Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1223–1234

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3982

ν,μ DICHOTOMY AND BOUNDED SOLUTIONS OF DIFFERENTIAL EQUATIONS

OLEKSANDR BOICHUK, DMYTRO BIHUN, AND OLEKSANDR POKUTNYI

Received 01 November, 2021

Abstract. For a μ,ν dichotomic systems generalization of Palmer's lemma was proved. Necessary and sufficient conditions of the existence of bounded on the whole axis solutions and quasisolutions that minimize the residual norm were obtained. Index of the corresponding operator was found.

2010 *Mathematics Subject Classification:* 34B27, 34C11, 34D09, 34G10, 34H10, 47A52, 47A53

Keywords: optimization, exponential dichotomy, Moore-Penrose pseudoinverse operator, ill-posed problems, e-trichotomy, dynamical systems, quasisolution

1. INTRODUCTION

It is well known Palmer lemma in finite-dimensional space (see [20], [21]), which connects the notion of an exponential dichotomy on the axes of the corresponding equation and the noetherianity of the corresponding differential operator. Namely, such a result, proved in the work of Palmer K. [20] in 1984 is hold.

Theorem 1 ([20]). *If the linear system*

$$x'(t) = A(t)x(t)$$

admits an exponential dichotomy on both semi axes $[0;+\infty)$ *and* $(-\infty;0]$*, then the operator*

$$L : BC^1(-\infty,+\infty) \to BC(-\infty,+\infty)$$

which defines by the following rule

$$(Lx)(t) = x'(t) - A(t)x(t)$$

is Noetherian.

Later [21], in 1988 he proved converse assertion.

The authors acknowledge a partial financial support due to the project "Mathematical modelling of complex dynamical systems and processes caused by the state security" (Reg. No. 0123U100853).

The work was supported by the project of young scientists of NAS of Ukraine No. 01-01/03-2021.

Theorem 2 ([21]). *Suppose that $A(t)$ - $n\times n$ matrix valued function, bounded and continuous on the interval J, where $J=(-\infty,+\infty)$, $[0,+\infty)$ or $(-\infty,0]$. Suppose that the operator*

$$L:BC^1(J)\to BC(J),$$

defined below, is semi-Fredholm. Then if J semi axe, then homogeneous system admits an exponential dichotomy on J and if $J=(-\infty,+\infty)$ admits an exponential dichotomy on both semi axes $[0;+\infty)$ and $(-\infty,0]$.

For difference equations in Banach space such assertion contains in [15] (see also [1]). Fredholm differential operators with unbounded operators were considered in [17].

Recently, various different kinds of nonuniform dichotomy are proposed (see [4], [7], [14], [18], [19], [22]), nonuniform polynomial dichotomy [3], [6], [8], ρ-nonuniform exponential dichotomy [5], nonuniform (μ, ν)-dichotomy (see [2], [9], [10], [12], [13], [25]).

In the present article we are going to prove Palmer lemma for the equation that admits the so called μ, ν-dichotomy in the space endowed with the Frechet topology (the topology of the corresponding space is generated by the system of semi-norms). It should be noted that exponential dichotomy for the differential equations in the Frechet spaces was developed in the paper [11].

In the paper [16] the author give an example of a semigroup of bounded operators $\{e^{m\Delta}: m\in\mathbb{N}\}$ in the Frechet space that has dichotomy but not in a Banach space, where Δ is the Laplace operator in unbounded domain.

On the other hand H.O. Walther [23], [24], recently, investigated the delay equation in the space $C((-\infty,0];\mathbb{R}^n)$ which is the Frechet space. The Frechet space $C((-\infty,0];\mathbb{R}^n)$ has the advantage that it contains all histories $x_t=x(t+\cdot),t\in\mathbb{R}$ of every solution of the differential equation $x'(t)=f(x_t)$ in contrast to a Banach space.

The exponential dichotomy for operators on the distribution space also requires the use of Frechet spaces instead of Banach spaces.

2. Statement of the problem

In the space $\mathbb{R}^n$ we consider a system of differential equation of the following form:

$$x'(t)+A(t)x(t)=f(t), \tag{2.1}$$

where vector-function $f\in L^m_\infty(\mathbb{R}^n)$, which means it has bounded semi-norms $||f||_m$,

$$L^m_\infty(\mathbb{R}^n)=\left\{f:\mathbb{R}\to\mathbb{R}^n:||f||_m=\sup_{t\in[-m;m]}||f(t)||<+\infty, m\in\mathbb{N}\right\},$$

matrix-valued function $A(t)\in\mathcal{L}(\mathbb{R}^n)$ which is strongly continuous and $||A||_m<+\infty$ for any m:

$$||A||_m=\sup_{t\in[-m,m]}||A(t)||$$

and the homogeneous system

$$x'(t) + A(t)x(t) = 0 \tag{2.2}$$

allows μ, ν dichotomy on the semi-axes $\mathbb{R}^s_+ = [s;+\infty)$ and $\mathbb{R}^s_- = (-\infty;s]$ with matrix projector-valued functions $P_+(t), t \geq s$, $P_-(t), t \leq s$. We denote by $X(t,\tau)$ fundamental matrix (2.2) normalized at $t = \tau$. Let us recall the corresponding definition of ν, μ dichotomy [7].

Definition 1. System (2.2) allows μ, ν dichotomy on the interval J, if projector-valued function $P(t) = P^2(t), t \in J$ exist, such that

i) $X(t,s)P(s) = P(t)X(t,s)$; constants $\alpha, \beta, d > 0$ and $\varepsilon \geq 0$ exist, such that

ii) $||X(t,s)P(s)|| \leq d\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} \nu(|s|)^{\varepsilon}, t \geq s, t, s \in J$

and

iii) $||X(t,s)Q(s)|| \leq d\left(\frac{\mu(s)}{\mu(t)}\right)^{-\beta} \nu(|s|)^{\varepsilon}, s \geq t, t, s \in J$ where $Q(t) = I - P(t)$.

3. Main results

Consider the case when $f \in L^m_\infty(\mathbb{R})$ and dichotomy from the point $s = 0$ on the semi-axes $\mathbb{R}_+$, $\mathbb{R}_-$. In fact, we can consider a family of equations

$$x'_\varepsilon(t) + A(t)x_\varepsilon(t) = f_\varepsilon(t) \tag{3.1}$$

instead of the equation (2.1), where $f_\varepsilon(t) = e^{-\varepsilon|t|^b} f(t)$. It is easy to see that

$$||f - f_\varepsilon||_m = \sup_{t \in [-m;m]} |(1 - e^{-\varepsilon|t|^b}) f(t)| \leq ||f||_{L^m_\infty}(1 - e^{-\varepsilon m b}) \to 0, \varepsilon \to 0.$$

Thus, the sequence f_ε converges to function f in the corresponding Frechet space. Using projectors that correspond for the dichotomy on the semi-axes $P_+(t), t \geq 0$ and $P_-(t), t \leq 0$ we introduce such matrix

$$D = P_+(0) - (I - P_-(0))$$

and projectors $P_{N(D)}$ and $P_{N(D^*)}$ on the kernel and cokernel of the matrix D, respectively. First, let us explore the question concerning Lyapunov's terms of the homogeneous equation.

Further, for simplicity, we would consider the case when $\mu(t) = e^{at}, \nu(t) = e^{bt}$. Note that the set of bounded solutions of the homogeneous equation on the semi-axes looks as follows

$$x(t, \xi_1, \xi_2) = \begin{cases} X(t,0)P_+(0)\xi_1, & t \geq 0 \\ X(t,0)(I - P_-(0))\xi_2, & t \leq 0. \end{cases}$$

Indeed

$$||X(t,0)P_+(0)\xi_1|| \leq d_1 e^{-\alpha_1 a t}||\xi_1||, t \geq 0.$$

Then

$$\sup_{t\in[0,m]} ||X(t,0)P_+(0)\xi_1|| \leq d_1||\xi_1||, t \geq 0.$$

Similarly

$$||X(t,0)(I-P_-(0))\xi_2|| \leq d_2 e^{\beta_2 at}||\xi_2||, t \leq 0.$$

$$\sup_{t\in[-m;0]} ||X(t,0)(I-P_-(0))\xi_2|| \leq d_2||\xi_2||, t \leq 0.$$

In order for this expression to define bounded on the entire axis solutions, it is necessary to unite them at zero. From the condition

$$x(0+,\xi_1,\xi_2) = x(0-,\xi_1,\xi_2)$$

we obtain such matrix equation

$$P_+(0)\xi_1 = (I-P_-(0))\xi_2.$$

It is easy to prove, that the set of solutions of such system coincides with the set of solutions of the equation

$$P_+(0)\xi = (I-P_-(0))\xi$$

which can be rewritten as follows

$$D\xi = 0.$$

It is known [1] that the set of solutions of such equation can be presented as

$$\xi = P_{N(D)}c, \ \ \forall c \in \mathbb{R}^n.$$

The dimensionality of projector $P_{N(D)}$ determines the linearly independent number of solutions of such system. If $r = dimP_{N(D)}$, then we can rewrite the set of solutions in the following form

$$\xi = P_{N(D)_r}c_r, \ \ \forall c_r \in \mathbb{R}^r \ \ (r = dimP_{N(D)}).$$

Note also that the definition of projector implies that

$$P_+(0)P_{N(D)} = (I-P_-(0))P_{N(D)}.$$

Then the set of bounded solutions of a homogeneous system can be represented in the following form

$$x(t,c_r) = X(t,0)P_+(0)P_{N(D)_r}c_r$$

or

$$x(t,c_r) = X(t,0)(I-P_-(0))P_{N(D)_r}c_r.$$

If we consider such limit as

$$\lim_{t\to+\infty}\frac{ln||x(t,c_r)||}{t} \leq \lim_{t\to+\infty}\frac{ln(d_1e^{-\alpha_1 at}||P_{N(D)_r}c_r||)}{t} = -\alpha_1 a.$$

Similarly

$$\lim_{t\to-\infty}\frac{ln||x(t,c_r)||}{t} \leq \lim_{t\to\infty}\frac{ln(d_2e^{\beta_2 at}||P_{N(D)_r}c_r||)}{t} = \beta_2 a.$$

Theorem 3. *Under the conditions μ, ν dichotomy, solutions of the system (2.1) for the right-hand side of $f \in L_\infty^m(\mathbb{R})$ exist if and only if the following solvability condition fulfills*

$$P_{N(D^*)}\left\{\int_0^\infty X(0,\tau)(I-P_+(\tau))f(\tau)d\tau + \int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau\right\} = 0. \tag{3.2}$$

Under condition (3.2) the set of bounded solutions has the following form

$$x(t,c_r) = X(t,0)P_+(0)P_{N(D)_r}c_r + (G[f])(t,0),$$

where $(G[f])(t,0)$ is generalized Green's operator:

$$(G[f])(t,0) = \begin{cases} -\int_t^{+\infty} X(t,\tau)(I-P_+(\tau))f(\tau)d\tau \\ \quad +\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau, & t \geq 0 \\ \int_{-\infty}^t X(t,\tau)P_-(\tau)f(\tau)d\tau \\ \quad -\int_t^0 X(t,\tau)(I-P_-(\tau))f(\tau)d\tau, & t \leq 0. \end{cases} \tag{3.3}$$

Moreover, if $f \in L_\infty^m(\mathbb{R})$, then $x, x' \in L_\infty^m(\mathbb{R})$.

Proof. For simplicity, we take $\mu(t) = e^{at}$, $\nu(t) = e^{bt}$, $a \geq b$. Bounded on semi-axes $\mathbb{R}^+$ and $\mathbb{R}^-$ solutions of the inhomogeneous family of equations (3.1), have the following form:

$$x(t,\xi) = \begin{cases} X(t,0)P_+(0)\xi - \int_t^{+\infty} X(t,\tau)(I-P_+(\tau))f(\tau)d\tau \\ \quad +\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau, & t \geq 0 \\ X(t,0)(I-P_-(0))\xi + \int_{-\infty}^t X(t,\tau)P_-(\tau)f(\tau)d\tau \\ \quad -\int_t^0 X(t,\tau)(I-P_-(\tau))f(\tau)d\tau, & t \leq 0 \end{cases} \tag{3.4}$$

Indeed:

$$||X(t,0)P_+(0)\xi|| \leq d_1 e^{-\alpha_1 at}||\xi||, t \geq 0,$$

$$\int_0^t ||X(t,\tau)P_+(\tau)f(\tau)||d\tau \leq d_1 \int_0^t e^{-\alpha_1 a(t-\tau)+\varepsilon b\tau}||f||_{L_\infty^m} d\tau$$
$$\leq \frac{d_1}{\alpha_1 a + \varepsilon b}(e^{\varepsilon bt} - e^{-\alpha_1 at})||f||_{L_\infty^m}.$$

Similarly

$$\left|\left|\int_t^{+\infty} X(t,\tau)(I-P_+(\tau))f(\tau)d\tau\right|\right| \leq \int_t^{+\infty} ||X(t,\tau)(I-P_+(\tau))f(\tau)||d\tau$$
$$\leq d_1 \int_t^{+\infty} e^{-\beta_1 a(\tau-t)+\varepsilon b\tau}||f||_{L_\infty^m} d\tau$$
$$\leq \frac{d_1 e^{\varepsilon bt}}{\beta_1 a - \varepsilon b}||f||_{L_\infty^m}.$$

Thus, adding the obtained inequalities, we have

$$||x(t,\xi)|| \leq d_1 e^{-\alpha_1 at}||\xi||$$

$$+\left(d_1 e^{\varepsilon bt}\left(\frac{1}{\alpha_1 a+\varepsilon b}+\frac{1}{\beta_1 a-\varepsilon b}\right)-\frac{d_1}{\alpha_1 a+\varepsilon b}e^{-\alpha_1 at}\right)||f||_{L_\infty^m}.$$

Finally we obtain

$$\sup_{t\in[0;m]}||x(t,\xi)||\leq d_1||\xi|| \tag{3.5}$$
$$+\left(d_1 e^{\varepsilon bm}\left(\frac{1}{\alpha_1 a+\varepsilon b}+\frac{1}{\beta_1 a-\varepsilon b}\right)-\frac{d_1}{\alpha_1 a+\varepsilon b}e^{-\alpha_1 am}\right)||f||_{L_\infty^m}.$$

□

It is also easy to obtain appropriate estimates for negative real numbers and the derivative of the solutions. The boundedness of other integrals is checked in the same way. Based on that

$$\frac{\partial(-\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau)}{\partial t}$$
$$=X(t,t)(I-P_+(t))f(t)-A(t)\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau$$
$$+X(t,t)P_+(t)f(t)+A(t)\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau$$
$$=f(t)+A(t)\left\{-\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau\right\}$$

which proves that expression (3.4) defines all bounded solutions of the system (2.1) on the semi-axes.

In order for the system (3.4) to define bounded solutions on the whole axis, it is necessary and sufficient that the next condition is fulfilled

$$x(0+,\xi)=x(0-,\xi).$$

This condition is equivalent to the solvability of the matrix equation

$$P_+(0)\xi-\int_0^{+\infty}X(0,\tau)(I-P_+(\tau))f(\tau)d\tau \tag{3.6}$$
$$=(I-P_-(0))\xi+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau.$$

If ξ is the solution of the equation (3.6), then substituting it into (3.4) leads us to the bounded on the whole axis solution of the system (2.1). Actually, based on the fact that matrix

$$D=P_+(0)-I+P_-(0)$$

always has a pseudo-inverse by Moore-Penrose, the set of the bounded on the whole axis solutions of the system (2.1) could be represented in the following form

$$x(t,\xi)=\begin{cases} X(t,0)P_+(0)\xi-\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau \\ \qquad +\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau, & t\geq 0 \\ X(t,0)(I-P_-(0))\xi+\int_{-\infty}^t X(t,\tau)P_-(\tau)f(\tau)d\tau- \\ \qquad -\int_t^0 X(t,\tau)(I-P_-(\tau))f(\tau)d\tau, & t\leq 0 \end{cases} \tag{3.7}$$

Let $g=\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau+\int_0^{+\infty}X(0,\tau)(I-P_+(\tau))f(\tau)d\tau$.

Condition $P_{N(D^*)}g=0$ [1] is necessary and sufficient for the solvability of the equation $D\xi=g$. Then we obtain that the condition for the existence of bounded on the whole axis solutions of the system (2.1) is equivalent to the solvability of the following matrix equation:

$$D\xi=\int_0^\infty X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau. \tag{3.8}$$

Since matrix D has pseudo-inverse by Moore-Penrose matrix, then equation (3.8) has solutions if and only if

$$P_{N(D^*)}\left\{\int_0^\infty X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau\right\}=0.$$

Under this condition, equation (3.8) has a set of solutions

$$\xi=D^+\left(\int_0^\infty X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau\right)+P_{N(D)}c, \tag{3.9}$$

where c is an arbitrary vector of the corresponding dimensionality. Substituting the obtained solutions into (3.4) we obtain a general view of the solutions bounded on the entire axis in this form

$$x(t,c_r)=X(t,0)P_+(0)P_{N(D)_r}c_r+(G[f])(t,0),$$

where $(G[f])(t,0)$ is generalized Green's operator:

$$(G[f])(t,0)=\begin{cases} -\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau \\ \qquad +\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau, & t\geq 0 \\ \int_{-\infty}^t X(t,\tau)P_-(\tau)f(\tau)d\tau \\ \qquad -\int_t^0 X(t,\tau)(I-P_-(\tau))f(\tau)d\tau, & t\leq 0. \end{cases} \tag{3.10}$$

Green's operator has the following property at 0 relative to the jump:

$$\begin{aligned}(G[f])(0+0)-(G[f])(0-0)&\\ =-\int_0^\infty X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+P_+(0)D^-g&\\ -\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau-(I-P_-(0))D^-g&\\ =-g+P_+(0)D^-g-D^-g+P_-(0)D^-g&\end{aligned}$$

$$= (P_+(0) - I + P_-(0))D^- g - g = DD^- g - g$$
$$= -(I - DD^-)g = -P_{N(D^*)}g = -\int_{-\infty}^{+\infty} H(t)f(t)dt = 0.$$

The corresponding solution is bounded. Thus, we obtain the statement of the theorem.

Remark 1. If the condition (3.2) isn't hold then expression (3.9) defines the set of quasisolutions of the system (3.8) (elements ξ from the set (3.9) give the minimum of the norm $||D\xi - g||$).

Corollary 1. *For any $f \in L_\infty^m(\mathbb{R}^n)$ there are constants d, $N_1(m) > 0$ such that the following estimates are satisfied*

$$||x||_{L_\infty^m} \le d||P_{N(D)}c|| + N_1(m)||f||_{L_\infty^m},$$
$$||G[f]||_{L_\infty^m} \le N_1(m)||f||_{L_\infty^m}.$$

For any $f_\varepsilon(t) = e^{-\varepsilon|t|b} f(t) \in L_\infty^m(\mathbb{R}^n)$ there are constants d, $N_2(m) > 0$ such that the following estimates are satisfied

$$||x_\varepsilon||_{L_\infty^m} \le d||P_{N(D)}c|| + N_2(m)||f||_{L_\infty^m}$$

or

$$||x_\varepsilon||_{L_\infty^m} \le d||P_{N(D)}c|| + N_2(m)e^{\varepsilon mb}||f_\varepsilon||_{L_\infty^m}$$
$$||G[f_\varepsilon]||_{L_\infty^m} \le N_2(m)||f||_{L_\infty^m}$$

or

$$||G[f_\varepsilon]||_{L_\infty^m} \le N_2(m)e^{\varepsilon mb}||f_\varepsilon||_{L_\infty^m}$$

Proof. Together with the inequality (3.5) we can obtain the following inequalities with the right hand side $f(t)$:

$$\sup_{t\in[0;m]} ||x(t,\xi)|| \le d_1||\xi|| + \left(d_1 e^{\varepsilon bm}\left(\frac{1}{\alpha_1 a + \varepsilon b} + \frac{1}{\beta_1 a - \varepsilon b}\right) - \frac{d_1}{\alpha_1 a + \varepsilon b} e^{-\alpha_1 am}\right)||f||_{L_\infty^m},$$
$$\sup_{t\in[-m;0]} ||x(t,\xi)|| \le d_2||\xi|| + \left(d_2 e^{\varepsilon bm}\left(\frac{1}{\alpha_2 a - \varepsilon b} + \frac{1}{\beta_2 a + \varepsilon b}\right) - \frac{d_2}{\beta_2 a + \varepsilon b} e^{-\beta_2 am}\right)||f||_{L_\infty^m}.$$

If $d = max\{d_1, d_2\}$,

$$K_1(m) = \max\left\{d_1 e^{\varepsilon bm}\left(\frac{1}{\alpha_1 a + \varepsilon b} + \frac{1}{\beta_1 a - \varepsilon b}\right), d_2 e^{\varepsilon bm}\left(\frac{1}{\alpha_2 a - \varepsilon b} + \frac{1}{\beta_2 a + \varepsilon b}\right)\right\},$$
$$L_1(m) = \min\left\{\frac{d_1 e^{-\alpha_1 am}}{\alpha_1 a + \varepsilon b}, \frac{d_2 e^{-\beta_2 am}}{\beta_2 a + \varepsilon b}\right\},$$

and $N_1(m) = K_1(m) - L_1(m)$, then

$$||x||_{L_\infty^m} \le d||P_{N(D)}c|| + N_1(m)||f||_{L_\infty^m}$$

For $f_\varepsilon(t) = e^{-\varepsilon b|t|} f(t)$ we have

$$\sup_{t\in[0;m]} ||x(t,\xi)|| \le d_1||\xi|| + \left(d_1\left(\frac{1}{\alpha_1 a}+\frac{1}{\beta_1 a}\right) - \frac{d_1}{\beta_1 a}e^{-\alpha_1 am}\right)||f||_{L^m_\infty}$$

$$\sup_{t\in[-m;0]} ||x(t,\xi)|| \le d_2||\xi|| + \left(d_2\left(\frac{1}{\alpha_2 a}+\frac{1}{\beta_2 a}\right) - \frac{d_2}{\beta_2 a}e^{-\beta_2 am}\right)||f||_{L^m_\infty}$$

If

$$K_2 = \max\left\{d_1\left(\frac{1}{\alpha_1 a}+\frac{1}{\beta_1 a}\right), d_2\left(\frac{1}{\alpha_2 a}+\frac{1}{\beta_2 a}\right)\right\}$$

$$L_2(m) = \min\left\{\frac{d_1}{\beta_1 a}e^{-\alpha_1 am}, \frac{d_2}{\beta_2 a}e^{-\beta_2 am}\right\}$$

and $N_2(m) = K_2 - L_2(m)$, then

$$||x_\varepsilon||_{L^m_\infty} \le d||P_{N(D)}c|| + N_2(m)||f||_{L^m_\infty}$$

or

$$||x_\varepsilon||_{L^m_\infty} \le d||P_{N(D)}c|| + N_2(m)e^{\varepsilon mb}||f_\varepsilon||_{L^m_\infty}$$

□

Corollary 2. *Operator*

$$\mathcal{L} := \frac{d}{dt} + A(t).$$

under conditions μ,ν dichotomy is Noetherian with index $r-d$, as the operator from the space

$$H^m_\infty(\mathbb{R}^n) = \{g \in L^m_\infty(\mathbb{R}^n),\ \textit{such that}\ g' \in L^m_\infty(\mathbb{R}^n)\}$$

into space $L^m_\infty(\mathbb{R}^n)$. Thus $\mathcal{L} : H^m_\infty(\mathbb{R}^n) \to L^m_\infty(\mathbb{R}^n)$.

Proof. Indeed, it follows from the proof of the previous theorem that the operator

$$\mathcal{L} := \frac{d}{dt} + A(t)$$

is Noetherian. The following numbers determine its index. Let us denote the number of linear independent solvability conditions

$$+P_{N(D^*)}\left\{\int_0^\infty T(0,\tau)(I-P_+(\tau))f_\varepsilon(\tau)d\tau + \int_{-\infty}^0 T(0,\tau)P_-(\tau)f_\varepsilon(\tau)d\tau\right\} = 0,$$

as d, and the number of linearly independent bounded solutions of the homogeneous equation of the form $P_{N(D)}c$ as r. Then the index of the operator $\mathcal{L}$ is determined as

$$ind\ \mathcal{L} = r - d.$$

□

Remark 2. Let us consider the case when this system is considered only on semi-axis. In this case, the system has a set of bounded solutions in the following form:

$$x(t,\xi_1) = T(t,0)P_+(0)\xi_1 - \int_t^{+\infty} T(t,\tau)(I-P_+(\tau))f_\varepsilon(\tau)d\tau$$
$$+\int_0^t T(t,\tau)P_+(\tau)f_\varepsilon(\tau)d\tau, \qquad t \geq 0$$

for the arbitrary heterogeneity and vector ξ_1. The boundedness of the solution from this set is proved in the same way as Theorem 1.

Example 1. Consider an example of a two-dimensional system

$$\frac{dx_1(t)}{dt} = th\, tx_1(t) + f_{1\varepsilon}(t),$$
$$\frac{dx_2(t)}{dt} = -th\, tx_2(t) + f_{2\varepsilon}(t).$$

Let us show that this system admits a uniform exponential dichotomy on the axes. In this case, we show that

$$X(t,\tau) = diag\left\{\frac{e^t+e^{-t}}{e^\tau+e^{-\tau}}, \frac{e^\tau+e^{-\tau}}{e^t+e^{-t}}\right\}.$$

Projectors

$$P_+(\tau) = diag\{0,1\}, P_-(\tau) = diag\{1,0\},$$
$$f(t) = (f_1(t), f_2(t))^T, f_\varepsilon(t) = (f_{1\varepsilon}(t), f_{2\varepsilon}(t))^T = e^{-\varepsilon|t|b}(f_1(t), f_2(t))^T$$

and $||f_\varepsilon||_\tau = e^{-\varepsilon|\tau|b}||f||_{L_\infty(\mathbb{R})}$. It is easy to show that the system allows uniform exponential dichotomy on the axes under such conditions. So let us write down the necessary and sufficient condition of solvability of such system. In this case, the matrix $D = P_+(0) - (I - P_-(0)) = 0$ and respectively $P_{N(D)} = P_{N(D^*)} = I$. The solvability condition takes the following form

$$\int_{-\infty}^{+\infty} \frac{f_{1\varepsilon}(\tau)}{e^\tau+e^{-\tau}} d\tau = 0$$

or

$$\int_{-\infty}^{+\infty} \frac{e^{-\varepsilon|\tau|b} f_1(\tau)}{e^\tau+e^{-\tau}} d\tau = 0$$

The set of bounded solutions looks like this

$$x(t,c) = \begin{pmatrix} 0 \\ \frac{2}{e^t+e^{-t}} c_2 \end{pmatrix} + (G[f_\varepsilon])(t,0), \forall c_2 \in \mathbb{R}$$

where

$$(G[f_\varepsilon])(t,0) = \begin{pmatrix} -\int_t^{+\infty} \frac{e^t+e^{-t}}{e^\tau+e^{-\tau}} f_{1\varepsilon}(\tau)d\tau, \tau \leq t \\ \int_0^t \frac{e^\tau+e^{-\tau}}{e^t+e^{-t}} f_{2\varepsilon}(\tau)d\tau, \tau \geq t \end{pmatrix}, \qquad t \geq 0$$

$$(G[f_\varepsilon])(t,0)=\begin{pmatrix} \int_{-\infty}^{t}\frac{e^t+e^{-t}}{e^\tau+e^{-\tau}}f_{1\varepsilon}(\tau)d\tau, \tau\le t \\ -\int_t^0\frac{e^\tau+e^{-\tau}}{e^t+e^{-t}}f_{2\varepsilon}(\tau)d\tau, \tau\ge t\end{pmatrix}, \qquad t\le 0.$$

or in the form

$$x(t,c)=\begin{pmatrix} 0 \\ \frac{2}{e^t+e^{-t}}c_2\end{pmatrix}+(G[f])(t,0), \forall c_2\in\mathbb{R}$$

where

$$(G[f])(t,0)=\begin{pmatrix} -\int_t^{+\infty}\frac{e^t+e^{-t}}{e^\tau+e^{-\tau}}e^{-\varepsilon|\tau|b}f_1(\tau)d\tau, \tau\le t \\ \int_0^t\frac{e^\tau+e^{-\tau}}{e^t+e^{-t}}e^{-\varepsilon|\tau|b}f_2(\tau)d\tau, \tau\ge t\end{pmatrix}, \qquad t\ge 0$$

$$(G[f])(t,0)=\begin{pmatrix} \int_{-\infty}^{t}\frac{e^t+e^{-t}}{e^\tau+e^{-\tau}}e^{-\varepsilon|\tau|b}f_1(\tau)d\tau, \tau\le t \\ -\int_t^0\frac{e^\tau+e^{-\tau}}{e^t+e^{-t}}e^{-\varepsilon|\tau|b}f_2(\tau)d\tau, \tau\ge t\end{pmatrix}, \qquad t\le 0.$$

REFERENCES

[1] A. A.Boichuk and A. M. Samoilenko, *Generalized inverse operators and Fredholm boundary-value problems.* Berlin: De Gruyter. 2nd edition, 2016. doi: 10.1515/9783110378443.

[2] L. Barreira, J. Chu, and C. Valls, "Lyapunov functions for general nonuniform dichotomies." *Milan J. Math.*, vol. 81, no. 1, pp. 153–169, 2013, doi: 10.1007/s00032-013-0198-y.

[3] L. Barreira, F. Meng, C. Valls, and J. Zhang, "Robustness of nonuniform polynomial dichotomies for difference equations." *Topol. Methods Nonlinear Anal.*, vol. 37, no. 2, pp. 357–376, 2011.

[4] L. Barreira and C. Valls, *Stability of nonautonomous differential equations.* Berlin: Springer, 2008. doi: 10.1007/978-3-540-74775-8; 978-3-540-74775-8.

[5] L. Barreira and C. Valls, "Journal of functional analysis." *Topol. Methods Nonlinear Anal.*, vol. 257, no. 2, pp. 464–484, 2009, doi: 10.1016/j.jfa.2008.11.018.

[6] L. Barreira and C. Valls, "Polynomial growth rates." *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 71, no. 11, pp. 5208–5219, 2009, doi: 10.1016/j.na.2009.04.005.

[7] L. Barreira and C. Valls, "Admissibility for nonuniform exponential contractions." *J. Differ. Equations*, vol. 249, no. 11, pp. 2889–2904, 2010, doi: 10.1016/j.jde.2010.06.010.

[8] A. Bento and C. Silva, "Stable manifolds for nonuniform polynomial dichotomies." *J. Funct. Anal.*, vol. 257, no. 1, pp. 122–148, 2009, doi: 10.1016/j.jfa.2009.01.032.

[9] A. Bento and C. Silva, "Nonuniform (μ,ν)-dichotomies and local dynamics of difference equations." *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 75, no. 1, pp. 78–90, 2012, doi: 10.1016/j.na.2011.08.008.

[10] A. Bento and C. Silva, "Generalized nonuniform dichotomies and local stable manifolds." *J. Dyn. Differ. Equations*, vol. 25, no. 4, pp. 1139–1158, 2013, doi: 10.1007/s10884-013-9331-4.

[11] A. A. Boichuk and A. A. Pokutnyi, "Exponential dichotomy and bounded solutions of differential equations in the frechet space." *Ukr. Math. J.*, vol. 66, no. 12, pp. 1781–1792, 2015, doi: 10.1007/s11253-015-1051-y.

[12] X. Chang, J. Zhang, and Q. J., "Robustness of nonuniform (μ,ν)-dichotomies in banach spaces." *J. Math. Anal. Appl.*, vol. 387, no. 2, pp. 582–594, 2012, doi: 10.1016/j.jmaa.2011.09.026.

[13] J. Chu, "Robustness of nonuniform behavior for discrete dynamics." *Bull. Sci. Math.*, vol. 137, no. 8, pp. 1031–1047, 2013, doi: 10.1016/j.bulsci.2013.03.003.

[14] J. Chu, F. Liao, S. Siegmund, Y. Xia, and W. Zhang, "Nonuniform dichotomy spectrum and reducibility for nonautonomous equations." *Bull. Sci. Math*, vol. 139, no. 5, pp. 538–557, 2015, doi: 10.1016/j.bulsci.2014.11.002.

[15] I. D. Chueshov, *Introduction to the theory of infinite-dimensional dissipative systems*. Kharkiv: ACTA, 2002. doi: 10.1007/978-1-4614-6946-9.

[16] E. Costa, "An extension of the concept of exponential dishotomy on frechet space which is stable under perturbation." *Comm. Pure Applied Anal.*, vol. 18, no. 2, pp. 845–868, 2019, doi: 10.3934/cpaa.2019041.

[17] Y. Latushkin and Y. Tomilov, "Fredholm differential operators with unbounded coefficients." *J. Differ. Equations*, vol. 208, no. 2, pp. 388–429, 2005, doi: 10.1016/j.jde.2003.10.018.

[18] M. Megan, B. Sasu, and A. Sasu, "On nonuniform exponential dichotomy of evolution operators in banach spaces." *Integral Equations Oper. Theory*, vol. 44, no. 1, pp. 71–78, 2002, doi: 10.1007/BF01197861.

[19] A. Minda and M. Megan, "On (h,k)-stability of evolution operators in banach spaces." *Appl. Math. Lett.*, vol. 24, no. 1, pp. 44–48, 2011, doi: 10.1016/j.aml.2010.08.009.

[20] K. J. Palmer, "Exponential dichotomies and transversal homoclinic points." *Journ. of Diff. Eq.*, vol. 55, pp. 225–256, 1984, doi: 10.1016/0022-0396(84)90082-2.

[21] K. J. Palmer, "Exponential dichotomies and fredholm operators." *Proc. Am. Math. Soc.*, vol. 104, pp. 149–156, 1988, doi: 10.2507/2047477.

[22] C. Preda, P. Preda, and A. Craciunescu, "A version of a theorem of r. datko for nonuniform exponential contractions." *J. Math. Anal. Appl.*, vol. 385, no. 1, pp. 572–581, 2012, doi: 10.1016/j.jmaa.2011.06.082.

[23] H. O. Walther, "Local invariant manifolds for delay differential equations with state space in $C^1((-\infty,0],R^n)$." *Electron. J. Qual. Theory Differ. Equ.*, no. 85, p. 29, 2016, doi: 10.14232/ejqtde.2016.1.85.

[24] H. O. Walther, "Semiflows for differential equations with locally bounded delay on solution manifolds in the space $C^1((-\infty,0],R^n)$." *Topol. Methods Nonlinear Anal.*, vol. 48, no. 2, pp. 507–537, 2016, doi: 10.12775/TMNA.2016.056.

[25] J. Zhang, M. Fan, and H. Zhu, "Nonuniform (h,k,μ,ν)-dichotomy with applications to nonautonomous dynamical systems." *J. Math. Anal. Appl.*, vol. 452, no. 1, pp. 505–551, 2017, doi: 10.1016/j.jmaa.2017.02.064.

Authors' addresses

Oleksandr Boichuk
Institute of mathematics of NAS of Ukraine, Differential equations and oscillation theory department, Tereschenkivska str. 3, 01024 Kiev-4, Ukraine
E-mail address: `boichuk.aa@gmail.com`

Dmytro Bihun
Institute of mathematics of NAS of Ukraine, Laboratory of boundary value problems of differential equations theory, Tereschenkivska str. 3, 01024 Kiev-4, Ukraine
E-mail address: `dmytrobigun94@gmail.com`

Oleksandr Pokutnyi
(**Corresponding author**) Institute of mathematics of NAS of Ukraine, Laboratory of boundary value problems of differential equations theory, Tereschenkivska str. 3, 01024 Kiev-4, Ukraine
Current address: Institute of mathematics of NAS of Ukraine, Laboratory of boundary value problems of differential equations theory, Tereschenkivska str. 3, 01024 Kiev-4, Ukraine
E-mail address: `lenasas@gmail.com`