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VORONOVSKAJA-TYPE QUANTITATIVE DIFFERENCE ESTIMATES OF POSITIVE LINEAR OPERATORS WITH DIFFERENT BASIS FUNCTIONS

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Abstract. The present article is an extension of work [2] and provides Voronovskaja-type quantitative estimate for the difference of positive linear operators with different fundamental functions. We also present the applications of such results using Bernstein operators, Bernstein-Kantorovich operators, Bernstein-Durrmeyer operators, Generalized-Bernstein operators based on Pólya distribution and its Durrmeyer and Kantorovich-type modifications. We use Mathematica software to direct massive computations.

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1. INTRODUCTION

At the outset of the study of difference of positive linear operators, Lupaş [16] is the first to commence the study. Thenceforth, it became an important field of research in Approximation Theory. Aral et al. [6] generalized the work of A. Lupaş [16] on polynomial differences. Gupta [9] studied quantitative difference estimates of Lupaş operators with Lupaş-Szász operators and Lupaş-Kantorovich operators using modulus of continuity. Although researchers have addressed several convergence estimates in various forms, in this context, we highlight some of the seminal contributions of authors in this regard, including the linear preservance of general Srivastava-Gupta operators [13], the rate of convergence for the exponential operators [11], introduction and convergence estimation in weighted statistical convergence [7, 8], the construction of the α-Schurer-Kantorovich operators and the establishment of its approximation results [18]. Several studies were attracted to the operators due to Srivastava and collaborators [21, 22] in order to further extend the results on this operator, for instance [1, 4, 15, 19, 24].

The authors in [10] studied some generic estimates for the difference of operators associated with different basis functions. Very latterly, Acu et al. [2] established Voronovskaja-type quantitative results for the difference of positive linear operators having the same fundamental functions. We consider here Voronovskaja-type quantitative difference estimates of positive linear operators having different fundamental functions.

We denote $J \subset \mathbb{R}$ as an interval and $\tilde{E}(J)$ as a space of continuous real valued functions determined on J containing the polynomials. Let us deal with two positive linear operators M_n and N_n having different basis functions as follows:

$$M_n(h,u) = \sum_{i \in K} F_i(h) p_{n,i}(u)$$

and

$$N_n(h,u) = \sum_{i \in K} G_i(h) q_{n,i}(u),$$

where $K = \mathbb{Z}^+ \cup \{0\}$, $u \in J$, F_i and G_i are positive linear functionals and $p_{n,i}$, $q_{n,i}$ denote the basis functions satisfying $\sum_{i \in K} p_{n,i} = e_0$ and $\sum_{i \in K} q_{n,i} = e_0$.

Raşa [20] associated the positive linear integral operators to discrete operators and considered the link between the two. Since it is simple to work with the discrete operators associated with positive linear operator. Let D_M and D_N defines the discrete operators associated to positive linear operators M_n and N_n given by

$$D_M(h,u) = \sum_{i \in K} h(b^{F_i}) p_{n,i}(u)$$

and

$$D_N(h,u) = \sum_{i \in K} h(b^{G_i}) q_{n,i}(u),$$

where $b^F = F(e_1)$. To prove the main result of the paper, we need the following lemma.

2. Auxiliary Result

Lemma 1 ([2]). *For arbitrary $u \in J$ and $h'' \in \tilde{E}(J)$ such that $\|h''\| := \sup_{u \in J} |h(u)| < \infty$, we have*

$$((M - D_M)h)(u) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!} ((M - D_M)(e_1 - ue_0)^j)(u)$$
$$= \sum_{i \in K} \left\{ F_i(R_2(h, b^{F_i}, .)) + \frac{1}{2} \left[h''(b^{F_i}) - h''(u) \right] \vartheta_2^{F_i} \right\} p_i(u),$$

where $R_2(h, b^{F_i}, .)$ is the remainder of Taylor's formula given by

$$R_2(h, b^F, u) = h(u) - h(b^F) - h'(b^F)(u - b^F) - \frac{h''(b^F)}{2}(u - b^F)^2$$
$$= \frac{(u - b^F)^2}{2} \left(h''(\eta) - h''(b^F) \right); \eta \in (u, b^F), \tag{2.1}$$

and $\vartheta_2^{F_i}$ *is governed through* $\vartheta_k^F = F(e_1 - b^F e_0)^k$, $k \in \mathbb{N}$.

Remark 1. For a positive linear functional F on $\tilde{E}(J)$ and $R_2(h, b^F, .)$ as the remainder of Taylor's formula given by (2.1), we have

$$\left|R_2\left(h, b^F, u\right)\right| \leq \frac{(u-b^F)^2}{2}\omega\left(h'', |u-b^F|\right),$$

where $\omega(h, .)$ denotes the first order modulus of continuity.

Furthermore, since $\vartheta_1^F := F(e_1 - b^F e_0) = 0$, we have

$$F(R_2(h, b^F, .)) = F(h) - h(b^F) - \frac{h''(b^F)}{2}\vartheta_2^F.$$

Remark 2. Taking into account Remark 1 and the well known inequality for $\varepsilon > 0$ given by

$$\omega(f, |t-u|) \leq \left(1 + \frac{(t-u)^2}{\varepsilon^2}\right)\omega(f, \varepsilon),$$

we have

$$\left|R_2\left(h, b^F, u\right)\right| \leq \left((u-b^F)^2 + \frac{(u-b^F)^4}{\varepsilon^2}\right)\frac{\omega(h'', \varepsilon)}{2},$$

implying

$$\left|F\left(R_2\left(h, b^F, .\right)\right)\right| \leq \left(\vartheta_2^F + \frac{\vartheta_4^F}{\varepsilon^2}\right)\frac{\omega(h'', \varepsilon)}{2}.$$

Next, we shall study the quantitative Voronovskaja theorem for the difference $M_n - N_n$.

3. Main Theorem

Theorem 1. *For* $u \in J$ *and* $h'' \in \tilde{E}(J)$ *such that* $\|h''\| := \sup_{u \in J} |h(u)| < \infty$, *we have*

$$\left|((M_n - N_n)h)(u) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}((M_n - N_n)(e_1 - ue_0)^j)(u)\right|$$
$$\leq \frac{[1+\alpha(u)]}{2}\omega(h'', \sqrt{\beta(u)}) + \frac{[1+((M_n+N_n)(e_1-ue_0)^2)(u)]}{2}\omega(h'', \sqrt{\gamma(u)}),$$

where

$$\alpha(u) = \sum_{i \in K}\left(\vartheta_2^{F_i} p_{n,i}(u) + \vartheta_2^{G_i} q_{n,i}(u)\right),$$
$$\beta(u) = \sum_{i \in K}\left(\vartheta_4^{F_i} p_{n,i}(u) + \vartheta_4^{G_i} q_{n,i}(u)\right),$$
$$\gamma(u) = \sum_{i \in K}\left((b^{F_i} - u)^2 \vartheta_2^{F_i}(u) p_{n,i}(u) + (b^{G_i} - u)^2 \vartheta_2^{G_i}(u) q_{n,i}(u)\right)$$

and $\vartheta_k^F(u) = F(e_1 - ue_0)^k$, $k \in \mathbb{N}$.

Proof. Let $u \in J$ be arbitrary fixed point. Then, we have

$$\begin{aligned}
&((M_n - N_n)h)(u) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}((M_n - N_n)(e_1 - ue_0)^j)(u) \\
&= \sum_{i\in K}\left\{ \left(F_i(h) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}(F_i(e_1 - ue_0)^j\right) p_{n,i}(u) \right. \\
&\quad \left. - \left(G_i(h) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}(G_i(e_1 - ue_0)^j\right) q_{n,i}(u) \right\} \\
&= \sum_{i\in K}\left\{ \left(F_i(h) - h(b^{F_i}) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}(F_i(e_1 - ue_0)^j\right) p_{n,i}(u) \right. \\
&\quad \left. - \left(G_i(h) - h(b^{G_i}) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}(G_i(e_1 - ue_0)^j\right) q_{n,i}(u) \right\} \\
&\quad + \sum_{i\in K}\left(h(b^{F_i})p_{n,i}(u) - h(b^{G_i})q_{n,i}(u)\right).
\end{aligned}$$

Using the fact that $\sum_{i\in K} p_{n,i} = e_0$ and $\sum_{i\in K} q_{n,i} = e_0$, we get

$$\begin{aligned}
&((M_n - N_n)h)(u) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}((M_n - N_n)(e_1 - ue_0)^j)(u) \\
&= \sum_{i\in K}\left\{ \left[F_i(h) - h(b^{F_i})\right] p_{n,i}(u) - \left[G_i(h) - h(b^{G_i})\right] q_{n,i}(u) \right. \\
&\quad + \left[h(b^{F_i}) - h(u) - h'(u)F_i(e_1 - ue_0) - \frac{h''(u)}{2}F_i(e_1 - ue_0)^2\right] p_{n,i}(u) \\
&\quad \left. - \left[h(b^{G_i}) - h(u) - h'(u)G_i(e_1 - ue_0) - \frac{h''(u)}{2}G_i(e_1 - ue_0)^2\right] q_{n,i}(u) \right\} \\
&= \sum_{i\in K}\left\{ \left[F_i(h) - h(b^{F_i})\right] p_{n,i}(u) - \left[G_i(h) - h(b^{G_i})\right] q_{n,i}(u) \right. \\
&\quad + \left[h(b^{F_i}) - h(u) - h'(u)(b^{F_i} - u) - \frac{h''(u)}{2}\vartheta_2^{F_i}(u)\right] p_{n,i}(u) \\
&\quad \left. - \left[h(b^{G_i}) - h(u) - h'(u)(b^{G_i} - u) - \frac{h''(u)}{2}\vartheta_2^{G_i}(u)\right] q_{n,i}(u) \right\}. \qquad (3.1)
\end{aligned}$$

Notice that, for a positive linear functional F, we have

$$\begin{aligned}
\vartheta_2^F(u) &= F(e_1 - ue_0)^2 = F(e_2) - 2ub^F + u^2F(e_0) \\
&= F(e_2) - (b^F)^2 + (b^F)^2 - 2ub^F + u^2 = \vartheta_2^F + (b^F - u)^2. \qquad (3.2)
\end{aligned}$$

Using (3.2) in (3.1) and taking into account (2.1), for fixed $u \in J$, we obtain

$$
\begin{aligned}
&((M_n - N_n)h)(u) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}((M_n - N_n)(e_1 - ue_0)^j)(u) \\
&= \sum_{i\in K} \left\{ \left[F_i(h) - h(b^{F_i}) - \frac{h''(u)}{2}\vartheta_2^{F_i} \right] p_{n,i}(u) - \left[G_i(h) - h(b^{G_i}) - \frac{h''(u)}{2}\vartheta_2^{G_i} \right] q_{n,i}(u) \right. \\
&\quad \left. + R_2\left(h,u,b^{F_i}\right) p_{n,i}(u) - R_2\left(h,u,b^{G_i}\right) q_{n,i}(u) \right\} \\
&= \sum_{i\in K} \left\{ \left[F_i(h) - h(b^{F_i}) - \frac{h''(b^{F_i})}{2}\vartheta_2^{F_i} \right] p_{n,i}(u) - \left[G_i(h) - h(b^{G_i}) - \frac{h''(b^{G_i})}{2}\vartheta_2^{G_i} \right] q_{n,i}(u) \right. \\
&\quad + \left[\frac{h''(b^{F_i}) - h''(u)}{2} \right] \vartheta_2^{F_i} p_{n,i}(u) - \left[\frac{h''(b^{G_i}) - h''(u)}{2} \right] \vartheta_2^{G_i} q_{n,i}(u) \\
&\quad \left. + R_2\left(h,u,b^{F_i}\right) p_{n,i}(u) - R_2\left(h,u,b^{G_i}\right) q_{n,i}(u) \right\} \\
&= \sum_{i\in K} \left\{ F_i\left(R_2(h,b^{F_i},.)\right) p_{n,i}(u) - G_i\left(R_2(h,b^{G_i},.)\right) q_{n,i}(u) + \left[\frac{h''(b^{F_i}) - h''(u)}{2} \right] \vartheta_2^{F_i} p_{n,i}(u) \right. \\
&\quad \left. - \left[\frac{h''(b^{G_i}) - h''(u)}{2} \right] \vartheta_2^{G_i} q_{n,i}(u) + R_2\left(h,u,b^{F_i}\right) p_{n,i}(u) - R_2\left(h,u,b^{G_i}\right) q_{n,i}(u) \right\}.
\end{aligned}
$$

Using Remark 2, we have

$$
\begin{aligned}
&\left| ((M_n - N_n)h)(u) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}((M_n - N_n)(e_1 - ue_0)^j)(u) \right| \\
&\le \sum_{i\in K} \left\{ \left(\vartheta_2^{F_i} + \frac{\vartheta_4^{F_i}}{\beta(u)} \right) \frac{\omega\left(h'', \sqrt{\beta(u)}\right)}{2} p_{n,i}(u) + \left(\vartheta_2^{G_i} + \frac{\vartheta_4^{G_i}}{\beta(u)} \right) \frac{\omega\left(h'', \sqrt{\beta(u)}\right)}{2} q_{n,i}(u) \right. \\
&\quad + \frac{1}{2}\left[\omega\left(h'', |b^{F_i} - u|\right) \vartheta_2^{F_i} p_{n,i}(u) + \omega\left(h'', |b^{G_i} - u|\right) \vartheta_2^{G_i} q_{n,i}(u) \right] \\
&\quad \left. + (b^{F_i} - u)^2 \frac{\omega\left(h'', |b^{F_i} - u|\right)}{2} p_{n,i}(u) + (b^{G_i} - u)^2 \frac{\omega\left(h'', |b^{G_i} - u|\right)}{2} q_{n,i}(u) \right\} \\
&= \frac{1}{2} \sum_{i\in K} \left\{ \left[\left(\vartheta_2^{F_i} + \frac{\vartheta_4^{F_i}}{\beta(u)} \right) p_{n,i}(u) + \left(\vartheta_2^{G_i} + \frac{\vartheta_4^{G_i}}{\beta(u)} \right) q_{n,i}(u) \right] \omega\left(h'', \sqrt{\beta(u)}\right) \right. \\
&\quad \left. + \vartheta_2^{F_i}(u)\omega\left(h'', |b^{F_i} - u|\right) p_{n,i}(u) + \vartheta_2^{G_i}(u)\omega\left(h'', |b^{G_i} - u|\right) q_{n,i}(u) \right\} \\
&\le \frac{\omega\left(h'', \sqrt{\beta(u)}\right)}{2} \sum_{i\in K} \left\{ \vartheta_2^{F_i} p_{n,i}(u) + \vartheta_2^{G_i}.q_{n,i}(u) + \frac{1}{\beta(u)} \left(\vartheta_4^{F_i} p_{n,i}(u) + \vartheta_4^{G_i} q_{n,i}(u) \right) \right\} \\
&\quad + \frac{\omega\left(h'', \sqrt{\gamma(u)}\right)}{2} \sum_{i\in K} \left\{ \left(1 + \frac{(b^{F_i} - u)^2}{\gamma(u)} \right) \vartheta_2^{F_i}(u) p_{n,i}(u) + \left(1 + \frac{(b^{G_i} - u)^2}{\gamma(u)} \right) \vartheta_2^{G_i}(u) q_{n,i}(u) \right\}
\end{aligned}
$$

$$= \frac{\omega\left(h'', \sqrt{\beta(u)}\right)}{2}[1+\alpha(u)] + \frac{\omega\left(h'', \sqrt{\gamma(u)}\right)}{2}\left[1+\sum_{i\in K}\left(\vartheta_2^{F_i}(u)p_{n,i}(u)+\vartheta_2^{G_i}(u)q_{n,i}(u)\right)\right].$$

Further, since we have

$$(M_n(e_1-ue_0)^k)(u) = \sum_{i\in K}\vartheta_k^{F_i}(u)p_{n,i}(u)$$
$$(N_n(e_1-ue_0)^k)(u) = \sum_{i\in K}\vartheta_k^{G_i}(u)q_{n,i}(u),$$

the theorem is proved. □

4. Illustrations

Now, we shall utilise Theorem 1 to estimate the Voronovskaja-type quantitative differences between some of the most often used positive linear operators in Approximation Theory, *viz.* Bernstein operators B_n, Bernstein Durrmeyer operators D_n, Bernstein Kantorovich operators K_n, Generalized-Bernstein operators based on Pólya distribution $P_n^{(1/n)}$, its Durrmeyer type integral modification $D_n^{(1/n)}$ and Kantorovich modification $K_n^{(1/n)}$.

Consider the well known Bernstein operators $B_n : C[0,1] \to C[0,1]$ given by

$$(B_n h)(u) = \sum_{i=0}^{n} p_{n,i}(u)h\left(\frac{i}{n}\right), \tag{4.1}$$

where $p_{n,i}(u) = \binom{n}{i}u^i(1-u)^{n-i},\ u \in [0,1]$ and $n \in \mathbb{N}$.

The Durrmeyer type integral modification of operators (4.1) for $h \in C[0,1]$ introduced by [3] is given by

$$(D_n h)(u) = (n+1)\sum_{i=0}^{n} p_{n,i}(u)\int_0^1 p_{n,i}(\varsigma)h(\varsigma)d\varsigma. \tag{4.2}$$

And the Bernstein-Kantorovich operators $K_n : L_1[0,1] \to C[0,1]$ proposed by [14] are defined as

$$(K_n h)(u) = (n+1)\sum_{i=0}^{n} p_{n,i}(u)\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(\varsigma)d\varsigma. \tag{4.3}$$

As considered by [17], the Generalized-Bernstein operators $P_n^{(1/n)} : C[0,1] \to C[0,1]$ based on Pólya distributions are given by

$$(P_n^{(1/n)}h)(u) = \sum_{i=0}^{n} p_{n,i}^{(1/n)}(u)h\left(\frac{i}{n}\right), \tag{4.4}$$

where $p_{n,i}^{(1/n)}(u) = \frac{2(n)!}{(2n)!}\binom{n}{i}(nu)_i(n-nu)_{n-i}$ and $(n)_i$ is the rising factorial given by $(n)_i = n(n+1)...(n+i-1)$.

The Durrmeyer type integral modification of operators (4.4) for $h \in C[0,1]$ introduced by [12] is given by

$$(D_n^{(1/n)}h)(u) = (n+1)\sum_{i=0}^{n} p_{n,i}^{(1/n)}(u)\int_0^1 p_{n,i}(\varsigma)h(\varsigma)d\varsigma. \tag{4.5}$$

Influenced by the work of [12], Agrawal et al. [5] proposed the Kantorovich modification of the operators (4.4) with $h \in C[0,1]$ as

$$(K_n^{(1/n)}h)(u) = (n+1)\sum_{i=0}^{n} p_{n,i}^{(1/n)}(u)\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(\varsigma)d\varsigma. \tag{4.6}$$

4.1. *Difference estimates between operators B_n and $P_n^{(1/n)}$*

Proposition 1. *Let $h'' \in C[0,1]$. Then, for the Voronovskaja-type difference between B_n and $P_n^{(1/n)}$, we have*

$$\left|((B_n - P_n^{(1/n)})h)(u) - \sum_{j=1}^{2}\frac{h^{(j)}(u)}{j!}((B_n - P_n^{(1/n)})(e_1 - ue_0)^j)(u)\right|$$
$$\le \frac{1}{2}\left[1 + \frac{u(1-u)(3n+1)}{n(n+1)}\right]\omega\left(h'', \sqrt{\gamma(u)}\right),$$

where

$$\gamma(u) = \frac{u(1-u)}{n^3(n+1)(n+2)(n+3)} \cdot \left[3u(1-u)(5n^4 - 24n^3 - n^2 - 16n - 12) + (27n^3 + 4n^2 + 11n + 6)\right].$$

Proof. If we indicate $F_i(h) = h\left(\frac{i}{n}\right)$, then we can express the operators (4.1) and (4.4) as

$$(B_n h)(u) = \sum_{i=0}^{n} F_i(h)p_{n,i}(u)$$

and

$$(P_n^{(1/n)}h)(u) = \sum_{i=0}^{n} F_i(h)p_{n,i}^{(1/n)}(u).$$

Thus, we have

$$b^{F_i} = \frac{i}{n}, \quad \vartheta_2^{F_i} = 0 = \vartheta_4^{F_i}, \quad \vartheta_2^{F_i}(u) = \left(\frac{i}{n} - u\right)^2.$$

By simple computations, we get $\alpha(u) = 0$, $\beta(u) = 0$, and

$$\gamma(u) = \frac{u(1-u)}{n^3(n+1)(n+2)(n+3)} \cdot \left[3u(1-u)(5n^4 - 24n^3 - n^2 - 16n - 12) + (27n^3 + 4n^2 + 11n + 6)\right].$$

Also, we have

$$[1+((B_n+P_n^{(1/n)})(e_1-ue_0)^2)(u)] = [1+\vartheta_2^{F_i}(u)p_{n,i}(u)+\vartheta_2^{F_i}(u)p_{n,i}^{(1/n)}(u)]$$
$$= 1+\frac{u(1-u)(3n+1)}{n(n+1)}.$$

The adoption of Theorem 1 leads us to the proof of the proposition. □

4.2. *Difference estimates between operators B_n and $D_n^{(1/n)}$*

Proposition 2. *Let $h'' \in C[0,1]$. Then, for the difference between B_n and $D_n^{(1/n)}$, we have*

$$\left|((B_n-D_n^{(1/n)})h)(u)-\sum_{j=1}^{2}\frac{h^{(j)}(u)}{j!}((B_n-D_n^{(1/n)})(e_1-ue_0)^j)(u)\right|$$
$$\leq \frac{1}{2}(1+\alpha(u))\,\omega\left(h'',\sqrt{\beta(u)}\right)$$
$$+\frac{1}{2}\left[\frac{u(1-u)(4n^3+n^2+5n+6)+n(n+1)(n^2+5n+8)}{n(n+1)(n+2)(n+3)}\right]\omega\left(h'',\sqrt{\gamma(u)}\right),$$

where

$$\alpha(u)=\frac{n^2(n-1)u(1-u)+(n+1)^2}{(n+1)(n+2)^2(n+3)},$$
$$\beta(u)=\frac{3}{(n+1)(n+2)^5(n+3)^2(n+4)(n+5)}$$
$$\cdot\left[n^4u^3(u-2)(n-1)(n-2)(n-3)(n-4)+n^3u^2(n-1)(n^4-15n^3\right.$$
$$+12n^2-60n-4)+2n^3u(n-1)(3n^3+7n^2+18n+2)$$
$$\left.+(n+1)^2(n+2)(n+3)(3n^2+5n+4)\right],$$
$$\gamma(u)=\frac{(u-1)}{n^3(n+1)^3(n+2)^2(n+3)^2}$$
$$\cdot\left[u^3\left(17n^8-176n^7-334n^6-17n^5-141n^4-1011n^3-1446n^2-900n-216\right)\right.$$
$$-u^2\left(17n^8-203n^7-440n^6-76n^5-81n^4-975n^3-1446n^2-900n-216\right)$$
$$-u\left(76n^7+189n^6+141n^5+120n^4+289n^3+325n^2+168n+36\right)$$
$$\left.-2n^3(n+1)(n+2)(n+3)\right].$$

Proof. If we denote $H_i(h)=(n+1)\int_0^1 p_{n,i}(\varsigma)h(\varsigma)d\varsigma$, then by using the same functional notation as available in Proposition 1, we can express the operators (4.1) and

(4.5) as

$$(B_nh)(u)=\sum_{i=0}^{n}F_i(h)p_{n,i}(u)$$

and

$$(D_n^{(1/n)}h)(u)=\sum_{i=0}^{n}H_i(h)p_{n,i}^{(1/n)}(u).$$

Thus, we require the following extra parameters given below:

$$b^{H_i}=\frac{i+1}{n+2},\quad \vartheta_2^{H_i}=\frac{(i+1)(n-i+1)}{(n+2)^2(n+3)},$$
$$\vartheta_4^{H_i}=\frac{3(i+1)(n-i+1)\left(i(n-4)(n-i)+3n^2+5n+4\right)}{(n+2)^4(n+3)(n+4)(n+5)},$$
$$\vartheta_2^{H_i}(u)=u^2-2u\left(\frac{i+1}{n+2}\right)+\frac{(i+1)(i+2)}{(n+2)(n+3)}.$$

Implication of Theorem 1 provides us the following:

$$\alpha(u)=\frac{n^2(n-1)u(1-u)+(n+1)^2}{(n+1)(n+2)^2(n+3)},$$
$$\begin{aligned}\beta(u)=&\frac{3}{(n+1)(n+2)^5(n+3)^2(n+4)(n+5)}\\ &\cdot\big[n^4u^3(u-2)(n-1)(n-2)(n-3)(n-4)+n^3u^2(n-1)(n^4-15n^3\\ &+12n^2-60n-4)+2n^3u(n-1)(3n^3+7n^2+18n+2)\\ &+(n+1)^2(n+2)(n+3)(3n^2+5n+4)\big],\end{aligned}$$
$$\begin{aligned}\gamma(u)=&\frac{(u-1)}{n^3(n+1)^3(n+2)^2(n+3)^2}\\ &\cdot\Big[u^3\left(17n^8-176n^7-334n^6-17n^5-141n^4-1011n^3-1446n^2-900n-216\right)\\ &-u^2\left(17n^8-203n^7-440n^6-76n^5-81n^4-975n^3-1446n^2-900n-216\right)\\ &-u\left(76n^7+189n^6+141n^5+120n^4+289n^3+325n^2+168n+36\right)\\ &-2n^3(n+1)(n+2)(n+3)\Big].\end{aligned}$$

Also, we have

$$\begin{aligned}[1+((B_n+D_n^{(1/n)})(e_1-ue_0)^2)(u)]&=[1+\vartheta_2^{F_i}(u)p_{n,i}(u)+\vartheta_2^{H_i}(u)p_{n,i}^{(1/n)}(u)]\\ &=\frac{u(1-u)(4n^3+n^2+5n+6)+n(n+1)(n^2+5n+8)}{n(n+1)(n+2)(n+3)}.\end{aligned}$$

Thus, Proposition 2 is proved. □

4.3. *Difference estimates between operators B_n and $K_n^{(1/n)}$*

Proposition 3. *Let $h'' \in C[0,1]$. Then, for the difference between B_n and $K_n^{(1/n)}$, we have*

$$\left|((B_n - K_n^{(1/n)})h)(u) - \sum_{j=1}^{2} \frac{h^{(j)}(u)}{j!}((B_n - K_n^{(1/n)})(e_1 - ue_0)^j)(u)\right|$$
$$\leq \frac{1}{2}\left(1 + \frac{1}{12(n+1)^2}\right)\omega\left(h'', \frac{1}{4\sqrt{5}(n+1)^2}\right)$$
$$+\frac{1}{2}\left[\frac{3u(1-u)(3n^3+2n^2+2n+1)+n(n+1)(3n^2+6n+4)}{3n(n+1)^3}\right]\omega\left(h'', \sqrt{\gamma(u)}\right),$$

where

$$\gamma(u) = \frac{1}{12n^3(n+1)^5(n+2)(n+3)}$$
$$\cdot\big[12u^4(n-1)(4n^9+44n^8+223n^7+463n^6+783n^5+1085n^4+1041n^3$$
$$+639n^2+228n+36) - 24u^3(15n^8-120n^7-140n^6-42n^5-184n^4-426n^3$$
$$-411n^2-192n-36) - u^2(48n^10+480n^9+1788n^8+6266n^7+7911n^6+5310n^5$$
$$+4891n^4+6738n^3+5964n^2+2724n+504) + u(506n^7+711n^6+678n^5+1003n^4$$
$$+1338n^3+1032n^2+420n+72) + n^3(n+1)(n+2)(n+3)\big].$$

Proof. Denoting $G_i(h) = (n+1)\int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(\varsigma)d\varsigma$ and making use of the same functional notations wherever required, we can express the operators (4.1) and (4.6) as follows:

$$(B_n h)(u) = \sum_{i=0}^{n} F_i(h)p_{n,i}(u),$$

and

$$(K_n^{(1/n)}h)(u) = \sum_{i=0}^{n} G_i(h)p_{n,i}^{(1/n)}(u).$$

Taking into account Proposition 1, we need the following additional values:

$$b^{G_i} = \frac{2i+1}{2(n+1)}, \quad \vartheta_2^{G_i} = \frac{1}{12(n+1)^2},$$
$$\vartheta_4^{G_i} = \frac{1}{80(n+1)^4}, \quad \vartheta_2^{G_i}(u) = \frac{1}{12(n+1)^2} + \left(\frac{2i+1}{2(n+1)} - u\right)^2.$$

By simple calculations on Theorem 1, we obtain

$$\alpha(u) = \frac{1}{12(n+1)^2}, \quad \beta(u) = \frac{1}{80(n+1)^4},$$

and

$$\gamma(u) = \frac{1}{12n^3(n+1)^5(n+2)(n+3)}$$
$$\cdot \big[12u^4(n-1)(4n^9+44n^8+223n^7+463n^6+783n^5+1085n^4+1041n^3$$
$$+639n^2+228n+36)-24u^3(15n^8-120n^7-140n^6-42n^5-184n^4-426n^3$$
$$-411n^2-192n-36)-u^2(48n^10+480n^9+1788n^8+6266n^7+7911n^6+5310n^5$$
$$+4891n^4+6738n^3+5964n^2+2724n+504)+u(506n^7+711n^6+678n^5+1003n^4$$
$$+1338n^3+1032n^2+420n+72)+n^3(n+1)(n+2)(n+3)\big].$$

Also, we have

$$[1+((B_n+K_n^{(1/n)})(e_1-ue_0)^2)(u)]$$
$$=[1+\vartheta_2^{F_i}(u)p_{n,i}(u)+\vartheta_2^{G_i}(u)p_{n,i}^{(1/n)}(u)]$$
$$=\left[\frac{3u(1-u)(3n^3+2n^2+2n+1)+n(n+1)(3n^2+6n+4)}{3n(n+1)^3}\right].$$

This completes the proof of the proposition. □

4.4. *Difference between operators $P_n^{(1/n)}$ and K_n*

Proposition 4. *Let $h'' \in C[0,1]$. Then the difference between the positive linear operators $P_n^{(1/n)}$ and K_n is given by*

$$\left|((P_n^{(1/n)}-K_n)h)(u)-\sum_{j=1}^{2}\frac{h^{(j)}(u)}{j!}((P_n^{(1/n)}-K_n)(e_1-ue_0)^j)(u)\right|$$
$$\leq \frac{1}{2}\left(1+\frac{1}{12(n+1)^2}\right)\omega\left(h'',\frac{1}{4\sqrt{5}(n+1)^2}\right)$$
$$+\frac{1}{2}\left[\frac{3u(1-u)(3n+1)+(3n^2+6n+4)}{3(n+1)^2}\right]\omega\left(h'',\sqrt{\gamma(u)}\right),$$

where

$$\gamma(u) = \frac{1}{12n(n+1)^4(n+2)(n+3)}$$
$$\cdot\big[12nu^3(u-2)(15n^4-53n^3-297n^2-355n-78)+u^2(180n^5-1003n^4$$
$$-4744n^3-5419n^2-1134n+24)$$
$$+u(367n^4+1180n^3+1159n^2+198n-24)+n(n+2)(n+3)\big].$$

Proof. Using the same functional notations as available, we can express the operators (4.4) and (4.3) in the following manner:

$$(P_n^{(1/n)}h)(u)=\sum_{i=0}^{n}F_i(h)p_{n,i}^{(1/n)}(u),$$

and

$$(K_nh)(u)=\sum_{i=0}^{n}G_i(h)p_{n,i}(u).$$

Thus, we have $\alpha(u)=\frac{1}{12(n+1)^2}$, $\beta(u)=\frac{1}{80(n+1)^4}$, and

$$\begin{aligned}\gamma(u)=&\frac{1}{12n(n+1)^4(n+2)(n+3)}\\&\cdot\big[12nu^3(u-2)(15n^4-53n^3-297n^2-355n-78)+u^2(180n^5-1003n^4\\&-4744n^3-5419n^2-1134n+24)\\&+u(367n^4+1180n^3+1159n^2+198n-24)+n(n+2)(n+3)\big].\end{aligned}$$

In addition, we have

$$\begin{aligned}[1+((P_n^{(1/n)}+K_n)(e_1-ue_0)^2)(u)]&=[1+\vartheta_2^{F_i}(u)p_{n,i}^{(1/n)}(u)+\vartheta_2^{G_i}(u)p_{n,i}(u)]\\&=\left[\frac{3u(1-u)(3n+1)+(3n^2+6n+4)}{3(n+1)^2}\right].\end{aligned}$$

Hence the proof follows using Theorem 1. □

4.5. *Difference between operators $P_n^{(1/n)}$ and D_n*

Proposition 5. *Let $h''\in C[0,1]$. Then, for the difference between $P_n^{(1/n)}$ and D_n, we have*

$$\begin{aligned}&\left|((P_n^{(1/n)}-D_n)h)(u)-\sum_{j=1}^{2}\frac{h^{(j)}(u)}{j!}((P_n^{(1/n)}-D_n)(e_1-ue_0)^j)(u)\right|\\&\le\frac{1}{2}(1+\alpha(u))\,\omega\left(h'',\sqrt{\beta(u)}\right)\\&\quad+\frac{1}{2}\left[\frac{2u(1-u)(2n^2+3n+3)+(n+1)(n^2+5n+8)}{(n+1)(n+2)(n+3)}\right]\omega\left(h'',\sqrt{\gamma(u)}\right),\end{aligned}$$

where

$$\begin{aligned}\alpha(u)&=\frac{n(n-1)u(1-u)+(n+1)}{(n+2)^2(n+3)},\\\beta(u)&=\frac{3}{(n+2)^4(n+3)(n+4)(n+5)}\\&\quad\cdot\big[nu^3(u-2)(n-1)(n-2)(n-3)(n-4)+nu^2(n-1)(n^3-14n^2\end{aligned}$$

$$+29n-28)+nu(n-1)(5n^2-3n+4)+(n+1)(3n^2+5n+4)\big],$$

$$\gamma(u)=\frac{2}{n(n+1)(n+2)^2(n+3)}$$
$$\cdot\big[4nu^3(u-2)(2n^3-12n^2-41n-39)+u^2(8n^4-69n^3-216n^2$$
$$-197n+4)+u(21n^3+52n^2+41n-4)+n(n+1)\big].$$

Proof. If we use the above-mentioned functional notations to represent the operators (4.4) and (4.2), then we can express them as

$$(P_n^{(1/n)}h)(u)=\sum_{i=0}^{n}F_i(h)p_{n,i}^{(1/n)}(u),$$

and

$$(D_nh)(u)=\sum_{i=0}^{n}H_i(h)p_{n,i}(u).$$

By basic calculation and taking Proposition 1 and Proposition 2 into consideration, we have the following:

$$\alpha(u)=\frac{n(n-1)u(1-u)+(n+1)}{(n+2)^2(n+3)},$$

$$\beta(u)=\frac{3}{(n+2)^4(n+3)(n+4)(n+5)}$$
$$\cdot\big[nu^3(u-2)(n-1)(n-2)(n-3)(n-4)+nu^2(n-1)(n^3-14n^2$$
$$+29n-28)+nu(n-1)(5n^2-3n+4)+(n+1)(3n^2+5n+4)\big],$$

$$\gamma(u)=\frac{2}{n(n+1)(n+2)^2(n+3)}$$
$$\cdot\big[4nu^3(u-2)(2n^3-12n^2-41n-39)+u^2(8n^4-69n^3-216n^2$$
$$-197n+4)+u(21n^3+52n^2+41n-4)+n(n+1)\big].$$

Also, we have

$$[1+((P_n^{(1/n)}+D_n)(e_1-ue_0)^2)(u)]=[1+\vartheta_2^{F_i}(u)p_{n,i}^{(1/n)}(u)+\vartheta_2^{H_i}(u)p_{n,i}(u)]$$
$$=\frac{2u(1-u)(2n^2+3n+3)+(n+1)(n^2+5n+8)}{(n+1)(n+2)(n+3)}.$$

Hence the application of Theorem 1 leads to the proof of the proposition. □

Remark 3. Here, we have studied the difference of Bernstein polynomials with generalized Bernstein based on Pólya, its Durrmeyer and Kantorovich variant. We also investigated the difference estimates between the operator having Pólya basis to Durrmeyer and Kantorovich operators having Bernstein basis. Similar efforts can be made to address the differences of a specific class of operators defined in [22, 23].

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