \int_0^1 L \cdot |x(s) - y(s)|\, d_q s \\
&\leq ML \cdot \|x - y\|.
\end{aligned}$$

Then, we obtain

$$\|Tx - Ty\| = \max_{t\in[0,1]} |Tx(t) - Ty(t)| \leq ML \cdot \|x - y\| = \lambda \|x - y\|,$$

where $\lambda = ML \in (0,1)$. Hence, T is a contraction mapping and the theorem is proved. □

We are now in a position to state and prove the existence result for the problem (1.3)-(1.4).

Lemma 4. *Suppose that* (H_1) *holds. Let* $g \in C[0,1]$, $1 < \alpha, \ \beta \leq 2$, $0 \leq c < 1$, $0 < \eta < 1$. *Then*

$$D_q^\beta \left(\varphi_p \left(D_q^\alpha x(t)\right)\right) = g(t)$$

$$x(0) = a \int_0^1 f_1(t) x(t) d_q t, \qquad x(1) = b \int_0^1 f_2(t) x(t) d_q t,$$

$$D_q^\alpha x(0) = 0, \qquad D_q^\alpha x(1) = c D_q^\alpha x(\eta),$$

is equivalent to

$$x(t) = \int_0^1 G(t,qs)\varphi_r \left(\int_0^1 H(s,q\tau) g(\tau) d_q\tau\right) d_q s,$$

where

$$H(t,s) = m(t,s) + \frac{c^{p-1} t^{\beta-1}}{1 - c^{p-1}\eta^{\beta-1}} m(\eta,s),$$

$$m(t,s) = \frac{1}{\Gamma_q(\beta)} \begin{cases} (t(1-s))^{(\beta-1)} - (t-s)^{(\beta-1)}, & 0 \leq s \leq t \leq 1 \\ (t(1-s))^{(\beta-1)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

and $G(t,s)$ *is as given in Lemma 3.*

Proof. This lemma can be proven similarly to Lemma 2.6 in [25]. □

Remark 2. We can easily get

$$0 \leq H(t,qs) \leq \frac{1}{\Gamma_q(\beta)} t^{\beta-1}(1-qs)^{(\beta-1)} \leq \frac{1}{\Gamma_q(\beta)}, \qquad t,s \in [0,1] \times [0,1].$$

Theorem 6. *Assume that the function* $f(t,x)$ *is continuous with respect to* $x \in \mathbb{R}$. *If* $N > 0$ *satisfies* $M \cdot Q^{r-1} \leq N \cdot (\Gamma_q(\beta))^{r-1}$, *where* $Q > 0$ *satisfies*

$$Q \geq \max_{\|x\| \leq N} |f(t,x)|, \quad \text{for } t \in [0,1]$$

then the problem (1.3)-(1.4) has a solution $x(t)$.

Proof. Let $K = \{x \in B : \ \|x\| \leq N\}$. Note that K is closed, bounded and convex subset of B to which Schauder fixed point theorem is applicable. Define $A : K \to B$ by

$$Ax(t) := \int_0^1 G(t,qs)\varphi_r \left(\int_0^1 H(s,q\tau) f(\tau, x(\tau)) d_q\tau\right) d_q s,$$

for $t \in [0,1]$. Obviously the solutions of problem (1.3)-(1.4) are the fixed points of operator A. It can be shown that $A : K \to B$ is continuous.

Claim that $A : K \to K$. Let $x \in K$. Consider

$$|Ax(t)| = \left| \int_0^1 G(t,qs)\varphi_r \left(\int_0^1 H(s,q\tau) f(\tau, x(\tau)) d_q\tau\right) d_q s \right|$$

$$\begin{aligned}
&\leq \int_0^1 |G(t,qs)| \left| \varphi_r \left(\int_0^1 H(s,q\tau) f(\tau, x(\tau)) d_q\tau \right) \right| d_q s \\
&\leq \int_0^1 M \cdot \left[\int_0^1 H(s,q\tau) d_q\tau \cdot Q \right]^{r-1} d_q s \\
&= M \cdot \left(\frac{Q}{\Gamma_q(\beta)} \right)^{r-1} \leq N,
\end{aligned}$$

for every $t \in [0,1]$. This implies $\|Ax\| \leq N$.

It can be shown that $A\colon K \to K$ is a compact operator by the Arzela-Ascoli theorem. Hence A has a fixed point in K by Schauder fixed point theorem. □

Corollary 1. *If f is continuous and bounded on $[0,1]\times\mathbb{R}$, then the problem (1.3)-(1.4) has a solution.*

Proof. Since the function $f(t,x)$ is bounded, it has a supremum for $t \in [0,1]$ and $x \in \mathbb{R}$. Let us choose $P > sup\{|f(t,x)| : (t,x) \in [0,1]\times\mathbb{R}\}$. Pick N large enough such that $P < N$. Then there is a number Q such that $P > Q$, where $Q \geq max\{|f(t,x)| : t \in [0,1], \ |x| \leq N\}$. Hence

$$1 < \frac{N}{P} \leq \frac{N}{Q},$$

and thus the problem (1.3)-(1.4) has a solution by Theorem 6. □

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