Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 119–127
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3984

BEST PROXIMITY POINTS OF MULTIVALUED GERAGHTY CONTRACTIONS

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Received 02 November, 2021

Abstract. There are situations when we have to resort to the approximate optimal solution of equations of the type $g(t) = t$ when g is not a self-map, because exact solution of that equation does not exist. The existence of such optimal solutions are ensured by best proximity point theorems. In this paper, we define multivalued Geraghty contraction (MVGC) in a complete metric space and establish the corresponding best proximity point (BPP) result. Our result extends the famous result due to Geraghty on fixed points.

2010 *Mathematics Subject Classification:* 47H10; 54H25; 54E50

Keywords: best proximity point, fixed point, Geraghty contraction, complete metric space, multivalued map, optimization

1. PRELIMINARIES

In 1969, Nadler [13] presented some very interesting fixed point results for multivalued maps by considering the distance between two arbitrary sets.

Let $(\mathcal{W},\eta)$ be a complete metric space (MS) and let $\Lambda(\mathcal{W})$ be the collection of all nonempty closed and bounded subsets of $\mathcal{W}$. Then for $\mathcal{X},\mathcal{Y} \in \Lambda(\mathcal{W})$, define the map $\mathcal{H} : \Lambda(\mathcal{W}) \times \Lambda(\mathcal{W}) \to [0,\infty)$ by

$$\mathcal{H}(\mathcal{X},\mathcal{Y}) = \max\{\sup_{\xi\in\mathcal{Y}} \Delta(\xi,\mathcal{X}), \sup_{\delta\in\mathcal{X}} \Delta(\delta,\mathcal{Y})\},$$

where $\Delta(\delta,\mathcal{Y}) = \inf_{\xi\in\mathcal{Y}} \eta(\delta,\xi)$. Consequently, $(\Lambda(\mathcal{W}),\mathcal{H})$ is an MS induced by η.

Let $\mathcal{X},\mathcal{Y}$ be two non-empty subsets of the MS $(\mathcal{W},\eta)$. The following notations will be used throughout:

$$\mathcal{X}_{\mathcal{Y}} = \{\theta \in \mathcal{X} : \eta(\theta,\xi) = \eta(\mathcal{X},\mathcal{Y}) \text{ for some } \xi \in \mathcal{Y}\},$$

$$\mathcal{Y}_{\mathcal{X}} = \{\xi \in \mathcal{Y} : \eta(\theta,\xi) = \eta(\mathcal{X},\mathcal{Y}) \text{ for some } \theta \in \mathcal{X}\},$$

where $\eta(\mathcal{X},\mathcal{Y}) = \inf\{\eta(\theta,\xi) : \theta \in \mathcal{X}, \xi \in \mathcal{Y}\}$.

This research is supported by UGC (Ministry of HRD, Govt. of India) through UGC-BSR Start-Up Grant vide letter No. F.30-452/2018(BSR) dated 12 Feb 2019.

For $\mathcal{X},\mathcal{Y} \in \Lambda(\mathcal{W})$, we have

$$\eta(\mathcal{X},\mathcal{Y}) \leq H(\mathcal{X},\mathcal{Y}).$$

$\theta \in \mathcal{W}$ is called a BPP of the multivalued map $\Phi : \mathcal{W} \to \Lambda(\mathcal{W})$ if $\Delta(\theta,\Phi\theta) = \eta(\mathcal{X},\mathcal{Y})$. $\upsilon \in \mathcal{W}$ is said to be a fixed point of Φ if $\upsilon \in \Phi\upsilon$.

Remark 1.
(1) In the MS $(\Lambda(\mathcal{W}),\mathcal{H})$, $\upsilon \in \mathcal{W}$ is a fixed point of Φ if and only if $\Delta(\upsilon,\Phi\upsilon) = 0$.
(2) If $\mathcal{X},\mathcal{Y}$ are two closed sets with $\mathcal{X} \cap \mathcal{Y} \neq \phi$, we obtain $\eta(\mathcal{X},\mathcal{Y}) = 0$. Then a fixed point and a BPP coincide.
(3) The function Δ is continuous, for if $\theta_n \to \theta$ as $n \to \infty$, then $\Delta(\theta_n,\mathcal{X}) \to \Delta(\theta,\mathcal{X})$ as $n \to \infty$ for any $\mathcal{X} \subseteq \mathcal{W}$.

The next two lemmas are important for the sequel.

Lemma 1 ([3,5])**.** *Let $(\mathcal{W},\eta)$ be a MS and $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \Lambda(\mathcal{W})$. Then*
(1) $\Delta(\theta,\mathcal{Y}) \leq \eta(\theta,\xi)$ *for any* $\xi \in \mathcal{Y}$ *and* $\theta \in \mathcal{W}$*;*
(2) $\Delta(\theta,\mathcal{Y}) \leq \mathcal{H}(\mathcal{X},\mathcal{Y})$ *for any* $\theta \in \mathcal{X}$*.*

Lemma 2 ([13])**.** *Let $\mathcal{X},\mathcal{Y} \in \Lambda(\mathcal{W})$ and let $\theta \in \mathcal{X}$, then for any $s > 0$, there exists $\xi \in \mathcal{Y}$ satisfying*

$$\eta(\theta,\xi) \leq \mathcal{H}(\mathcal{X},\mathcal{Y}) + s.$$

A point $\xi \in \mathcal{Y}$ may not exist satisfying

$$\eta(\theta,\xi) \leq \mathcal{H}(\mathcal{X},\mathcal{Y}).$$

If $\mathcal{Y}$ is compact, then we obtain a point ξ satisfying $\eta(\theta,\xi) \leq \mathcal{H}(\mathcal{X},\mathcal{Y})$.

The concept of $\mathcal{H}$-continuity plays a significant role as discussed next.

Definition 1 ([7])**.** Let $(\mathcal{W},\eta)$ be a MS. A multivalued map $\Phi : \mathcal{W} \to \Lambda(\mathcal{W})$ is said to be $\mathcal{H}$-continuous at a point θ_0, if for each sequence $\{\theta_n\} \subset \mathcal{W}$, such that $\lim_{n\to\infty} \eta(\theta_n,\theta_0) = 0$, we have $\lim_{n\to\infty} \mathcal{H}(\Phi\theta_n,\Phi\theta_0) = 0$ (i.e., if $\theta_n \to \theta_0$, then $\Phi\theta_n \to \Phi\theta_0$ as $n \to \infty$).

Definition 2 ([13])**.** Let $\Phi : \mathcal{W} \to \Lambda(\mathcal{W})$ be a multivalued map. Φ is said to be a multivalued contraction if $\mathcal{H}(\Phi\theta,\Phi\xi) \leq \kappa\eta(\theta,\xi)$ for all $\theta,\xi \in \mathcal{W}$, where $\kappa \in [0,1)$.

Remark 2.
(1) When Φ is continuous on every point of $\mathcal{X}$, it is said to be $\mathcal{H}$-continuous on $\mathcal{X}$.
(2) If Φ is a multivalued contraction, then it is $\mathcal{H}$-continuous.

Sankar Raj [14] put forward the notion of P-property. The idea of weak P property was introduced by Zhang et al. [19] which enhanced the results of Caballero et al. [4] on Geraghty-contractions.

Definition 3 ([14])**.** Let $(\mathcal{W},\eta)$ be a MS and $\mathcal{X},\mathcal{Y}$ be two non-empty subsets of $\mathcal{W}$ such that $\mathcal{X}_{\mathcal{Y}} \neq \phi$. The pair $(\mathcal{X},\mathcal{Y})$ satisfies the P-property if and only if $\eta(\theta_1,\xi_1) = \eta(\mathcal{X},\mathcal{Y}) = \eta(\theta_2,\xi_2)$ implies $\eta(\theta_1,\theta_2) = \eta(\xi_1,\xi_2)$, where $\theta_1,\theta_2 \in \mathcal{X}_{\mathcal{Y}}$ and $\xi_1,\xi_2 \in \mathcal{Y}_{\mathcal{X}}$.

Definition 4 ([19])**.** Let $(\mathcal{W},\eta)$ be a MS and $\mathcal{X},\mathcal{Y}$ be two non-empty subsets of $\mathcal{W}$ such that $\mathcal{X}_{\mathcal{Y}} \neq \phi$. The pair $(\mathcal{A},\mathcal{Y})$ satisfies the weak P-property if and only if $\eta(\theta_1,\theta_1) = \eta(\mathcal{X},\mathcal{Y}) = \eta(\theta_2,\xi_2)$ implies $\eta(\theta_1,\theta_2) \leq \eta(\xi_1,\xi_2)$, where $\theta_1,\theta_2 \in \mathcal{X}_{\mathcal{Y}}$ and $\xi_1,\xi_2 \in \mathcal{Y}_{\mathcal{A}}$.

Srivastava et al. [15,16] presented Krasnosel'skii type hybrid fixed point theorems which significantly improved the study of fractional integral equations. Recently, Debnath and Srivastava [9] studied common BPPs for multivalued contractive pairs of mappings. Debnath and Srivastava [10] also proved new extensions of Kannan's and Reich's theorems. Another Kannan-type contraction for multivalued asymptotic regular maps were presented by Debnath et al. [8]. Further, a very significant application of fixed points of $F(\psi,\varphi)$-contractions to fractional differential equations was recently provided by Srivastava et al. [18]. Srivastava et al. [17] studied implicit functional differential inclusions of arbitrary fractional order.

Geraghty [11] used a particular family of functions that generalized the Banach's fixed point theorem. Let $\mathcal{G}$ be the collection of mappings $g:[0,\infty)\to[0,1)$ satisfying the condition: $g(t_n)\to 1$ implies $t_n \to 0$. An example of such a map is $g(t) = (1+t)^{-1}$ for all $t>0$ and $g(0)\in[0,1)$.

Many authors have studied different generalized Geraghty-type contractions [1,2,12], but the exact multivalued analogue of Geraghty contraction [11] has not been established yet. Recently, Debnath [6] presented a new technique of studying BPPs for multivalued F-contractions, which generalized and extended several existing results in literature. In the current paper, we observe that this new technique is also helpful for establishing the multivalued analogue of the Geraghty contraction.

2. Best proximity point of MVGC

In this section, we extend the famous result on fixed points by Geraghty to its multivalued analogue in terms of BPPs. However, additional assumptions such as weak P-property and compactness of the images of the multivalued map under consideration have been made.

First, we define a MVGC.

Definition 5. Let $(\mathcal{W},\eta)$ be a MS and $\mathcal{X},\mathcal{Y}$ be two non-empty subsets of $\mathcal{W}$. The mapping $\Phi:\mathcal{X}\to\Lambda(\mathcal{Y})$ is said to be a multivalued Geraghty contraction (MVGC) if there exists $g\in\mathcal{G}$ such that

$$\mathcal{H}(\Phi\theta,\Phi\xi) \leq g\big(\eta(\theta,\xi)\big)\eta(\theta,\xi)$$

for all $\theta,\xi\in\mathcal{X}$.

Remark 3. Since $g : [0,\infty) \to [0,1)$, it is easy to see that $\mathcal{H}(\Phi\theta,\Phi\xi) < \eta(\theta,\xi)$ for all $\theta,\xi \in \mathcal{X}$ with $\theta \neq \xi$. Therefore, every MVGC is a multivalued contractive mapping and hence $\mathcal{H}$-continuous.

Theorem 1. *Let $(\mathcal{W},\eta)$ be a complete MS and $\mathcal{X},\mathcal{Y}$ be two non-empty closed subsets of $\mathcal{W}$ such that $\mathcal{X}_{\mathcal{Y}} \neq \phi$ and that the pair $(\mathcal{X},\mathcal{Y})$ satisfies the weak P-property. Let $\Phi : \mathcal{X} \to \Lambda(\mathcal{Y})$ be a MVGC such that $\Phi\theta$ is compact for each $\theta \in \mathcal{X}$ and $\Phi\theta \subseteq \mathcal{Y}_{\mathcal{X}}$ for all $\theta \in \mathcal{X}_{\mathcal{Y}}$. Then Φ has a BPP.*

Proof. Fix $\theta_0 \in \mathcal{X}_{\mathcal{Y}}$ and choose $\xi_0 \in \Phi\theta_0 \subseteq \mathcal{Y}_{\mathcal{X}}$. By the definition of $\mathcal{Y}_{\mathcal{X}}$, we can select $\theta_1 \in \mathcal{X}_{\mathcal{Y}}$ such that

$$\eta(\theta_1,\xi_0) = \eta(\mathcal{X},\mathcal{Y}). \tag{2.1}$$

If $\xi_0 \in \Phi\theta_1$, then

$$\eta(\mathcal{X},\mathcal{Y}) \leq \Delta(\theta_1,\Phi\theta_1) \leq \eta(\theta_1,\xi_0) = \eta(\mathcal{X},\mathcal{Y}).$$

Thus $\eta(\mathcal{X},\mathcal{Y}) = \Delta(\theta_1,\Phi\theta_1)$, i.e., θ_1 is a BPP of Φ. Therefore, assume that $\xi_0 \notin \Phi\theta_1$.

Since $\Phi\theta_1$ is compact, by Lemma 2, there exists $\xi_1 \in \Phi\theta_1$ such that

$$0 < \eta(\xi_0,\xi_1) \leq \mathcal{H}(\Phi\theta_0,\Phi\theta_1) \leq g\big(\eta(\theta_0,\theta_1)\big)\eta(\theta_0,\theta_1). \tag{2.2}$$

Since $\xi_1 \in \Phi\theta_1 \subseteq \mathcal{Y}_{\mathcal{X}}$, there exists $\theta_2 \in \mathcal{X}_{\mathcal{Y}}$ such that

$$\eta(\theta_2,\xi_1) = \eta(\mathcal{X},\mathcal{Y}). \tag{2.3}$$

Using (2.1), (2.3) and the weak P-property, we have that

$$\eta(\theta_1,\theta_2) \leq \eta(\xi_0,\xi_1). \tag{2.4}$$

(2.2) and (2.4) imply that

$$\eta(\theta_1,\theta_2) \leq \eta(\xi_0,\xi_1) \leq \mathcal{H}(\Phi\theta_0,\Phi\theta_1) \leq g\big(\eta(\theta_0,\theta_1)\big)\eta(\theta_0,\theta_1). \tag{2.5}$$

If $\xi_1 \in \Phi\theta_2$, then

$$\eta(\mathcal{A},\mathcal{B}) \leq \Delta(\theta_2,\Phi\theta_2) \leq \eta(\theta_2,\xi_1) = \eta(\mathcal{A},\mathcal{B}),$$

i.e., $\eta(\mathcal{A},\mathcal{B}) = \Delta(\theta_2,\Phi\theta_2)$, and hence, θ_2 is a BPP of Φ. So, assume that $\xi_1 \notin \Phi\theta_2$.

Since $\Phi\theta_2$ is compact, by Lemma 2, there exists $\xi_2 \in \Phi\theta_2$ such that

$$0 < \eta(\xi_1,\xi_2) \leq \mathcal{H}(\Phi\theta_1,\Phi\theta_2) \leq g\big(\eta(\theta_1,\theta_2)\big)\eta(\theta_1,\theta_2). \tag{2.6}$$

Since $\xi_2 \in \Phi\theta_2 \subseteq \mathcal{Y}_{\mathcal{X}}$, there exists $\theta_3 \in \mathcal{X}_{\mathcal{Y}}$ such that

$$\eta(\theta_3,\xi_2) = \eta(\mathcal{X},\mathcal{Y}). \tag{2.7}$$

From (2.3) and (2.7) and using weak $P-$property, we have that

$$\eta(\theta_2,\theta_3) \leq \eta(\xi_1,\xi_2). \tag{2.8}$$

From (2.6) and (2.8), we have

$$\eta(\theta_2,\theta_3) \leq \eta(\xi_1,\xi_2) \leq \mathcal{H}(\Phi\theta_1,\Phi\theta_2) \leq g\big(\eta(\theta_1,\theta_2)\big)\eta(\theta_1,\theta_2). \tag{2.9}$$

Continuing in this manner, we obtain two sequences $\{\theta_n\}$ and $\{\xi_n\}$ in $\mathcal{X}_{\mathcal{Y}}$ and $\mathcal{Y}_{\mathcal{X}}$ respectively, satisfying

(A) $\xi_n \in \Phi\theta_n \subseteq \mathcal{Y}_{\mathcal{X}}$,
(B) $\eta(\theta_{n+1}, \xi_n) = \eta(\mathcal{X}, \mathcal{Y})$,
(C) $\eta(\theta_n, \theta_{n+1}) \leq \eta(\xi_{n-1}, \xi_n) \leq g\big(\eta(\theta_{n-1}, \theta_n)\big)\eta(\theta_{n-1}, \theta_n)$.

for each $n = 0, 1, 2, \ldots$.

From **(C)** we have that

$$\eta(\theta_n, \theta_{n+1}) < \eta(\theta_{n-1}, \theta_n) \text{ for all } n \in \mathbb{N}. \tag{2.10}$$

If $\eta(\theta_{n_0}, \theta_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$, then $\eta(\theta_{n_0+1}, \theta_{n_0+2}) < \eta(\theta_{n_0}, \theta_{n_0+1}) = 0$, which is a contradiction. Thus, $\eta(\theta_n, \theta_{n+1}) > 0$ for all $n \in \mathbb{N}$.

Hence, $\{\eta(\theta_n, \theta_{n+1})\}$ is a decreasing sequence of positive real numbers, and therefore, there exists $r \geq 0$ such that $\lim_{n\to\infty} \eta(\theta_n, \theta_{n+1}) = 0$.

We claim that $r = 0$.

If $r > 0$, then from **(C)** we have that

$$0 < \frac{\eta(\theta_n, \theta_{n+1})}{\eta(\theta_{n-1}, \theta_n)} \leq g\big(\eta(\theta_{n-1}, \theta_n)\big) < 1 \text{ for all } n \in \mathbb{N}. \tag{2.11}$$

Taking limit in (2.11) as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{\eta(\theta_n, \theta_{n+1})}{\eta(\theta_{n-1}, \theta_n)} = \frac{r}{r} = 1 \leq \lim_{n\to\infty} g\big(\eta(\theta_{n-1}, \theta_n)\big) \leq 1. \tag{2.12}$$

Therefore, $\lim_{n\to\infty} g\big(\eta(\theta_{n-1}, \theta_n)\big) = 1$.

Since $g \in$, we have $\lim_{n\to\infty} g\big(\eta(\theta_{n-1}, \theta_n)\big) = \lim_{n\to\infty} g\big(\eta(\theta_n, \theta_{n+1})\big) = r = 0$.

Next, we prove that $\{\theta_n\}$ is a Cauchy sequence in $\mathcal{X}_{\mathcal{Y}}$.

From **(C)**, for fixed $p, q \in \mathbb{N}$, we have that

$$\eta(\theta_{p+1}, \xi_p) = \eta(\mathcal{X}, \mathcal{Y})$$

and

$$\eta(\theta_{q+1}, \xi_q) = \eta(\mathcal{X}, \mathcal{Y}).$$

By the weak P-property of $(\mathcal{X}, \mathcal{Y})$, we have

$$\eta(\theta_{p+1}, \theta_{q+1}) \leq \eta(\xi_p, \xi_q).$$

Let $\{\theta_n\}$ be not Cauchy and that $\limsup_{m,n\to\infty} \eta(\theta_n, \theta_m) > 0$. Then we have

$$\begin{aligned}
\eta(\theta_n, \theta_m) &\leq \eta(\theta_n, \theta_{n+1}) + \eta(\theta_{n+1}, \theta_{m+1}) + \eta(\theta_{m+1}, \theta_m) \\
&\leq \eta(\theta_n, \theta_{n+1}) + \eta(\xi_n, \xi_m) + \eta(\theta_{m+1}, \theta_m) \\
&\leq \eta(\theta_n, \theta_{n+1}) + \mathcal{H}(\Phi\xi_n, \Phi\xi_m) + \eta(\theta_{m+1}, \theta_m), \\
&\qquad (\text{since } \xi_n \in \Phi\theta_n, \xi_m \in \Phi\theta_m) \\
&\leq \eta(\theta_n, \theta_{n+1}) + g(\eta(\theta_n, \theta_m))\eta(\theta_n, \theta_m) + \eta(\theta_{m+1}, \theta_m)
\end{aligned}$$

$$\implies \eta(\theta_n,\theta_m) \le \frac{\eta(\theta_n,\theta_{n+1})+\eta(\theta_m,\theta_{m+1})}{1-g(\eta(\theta_n,\theta_m))}.$$

Using the fact that $\limsup_{m,n\to\infty} g(\eta(\theta_n,\theta_m)) > 0$ and $r=0$, from the last inequality, we have that

$$\limsup_{m,n\to\infty}[1-g(\eta(\theta_n,\theta_m))] = 0.$$

Hence, $\limsup_{m,n\to\infty} g(\eta(\theta_n,\theta_m)) = 1$.

Since $g\in\mathcal{G}$, we have that $\limsup_{m,n\to\infty}\eta(\theta_n,\theta_m)=0$, which is a contradiction, and therefore, $\{\theta_n\}$ is Cauchy in $\mathcal{X}_{\mathcal{Y}}\subseteq\mathcal{X}$.

Since $(\mathcal{W},\eta)$ is complete and $\mathcal{X}$ is closed, we have $\lim_{n\to\infty}\theta_n=\theta$ for some $\theta\in\mathcal{X}$.

Since Φ is $\mathcal{H}$-continuous (for it is an MVGC), we have

$$\lim_{n\to\infty}\mathcal{H}(\Phi\theta_n,\Phi\theta)=0. \tag{2.13}$$

Exactly in the similar manner as above, using **(C)**, we can prove that $\{\xi_n\}$ is Cauchy in $\mathcal{Y}$ and since $\mathcal{Y}$ is closed, there exists $\xi\in\mathcal{Y}$ such that $\lim_{n\to\infty}\xi_n=\xi$.

Since $\eta(\theta_{n+1},\xi_n)=\eta(\mathcal{X},\mathcal{Y})$ for all $n\in\mathbb{N}$, we have $\lim_{n\to\infty}\eta(\theta_{n+1},\xi_n)=\eta(\theta,\xi)=\eta(\mathcal{X},\mathcal{Y})$.

We claim that $\xi\in\Phi\theta$. Indeed, since $\xi_n\in\Phi\theta_n$ for all $n\in\mathbb{N}$, we have

$$\lim_{n\to\infty}\Delta(\xi_n,\Phi\theta)\le\lim_{n\to\infty}\mathcal{H}(\Phi\theta_n,\Phi\theta)=0.$$

Therefore, $\Delta(\xi,\Phi\theta)=0$. Since $\Phi\theta$ is closed, we have $\xi\in\Phi\theta$.

Now,

$$\eta(\mathcal{X},\mathcal{Y})\le\Delta(\theta,\Phi\theta)\le\eta(\theta,\xi)=\eta(\mathcal{X},\mathcal{Y}).$$

Hence $\Delta(\theta,\Phi\theta)=\eta(\mathcal{X},\mathcal{Y})$, i.e., θ is a BPP of Φ. □

Example 1. Let $\mathcal{W}=\mathbb{R}$ and $\eta(\theta,\xi)=|\theta-\xi|$ for all $\theta,\xi\in\mathbb{R}$. Then $(\mathcal{W},\eta)$ is a complete MS. Let $\mathcal{X}=[7,8]$ and $\mathcal{Y}=[-8,-7]$. Then $\mathcal{X}_{\mathcal{Y}}=\{7\}$ and $\mathcal{Y}_{\mathcal{X}}=\{-7\}$.

Define $\Phi:\mathcal{X}\to\Lambda(\mathcal{Y})$ by

$$\Phi\theta=\big[\frac{-\theta-7}{2},-7\big] \text{ for all } \theta\in\mathcal{X}.$$

Also, consider the function $g:[0,\infty)\to[0,1)$ by $g(t)=\frac{1}{1+t}$ for all $t>0$ and $g(0)=0$.

Now, $\Phi(7)=\{-7\}$ (i.e., $\Phi\theta\subseteq\mathcal{Y}_{\mathcal{X}}$ for all $\theta\in\mathcal{X}_{\mathcal{Y}}$).

If $\theta\ne\xi$, then

$$\begin{aligned}\mathcal{H}(\Phi\theta,\Phi\xi)&=\mathcal{H}\big(\big[\frac{-\theta-7}{2},-7\big],\big[\frac{-\xi-7}{2},-7\big]\big)\\&=\big|(\frac{-\theta-7}{2})-(\frac{-\xi-7}{2})\big|\\&=\frac{|\xi-\theta|}{2}\end{aligned}$$

$$\leq \frac{1}{1+\eta(\theta,\xi)} \cdot \eta(\theta,\xi), (\text{ since } \sup\{\eta(\theta,\xi) : \theta,\xi \in [7,8]\} = 1)$$
$$= g(\eta(\theta,\xi))\eta(\theta,\xi).$$

Thus, Φ is an MVGC and all conditions of Theorem 1 are satisfied. We see that $\Delta(7,\Phi(7)) = \eta(\mathcal{X},\mathcal{Y}) = 14$. Hence 7 is a BPP of Φ.

Corollary 1. *Let $(\mathcal{W},\eta)$ be a complete MS and $\mathcal{X}$ be a non-empty closed subset of $\mathcal{W}$. Let $\Phi : \mathcal{X} \to \Lambda(\mathcal{X})$ be a MVGC such that $\Phi\theta$ is compact for each $\theta \in \mathcal{X}$. Then Φ has a fixed point.*

Proof. The proof follows if we assume $\mathcal{X} = \mathcal{Y}$ in Theorem 1. □

Remark 4. In Corollary 1, if we take $\mathcal{X} = \mathcal{W}$, we obtain the multivalued analogue of Geraghty's result [11].

Next, we present an example for Corollary 1.

Example 2. Let $\mathcal{W} = [0,1]$ and $\eta(\theta,\xi) = |\theta-\xi|$ for all $\theta,\xi \in \mathbb{R}$. Then $(\mathcal{W},\eta)$ is a complete MS.

Define $\Phi : \mathcal{W} \to \Lambda(\mathcal{W})$ by $\Phi\theta = \left[0, \frac{\theta}{11}\right]$ for all $\theta \in [0,1]$. Figure 1 describes the plot of the multivalued mapping $\Phi(\theta)$.

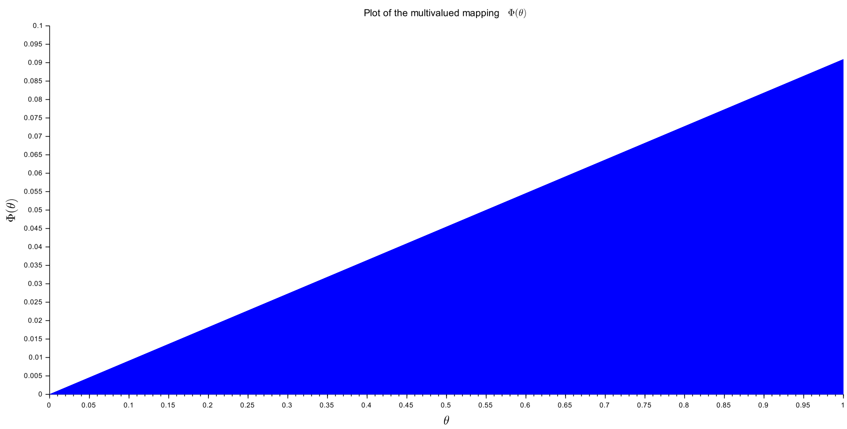

FIGURE 1. Plot of the multivalued mapping Φ.

If $\theta \neq \xi$, we have that

$$\mathcal{H}(\Phi\theta,\Phi\xi) = \left|\frac{\theta}{11} - \frac{\xi}{11}\right|$$
$$= \frac{|\xi-\theta|}{11} = \frac{1}{11}\eta(\theta,\xi)$$

$$\leq \frac{e^{-\eta(\theta,\xi)}}{2} \cdot \eta(\theta,\xi) = g(\eta(\theta,\xi)) \cdot \eta(\theta,\xi),$$

where $g(t) = \frac{e^{-t}}{2}$ for $t > 0$ and $g(0) = 0$ and hence $g \in \mathcal{G}$.

We observe that $0 \in \Phi(0)$, i.e., 0 is a fixed point of Φ.

Conclusion. We have proved the main result with a strong condition that images of the MVGC are compact sets. Relaxation of this compactness criterion is a suggested future work. Adopting a new technique, we extended the famous fixed point result due to Geraghty to its exact multivalued counterpart in terms of BPP. The corresponding fixed point theorem followed as a consequence.

ACKNOWLEDGEMENT

The author is very grateful to the learned referees for their valuable comments and suggestions towards correction of certain errors and improvement of the manuscript.

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