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CONVERGENCE THEOREM AND CONVERGENCE RATE OF A NEW FASTER ITERATION METHOD FOR CONTINUOUS FUNCTIONS ON AN ARBITRARY INTERVAL

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Abstract. The aim of this paper is to propose a new faster iterative method, called the MN-iteration process, for approximating a fixed point of continuous functions on an arbitrary interval. Then, a necessary and sufficient condition for the convergence of the MN-iteration of continuous functions on an arbitrary interval is established. We also compare the rate of convergence between the proposed iteration and some other iteration processes in the literature. Specifically, our main result shows that MN-iteration converges faster than NSP-iteration to the fixed point. We finally give numerical examples to compare the result with Mann, Ishikawa, Noor, SP and NSP iterations. Our findings improve corresponding results in the contemporary literature.

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1. INTRODUCTION

Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous function. A point $x \in C$ is called a fixed point of f if $f(x) = x$.

Iteration procedures are used in nearly every branch of applied mathematics. There are many iterative methods for finding a fixed point of f. In computational mathematics, it is important to compare the iterative schemes with regard to their rate of convergence.

The classical iteration process was introduced by Mann [7] which is formulated as follows:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n f(u_n) \tag{1.1}$$

for all $n \geq 1$, where $\alpha_n \in [0,1]$. Such an iteration process is known as *Mann iteration.* In 1991, Borwein and Borwein [3] proved the convergence theorem for a continuous function on the closed and bounded interval in the real line by using iteration (1.1).

The Ishikawa iterative scheme, usually called the two-step iteration method, due to Ishikawa [6] is given by $s_1 \in C$ and

$$\begin{aligned} t_n &= (1-\beta_n)s_n + \beta_n f(s_n), \\ s_{n+1} &= (1-\alpha_n)s_n + \alpha_n f(t_n) \end{aligned} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]. Such iterative method is called *Ishikawa iteration.* In 2006, Qing and Qihou [11] proved the convergence theorem of the sequence generated by iteration (1.2) for a continuous function on the closed interval in the real line (see also [15]).

In 2000, Noor [8] defined the following iterative scheme by $l_1 \in C$ and

$$\begin{aligned} m_n &= (1-\mu_n)l_n + \mu_n f(l_n), \\ v_n &= (1-\beta_n)l_n + \beta_n f(m_n), \\ l_{n+1} &= (1-\alpha_n)l_n + \alpha_n f(v_n) \end{aligned} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1], which is called *Noor iteration* for continuous functions on an arbitrary interval in the real line.

Clearly, the Mann and Ishikawa iteration processes are special cases of the Noor iteration. However, there are only a few articles concerning comparison of those iterative methods in order to establish which one converges faster. For more details, we orient the reader to [1, 2, 10, 12, 14] and references therein.

Rhoades [13] introduced the concept to compare iterative methods which one converges faster as follows.

Definition 1 ([9, Definition 3.1])**.** Let C be a closed interval on the real line and let $f: C \to C$ be a continuous mapping. Suppose that $\{x_n\}$ and $\{w_n\}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}$ is said to converge faster than $\{w_n\}$ if

$$|x_n - p| \leq |w_n - p|$$

for all $n \geq 1$.

In 2011, Phuengrattana and Suantai [9] introduced and studied the SP-iteration as follows: $h_1 \in C$ and

$$\begin{aligned} e_n &= (1-\mu_n)h_n + \mu_n f(h_n), \\ d_n &= (1-\beta_n)e_n + \beta_n f(e_n), \\ h_{n+1} &= (1-\alpha_n)d_n + \alpha_n f(d_n) \end{aligned} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1]. They showed that (1.4) converges to a fixed point of f. Moreover, the rate of convergence is better than those of Mann (1.1), Ishikawa (1.2) and Noor (1.3) in the sense of Rhoades [13].

Clearly Mann iteration is special cases of SP-iteration. Some interesting results concerning fixed point theory of continuous functions can be found in [5].

Recently, by combining the SP-iteration and Noor iteration, Cholamjiak and Pholasa [4] proposed the NSP-iteration as follows: $w_1 \in C$ and

$$\begin{aligned} r_n &= (1-\mu_n)w_n + \mu_n f(w_n), \\ q_n &= (1-\tau_n-\beta_n)w_n + \tau_n r_n + \beta_n f(r_n), \\ w_{n+1} &= (1-\gamma_n-\alpha_n)w_n + \gamma_n q_n + \alpha_n f(q_n) \end{aligned} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\tau_n\}$ and $\{\gamma_n\}$ are sequences in [0,1]. They proved some convergence theorems of such iterations for continuous functions on an arbitrary interval. Also, they compared the rate of convergence of Mann, Ishikawa, Noor and NSP iterations by numerical examples and concluded that NSP-iteration converges faster than all of them.

Inspired and motivated by these facts, we introduce and study a new modified Noor-iteration process for solving a fixed point problem for continuous function on an arbitrary interval in the real line. The scheme is defined as follows.

Let C be a closed interval on the real line and $f: C \to C$ given mapping. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)x_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned} \tag{1.6}$$

where, $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ are appropriate real sequences in $[0,1]$. The iterative scheme (1.6) is called the modified Noor iteration for continuous functions (abbreviate MN-iteration).

The first purpose of this article is to give a necessary and sufficient condition for the strong convergence of the MN-iteration of continuous functions on an arbitrary interval. The second purpose is to improve the rate of convergence compared to previous work. Specifically, our main result shows that MN-iteration converges faster than NSP-iteration to the fixed point. Numerical examples are also presented to compare the result with Mann, Ishikawa, Noor, SP and NSP iterations.

Consequently, we have that MN-iteration converges faster than the other schemes in the same category.

2. Convergence theorem

In this section, we provide the convergence theorem of MN-iteration (1.6) for continuous functions on an arbitrary closed interval. Now, we will give some crucial lemmas for proofs of our main results.

Lemma 1. *Let C be a closed interval on the real line (can be unbounded) and let $f: C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$ such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$

and $\sum_{n=1}^{\infty} \tau_n < \infty$. *From an arbitrary initial guess* $x_1 \in C$, *define the sequence* $\{x_n\}$ *using (1.6). If* $x_n \to a$, *then* a *is a fixed point of* f.

Proof. Let $x_n \to a$, and suppose $a \neq f(a)$. Then $\{x_n\}$ is bounded. So, $\{f(x_n)\}$ is bounded by the continuity of f. So are $\{y_n\}, \{z_n\}, \{f(y_n)\}$ and $\{f(z_n)\}$. Moreover, $z_n \to a$ since $x_n \to a$ and $\mu_n \to 0$. We also have $y_n \to a$ since $x_n \to a$, $\beta_n \to 0$ and $\tau_n \to 0$. From (1.6), we get

$$\begin{aligned} x_{n+1} &= (1-\gamma_n-\alpha_n)x_n+\gamma_n f(z_n)+\alpha_n f(y_n) \\ &= x_n+\gamma_n(f(z_n)-x_n)+\alpha_n(f(y_n)-x_n). \end{aligned} \tag{2.1}$$

Let $p_k = f(z_k)-x_k, q_k = f(y_k)-x_k$. Then, we have

$$\lim_{k\to\infty} p_k = \lim_{k\to\infty}(f(z_k)-x_k) = f(a)-a \neq 0,$$
$$\lim_{k\to\infty} q_k = \lim_{k\to\infty}(f(y_k)-x_k) = f(a)-a \neq 0.$$

From (2.1) we get

$$\begin{aligned} x_n &= x_1+\sum_{k=1}^{n}\gamma_k(f(z_k)-x_k)+\sum_{k=1}^{n}\alpha_k(f(y_k)-x_k) \\ &= x_1+\sum_{k=1}^{n}\gamma_k p_k+\sum_{k=1}^{n}\alpha_k q_k. \end{aligned}$$

It is worth noting here that $\sum_{k=1}^{\infty}\gamma_k p_k < \infty$ since $\lim_{k\to\infty} p_k \neq 0$ and $\sum_{k=1}^{\infty}\gamma_k < \infty$. This shows that $\{x_n\}$ is a divergent sequence since $\lim_{k\to\infty} q_k \neq 0$ and $\sum_{k=1}^{\infty}\alpha_k = \infty$. This contradicts to the convergence of $\{x_n\}$. Hence $f(a) = a$ and a is fixed point of f. □

Lemma 2. *Let* C *be a closed interval on the real line (can be unbounded) and let* $f\colon C \to C$ *be a continuous function. Let* $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ *and* $\{\tau_n\}$ *be sequences in* $[0,1]$ *such that* $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ *and* $\sum_{n=1}^{\infty}\tau_n < \infty$. *From an arbitrary initial guess* $x_1 \in C$, *define the sequence* $\{x_n\}$ *using (1.6). If* $\{x_n\}$ *is bounded, then* $\{x_n\}$ *is convergent.*

Proof. Suppose $\{x_n\}$ is not convergent. Let $a = \liminf_n x_n$ and $b = \limsup_n x_n$. Then $a < b$. We first show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m)-m > 0$. Since f is continuous, there exists δ with $0 < \delta < b-a$ such that for $|x-m| \leq \delta$,

$$f(x)-x > 0.$$

By continuity of f and $\{x_n\}$ is bounded we have that $\{f(x_n)\}$ is bounded, so $\{z_n\}$, $\{y_n\}$, $\{f(z_n)\}$ and $\{f(y_n)\}$ are bounded sequences. Using

$$\begin{aligned}
x_{n+1}-x_n &= \gamma_n(f(z_n)-x_n)+\alpha_n(f(y_n)-x_n),\\
y_n-x_n &= \tau_n(f(x_n)-x_n)+\beta_n(f(z_n)-x_n),\\
z_n-x_n &= \mu_n(f(x_n)-x_n),
\end{aligned}$$

we can easily show that $|z_n-x_n|\to 0$, $|y_n-x_n|\to 0$ and $|x_{n+1}-x_n|\to 0$. Thus, there exists a positive integer N such that

$$|x_{n+1}-x_n|<\frac{\delta}{2},\ |y_n-x_n|<\frac{\delta}{2},\ |z_n-x_n|<\frac{\delta}{2}, \forall n>N. \tag{2.2}$$

Since $b=\limsup_n x_n>m$, there exists $k_1>N$ such that $x_{n_{k_1}}>m$. Let $n_{k_1}=k$, then $x_k>m$. For x_k, there exist two cases as follows:

(i) $x_k>m+\frac{\delta}{2}$, then $x_{k+1}>x_k-\frac{\delta}{2}\geq m$ using (2.2). So, we have $x_{k+1}>m$.

(ii) $m<x_k<m+\frac{\delta}{2}$, then $m-\frac{\delta}{2}<y_k<m+\delta$ and $m-\frac{\delta}{2}<z_k<m+\delta$ by (2.2). So, we obtain $|x_k-m|<\frac{\delta}{2}<\delta, |y_k-m|<\delta, |z_k-m|<\delta$. Hence

$$f(x_k)-x_k>0,\ f(y_k)-y_k>0,\ f(z_k)-z_k>0. \tag{2.3}$$

From (2.1) and (2.3), we have

$$\begin{aligned}
x_{k+1} &= x_k+\gamma_k(f(z_k)-x_k)+\alpha_k(f(y_k)-x_k)\\
&= x_k+\gamma_k(f(z_k)-z_k)+\gamma_k(z_k-x_k)+\alpha_k(f(y_k)-y_k)+\alpha_k(y_k-x_k)\\
&= x_k+\gamma_k(f(z_k)-z_k)+\gamma_k\mu_k(f(x_k)-x_k)+\alpha_k(f(y_k)-y_k)\\
&\quad +\alpha_k\tau_k(f(x_k)-x_k)+\alpha_k\beta_k(f(z_k)-x_k)\\
&= x_k+\gamma_k(f(z_k)-z_k)+\gamma_k\mu_k(f(x_k)-x_k)+\alpha_k(f(y_k)-y_k)\\
&\quad +\alpha_k\tau_k(f(x_k)-x_k)+\alpha_k\beta_k(f(z_k)-z_k)+\alpha_k\beta_k(z_k-x_k)\\
&= x_k+\gamma_k(f(z_k)-z_k)+\gamma_k\mu_k(f(x_k)-x_k)+\alpha_k(f(y_k)-y_k)\\
&\quad +\alpha_k\tau_k(f(x_k)-x_k)+\alpha_k\beta_k(f(z_k)-z_k)+\alpha_k\beta_k\mu_k(f(x_k)-x_k)\\
&> x_k.
\end{aligned}$$

Thus $x_{k+1}>x_k>m$. This together with (i) and (ii), imply $x_{k+1}>m$. Similarly, we get that $x_{k+2}>m$, $x_{k+3}>m,\dots$. Thus we have $x_n>m$ for all $n>k=n_{k_1}$. So $a=\lim_{k\to\infty}x_{n_k}\geq m$, which is a contradiction with $a<m$. Thus $f(m)=m$.

We next consider the following two cases.

(i) There exists x_M such that $a<x_M<b$. Then $f(x_M)=x_M$. It follows that

$$z_M=(1-\mu_M)x_M+\mu_M f(x_M)=x_M$$

and

$$\begin{aligned} y_M &= (1-\tau_M-\beta_M)z_M+\tau_M f(x_M)+\beta_M f(z_M) \\ &= (1-\tau_M-\beta_M)x_M+\tau_M f(x_M)+\beta_M f(x_M) \\ &= x_M. \end{aligned}$$

It follows that

$$\begin{aligned} x_{M+1} &= (1-\gamma_M-\alpha_M)y_M+\gamma_M f(z_M)+\alpha_M f(y_M) \\ &= (1-\tau_M-\gamma_M)x_M+\gamma_M f(x_M)+\alpha_M f(x_M) \\ &= x_M. \end{aligned}$$

Similarly, we obtain $x_M = x_{M+1} = x_{M+2} = \dots$. It clear that $x_n \to x_M$. Since there exists $x_{n_k} \to a, x_M = a$. This shows that $x_n \to a$, which is a contradiction.

(ii) For all $n, x_n \leq a$ or $x_n \geq b$. Since $b-a > 0$ and $\lim_{n\to\infty} |x_{n+1}-x_n| = 0$, there exists $\tilde{N}$ such that $|x_{n+1}-x_n| < \frac{(b-a)}{2}$ for $n > \tilde{N}$. So, it is seen that $x_n \leq a$ for $n > \tilde{N}$, or it is always that $x_n \geq b$ for $n > \tilde{N}$. If $x_n \leq a$ for $n > \tilde{N}$, then $b = \lim_{j\to\infty} x_{n_j} \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > \tilde{N}$, then $a = \lim_{k\to\infty} x_{n_k} \geq b$, which is a contradiction with $a < b$. Thus we conclude that $x_n \to a$. The proof is completed.

□

We are now ready to prove the main theorem.

Theorem 1. *Let C be a closed interval on the real line (can be unbounded) and let $f: C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. From an arbitrary initial guess $x_1 \in C$, define the sequence $\{x_n\}$ using (1.6). Then $\{x_n\}$ is bounded if and only if it converges to a fixed point of f.*

Proof. Sufficiency is obvious. It suffices to show that if $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a fixed point. Let $\{x_n\}$ be a bounded sequence. Using Lemma 2, we have $\{x_n\}$ is a convergent sequence. Hence, by Lemma 1, it converges to a fixed point of f. □

When $C = [a,b]$ in Theorem 1, we obtain the following result.

Corollary 1. *Let* $f\colon [a,b] \to [a,b]$ *be a continuous function. Let* $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ *and* $\{\tau_n\}$ *be sequences in* $[0,1]$. *Let* $\{x_n\}$ *be a sequence generated iteratively by* $x_1 \in [a,b]$ *and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)x_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \tau_n < \infty$. *Then* $\{x_n\}$ *converges to a fixed point of* f.

3. Rate of convergence

In this section, we provide a theoretical estimation proof of the rate of convergence of the sequence $\{x_n\}$ defined by (1.6). We compare the convergence rate of (1.6) with the NSP-iteration proposed in [4]. We show that the MN-iteration (1.6) converges faster than the NSP-iteration (1.5) for the class of continuous nondecreasing functions on an arbitrary interval in the sense of Rhoades [13].

We next prove some crucial lemmas which will be used in the sequel.

Lemma 3. *Let* C *be a closed interval on the real line and let* $f\colon C \to C$ *be a continuous and nondecreasing function. Let* $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ *and* $\{\tau_n\}$ *be sequences in* $[0,1)$. *Let* $\{w_n\}$ *and* $\{x_n\}$ *be sequences defined by (1.5) and (1.6), respectively. Then the following hold:*

(i) If $f(w_1) < w_1$, *then* $f(w_n) < w_n$ *for all* $n \geq 1$ *and* $\{w_n\}$ *is nonincreasing.*
(ii) If $f(w_1) > w_1$, *then* $f(w_n) > w_n$ *for all* $n \geq 1$ *and* $\{w_n\}$ *is nondecreasing.*
(iii) If $f(x_1) < x_1$, *then* $f(x_n) < x_n$ *for all* $n \geq 1$ *and* $\{x_n\}$ *is nonincreasing.*
(iv) If $f(x_1) > x_1$, *then* $f(x_n) > x_n$ *for all* $n \geq 1$ *and* $\{x_n\}$ *is nondecreasing.*

Proof.

(i) Let $f(w_1) < w_1$. Then $f(w_1) < r_1 \leq w_1$. Since f is nondecreasing, we have $f(r_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(r_1) < q_1 \leq w_1$. Thus $f(q_1) \leq f(w_1) < r_1 \leq w_1$. For q_1, we consider the following two cases.

Case 1: $f(r_1) < q_1 \leq r_1$. Then $f(q_1) \leq f(r_1) < q_1 \leq r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$, if $q_1 < w_2 \leq r_1$, then $f(w_2) \leq f(r_1) < q_1 < w_2$ and if $r_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < r_1 < w_2$. Thus we have $f(w_2) < w_2$.

Case 2: $r_1 < q_1 \leq w_1$. Then $f(q_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 < q_1 \leq w_1$. It follows that if $f(q_1) <$

$w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$ and if $q_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < q_1 < w_2$. Hence, we have $f(w_2) < w_2$.

In conclusion by Case 1 and Case 2, we have $f(w_2) < w_2$. By continuing in this way, we can show that $f(w_n) < w_n$ for all $n \geq 1$. This implies $r_n \leq w_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(r_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Thus $q_n \leq w_n$ for all $n \geq 1$, then $f(q_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Hence, we have $w_{n+1} \leq w_n$ for all $n \geq 1$, that is $\{w_n\}$ is nonincreasing.

(ii) By using the same argument as in (i), we obtain the desired result.

(iii) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we have $f(z_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(z_1) < y_1 \leq x_1$. Thus $f(y_1) \leq f(x_1) < z_1 \leq x_1$. For y_1, we consider the following two cases.

Case 1: $f(z_1) < y_1 \leq z_1$. Then $f(y_1) \leq f(z_1) < z_1 < x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$, if $y_1 < x_2 \leq z_1$, then $f(x_2) \leq f(z_1) < y_1 < x_2$ and if $z_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < z_1 < x_2$. Thus we have $f(x_2) < x_2$.

Case 2: $z_1 < y_1 \leq x_1$. Then $f(y_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(y_1) < x_2 \leq x_1$. Thus $f(x_2) \leq f(x_1) < z_1 < y_1 \leq x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$ and if $y_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < y_1 < x_2$. Hence, we have $f(x_2) < x_2$.

In conclusion by Case 1 and Case 2, we have $f(x_2) < x_2$. By continuing in this way, we can show that $f(x_n) < x_n$ for all $n \geq 1$. This implies $z_n \leq x_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(z_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Thus $y_n \leq x_n$ for all $n \geq 1$, then $f(y_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Hence, we have $x_{n+1} \leq x_n$ for all $n \geq 1$, that is $\{x_n\}$ is nonincreasing.

(iv) Following the proof line as in (iii), we obtain the desired result.

□

Lemma 4. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the NSP-iteration (1.5) and MN-iteration (1.6), respectively. Then the following are satisfied:*

(i) If $f(w_1) < w_1$, then $x_n \leq w_n$ for all $n \geq 1$.
(ii) If $f(w_1) > w_1$, then $x_n \geq w_n$ for all $n \geq 1$.

Proof.

(i) Let $f(w_1) < w_1$. Then $f(x_1) < x_1$ since $w_1 = x_1$. From (1.6), we get $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we obtain $f(z_1) \leq f(x_1) < z_1 \leq x_1$. Hence $f(z_1) < y_1 \leq z_1$. Using the NSP-iteration (1.5) and MN-iteration (1.6), we obtain the following estimation: $z_1 - r_1 = (1-\mu_1)(x_1 - w_1) + \mu_1(f(x_1) - f(w_1)) = 0$. So, $z_1 = r_1$, and so $y_1 - q_1 = (1-\tau_1-\beta_1)(x_1 - w_1) + \tau_1(f(x_1) - r_1) + \beta_1(f(z_1) - f(r_1)) \leq 0$. Hence, we have $y_1 \leq q_1$. Since

f is nondecreasing, we have $f(y_1) \leq f(q_1)$. We next obtain $x_2 - w_2 = (1 - \gamma_1 - \alpha_1)(x_1 - w_1) + \gamma_1(f(z_1) - q_1) + \alpha_1(f(y_1) - f(q_1)) \leq 0$, so, $x_2 \leq w_2$. Assume that $x_k \leq w_k$. Thus $f(x_k) \leq f(w_k)$. From Lemma 3 (i) and Lemma 3 (iii), we get $f(w_k) < w_k$ and $f(x_k) < x_k$. It follows that $f(x_k) < z_k \leq x_k$ and $f(z_k) \leq f(x_k) < z_k$. Thus $z_k - r_k = (1-\mu_k)(x_k - w_k) + \mu_k(f(x_k) - f(w_k)) \leq 0$. So, $z_k \leq r_k$. Since $f(z_k) \leq f(r_k)$, we have $y_k - q_k = (1 - \tau_k - \beta_k)(x_k - w_k) + \tau_k(f(x_k) - r_k) + \beta_k(f(z_k) - f(r_k)) \leq 0$, so, $y_k \leq q_k$, which yields $f(y_k) \leq f(q_k)$. In addition, $f(z_k) \leq f(x_k) < z_k \leq x_k$, using (1.6), we have

$f(z_k) - y_k = (1 - \tau_k - \beta_k)(f(z_k) - x_k) + \tau_k(f(z_k) - f(x_k)) + \beta_k(f(z_k) - f(z_k)) \leq 0$. So, $f(z_k) - q_k = (f(z_k) - y_k) + (y_k - q_k) \leq 0$. This shows that $x_{k+1} - w_{k+1} = (1 - \gamma_k - \alpha_k)(x_k - w_k) + \gamma_k(f(z_k) - q_k) + \alpha_k(f(y_k) - f(q_k)) \leq 0$, which gives, $x_{k+1} \leq w_{k+1}$. By induction, we conclude that $x_n \leq w_n$ for all $n \geq 1$.

(ii) From Lemma 3 (ii), Lemma 3 (iv) and the same argument as in (i), we can show that $x_n \geq w_n$ for all $n \geq 1$.

□

For convenience, we write algorithm (1.6) by $MN(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$.

Proposition 1. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in C\colon\, p = f(p)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. If $f(x_1) > x_1$, then $\{x_n\}$ defined by $NSP(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ and $MN(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3 $((ii),(iv))$, we know that $\{x_n\}$ is nondecreasing. Since the initial point $x_1 > \sup\{p \in C\colon\, p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Proposition 2. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in C\colon\, p = f(p)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. If $f(x_1) < x_1$, then $\{x_n\}$ defined by $NSP(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ and $MN(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3 $((i),(iii))$, we know that $\{x_n\}$ is nonincreasing. Since the initial point $x_1 < \inf\{p \in C\colon\, p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Next, we compare the rate of convergence of MN-iteration with NSP-iteration.

Theorem 2. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the NSP-iteration (1.5) and the MN-iteration (1.6),* *respectively. If the NSP-iteration $\{w_n\}$ converges to $p \in F(f)$, then the MN-iteration $\{x_n\}$ converges to p. Moreover, the MN-iteration (1.6) converges faster than the NSP-iteration (1.5).*

Proof. Assume that the NSP-iteration $\{w_n\}$ converges to $p \in F(f)$. Put $L = \inf\{p \in C\colon p = f(p)\}$ and $U = \sup\{p \in C\colon p = f(p)\}$. For $w_1 = x_1$, we divide our proof into the following three cases:

Case 1: $w_1 = x_1 > U$, Case 2: $w_1 = x_1 < L$, Case 3: $L \leq w_1 = x_1 \leq U$.

Case 1: $w_1 = x_1 > U$. By Proposition 1, we get $f(w_1) < w_1$ and $f(x_1) < x_1$. So, by Lemma 4 (i), we have $x_n \leq w_n$ for all $n \geq 1$. By induction, we can show that $U \leq x_n$ for all $n \geq 1$. Then, we have $0 \leq x_n - p \leq w_n - p$, which yields $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. This shows that $x_n \to p$. By Definition 1, we conclude that the MN-iteration $\{x_n\}$ converges faster than the NSP-iteration $\{w_n\}$.

Case 2: $w_1 = x_1 < L$. By Proposition 2, we get $f(w_1) > w_1$ and $f(x_1) > x_1$. This implies, by Lemma 4 (ii), that $x_n \geq w_n$ for all $n \geq 1$. So, by induction, we can show that $x_n \leq L$ for all $n \geq 1$. Then, we have $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. It follows that $x_n \to p$ and the MN-iteration $\{x_n\}$ converges faster than the NSP-iteration $\{w_n\}$.

Case 3: $L \leq w_1 = x_1 \leq U$. Suppose that $f(w_1) \neq w_1$. If $f(w_1) < w_1$, we have, by Lemma 3 (i), that $\{w_n\}$ is nonincreasing with limit p. Lemma 4 (i) gives $p \leq x_n \leq w_n$ for all $n \geq 1$. It follows that $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. Therefore $x_n \to p$ and the result follows. If $f(w_1) > w_1$, by Lemma 3 (ii) and Lemma 4 (ii), then we can also show that the result holds.

□

4. Numerical examples

In this section, some numerical examples are given to demonstrate the convergence of the algorithm defined in this paper. For convenience, we call the iteration (1.6) the MN-iteration.

Example 1. $f\colon [-1,4] \to [-1,4]$ defined by $f(x) = \frac{x^3+x-3}{19}$. The fixed point of the function is $p = -0.166925$. Initial point is $x_1 = 4$ and control conditions are $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{1.7}}$, $\mu_n = \frac{1}{(n+1)^{2.3}}$, $\gamma_n = \frac{1}{(n+1)^{1.5}}$ and $\tau_n = \frac{1}{(n+1)^{1.1}}$. The stopping criteria is $|x_n - p| < 10^{-8}$.

Example 2. $f\colon [1,\infty] \to [1,\infty]$ defined by $f(x) = x^{0.3} - (\sqrt{log(x+9)} - 1)^3$. The fixed point of the function is $p = 1$. Initial point is $x_1 = 9$ and control conditions are $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{2.0}}$, $\mu_n = \frac{1}{(n+1)^{3.6}}$, $\gamma_n = \frac{1}{(n+1)^{2.5}}$ and $\tau_n = \frac{1}{(n+1)^{1.1}}$. The stopping criteria is $|x_n - p| < 10^{-6}$.

n	Mann	Ishikawa	Noor	NSP	SP	MN-iteration	
	u_n	s_n	l_n	w_n	h_n	x_n	$\lvert x_n - p\rvert$
1	4	4	4	4	4	4	4.166925
5	1.393239	0.753340	0.628636	0.451696	0.293228	0.016801	0.183726
10	0.046198	-0.049731	-0.066200	-0.090005	-0.119193	-0.150974	0.015950
15	-0.120743	-0.141524	-0.145086	-0.150271	-0.157307	-0.163938	0.015950
20	-0.153846	-0.159732	-0.160740	-0.162211	-0.164309	-0.166147	0.000777
25	-0.162557	-0.164523	-0.164860	-0.165351	-0.166074	-0.166679	0.000245
30	-0.165290	-0.166026	-0.166152	-0.166336	-0.166612	-0.166836	0.000088
35	-0.166258	-0.166558	-0.166610	-0.166685	-0.166799	-0.166890	0.000034
No. of iterations	133	126	124	121	113	99	

TABLE 1. Mann, Ishikawa, Noor, NSP, SP and MN iterations for $f(x) = \frac{x^3+x-3}{19}$.

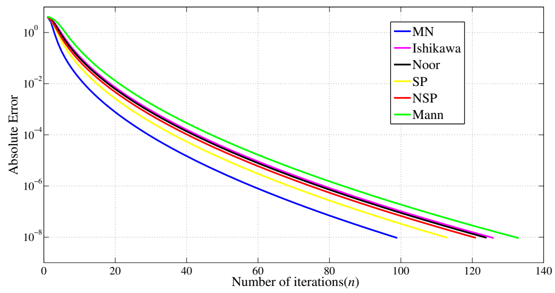

FIGURE 1. Error values obtained from MN, Ishikawa, Noor, SP, NSP and Mann iterations for $f(x) = \frac{x^3+x-3}{19}$.

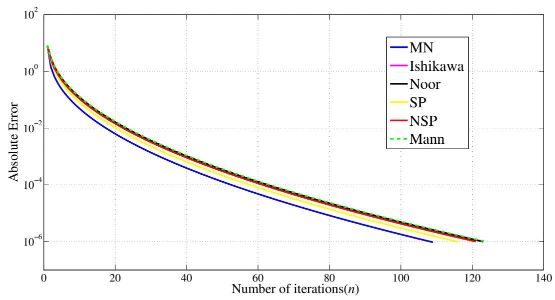

FIGURE 2. Mann, Ishikawa, Noor, SP, NSP and MN iterations for given $x_1 = 9$ of $f(x) = x^{0.3} - (\sqrt{log(x+9)} - 1)^3$.

n	Mann	Ishikawa	Noor	NSP	SP	MN-iteration	
	u_n	s_n	l_n	w_n	h_n	x_n	$\lvert x_n - p\rvert$
1	9	9	9	9	9	9	8
5	1.590135	1.567476	1.567411	1.493424	1.382242	1.235851	0.235851
10	1.127788	1.122291	1.122277	1.107154	1.079598	1.048813	0.048813
15	1.042340	1.040461	1.040456	1.035520	1.025965	1.015841	0.015841
20	1.017031	1.016264	1.016262	1.014287	1.010351	1.006290	0.006290
25	1.007706	1.007356	1.007355	1.006463	1.004656	1.002821	0.002821
30	1.003782	1.003609	1.003609	1.003172	1.002276	1.001375	0.001375
35	1.001972	1.001882	1.001882	1.001654	1.001183	1.000714	0.000714
40	1.001078	1.001029	1.001029	1.000904	1.000645	1.000389	0.000389
45	1.000613	1.000584	1.000584	1.000514	1.000366	1.000220	0.000220
No. of iterations	124	123	123	121	116	109	

TABLE 2. Mann, Ishikawa, Noor, NSP, SP and MN iterations for $x_1 = 9$ and $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$.

Table 1, Table 2, Figure 1 and Figure 2 show the behavior of six comparative methods consisting of Mann iteration, Ishikawa iteration, Noor iteration, NSP-iteration, SP-iteration and MN-iteration in converging to the fixed point of the numerical experiments. The results of the both examples indicates that the MN-iteration converges faster than the other methods.

Next, we will consider on the rate of convergence between the MN-iteration and the algorithm defined in this paper. The Definition 1 will be used to indicate the rate of convergence in the numerical aspects and results are scoped only on the Example 1 and Example 2.

n	Mann $\lvert u_n - p\rvert$	Ishikawa $\lvert s_n - p\rvert$	Noor $\lvert l_n - p\rvert$	NSP $\lvert w_n - p\rvert$	SP $\lvert h_n - p\rvert$	MN $\lvert x_n - p\rvert$
1	4.166925	4.166925	4.166925	4.166925	4.166925	4.166925
...	...	...	...	...	...	...
22	8.29793E-03	4.56306E-03	3.92351E-03	2.99014E-03	1.63974E-03	4.81195E-04
23	6.66600E-03	3.66550E-03	3.15176E-03	2.40187E-03	1.31039E-03	3.82269E-04
24	5.38267E-03	2.95970E-03	2.54489E-03	1.93931E-03	1.05299E-03	3.05476E-04
...	...	...	...	...	...	...
58	2.15202E-05	1.18252E-05	1.01680E-05	7.74561E-06	3.91320E-06	1.03354E-06
59	1.88782E-05	1.03734E-05	8.91967E-06	6.79466E-06	3.42936E-06	9.04423E-07
60	1.65801E-05	9.11051E-06	7.83376E-06	5.96744E-06	3.00896E-06	7.92411E-07

TABLE 3. The rate of convergence of Mann, Ishikawa, Noor, NSP, SP and MN iterations for $f(x) = \frac{x^3+x-3}{19}$ given in Example 1.

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 1 visually.

Rate of convergence between two sequences

n	$\frac{\lvert x_n-p\rvert}{\lvert u_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert s_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert l_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert w_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert h_n-p\rvert}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.1178	0.1996	0.2309	0.2970	0.3993
10	0.0748	0.1361	0.1584	0.2074	0.3342
20	0.0595	0.1081	0.1257	0.1650	0.2973
30	0.0540	0.0981	0.1141	0.1498	0.2823
40	0.0510	0.0928	0.1079	0.1417	0.2735
50	0.0491	0.0894	0.1040	0.1365	0.2676
60	0.0478	0.0870	0.1012	0.1328	0.2634

TABLE 4. Convergence comparison of sequences generated by Mann iteration, Ishikawa iteration, Noor iteration, NSP-iteration and SP-iteration with MN-iteration (see in Table 3) for numerical experiment of Example 1.

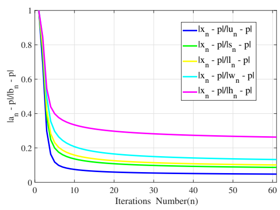

FIGURE 3. Convergence comparison of sequence generated by Mann iteration (u_n), Ishikawa iteration (s_n), Noor iteration (l_n), NSP-iteration (w_n) and SP-iteration (h_n) with MN-iteration (x_n) for Example 1.

n	Mann $\lvert u_n-p\rvert$	Ishikawa $\lvert s_n-p\rvert$	Noor $\lvert l_n-p\rvert$	NSP $\lvert w_n-p\rvert$	SP $\lvert h_n-p\rvert$	MN $\lvert x_n-p\rvert$
1	8	8	8	8	8	8
…	…	…	…	…	…	…
35	1.97255E-03	1.88222E-03	1.88202E-03	1.65419E-03	1.18342E-03	7.14044E-04
36	1.74222E-03	1.66239E-03	1.66221E-03	1.46101E-03	1.04470E-03	6.30133E-04
37	1.54156E-03	1.47089E-03	1.47074E-03	1.29272E-03	9.23939E-04	5.57109E-04
…	…	…	…	…	…	…
86	1.44878E-05	1.38175E-05	1.38160E-05	1.21448E-05	8.59461E-06	5.13795E-06
87	1.34005E-05	1.27804E-05	1.27791E-05	1.12333E-05	7.94884E-06	4.75143E-06
88	1.24005E-05	1.18267E-05	1.18254E-05	1.03950E-05	7.35501E-06	4.39603E-06

TABLE 5. The rate of convergence of Mann, Ishikawa, Noor, NSP, SP and MN iterations for $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$ given in Example 2.

	Rate of convergence between two sequences				
n	$\frac{\lvert x_n-p\rvert}{\lvert u_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert s_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert l_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert w_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert h_n-p\rvert}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.3997	0.4156	0.4157	0.4780	0.6170
10	0.3820	0.3992	0.3992	0.4555	0.6132
20	0.3694	0.3868	0.3868	0.4403	0.6077
40	0.3606	0.3779	0.3780	0.4300	0.6024
60	0.3570	0.3743	0.3744	0.4259	0.5998
80	0.3551	0.3551	0.3723	0.4236	0.5982
100	0.3538	0.3710	0.3710	0.4221	0.5971

TABLE 6. Convergence comparison of sequences generated by Mann iteration, Ishikawa iteration, Noor iteration, NSP-iteration and SP-iteration with MN-iteration (see in Table 5) for numerical experiment of Example 2.

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 2 visually.

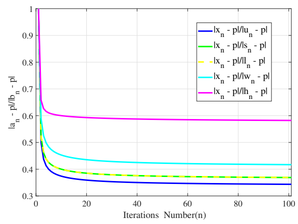

FIGURE 4. Convergence comparison of sequence generated by Mann iteration (u_n), Ishikawa iteration (s_n), Noor iteration (l_n), NSP-iteration (w_n) and SP-iteration (h_n) with MN-iteration (x_n) for Example 2.

Table 3 and 5 show the absolute errors of Mann, Ishikawa, Noor, NSP, SP and MN iterations of the Example 1 and Example 2, respectively. Table 4 and Table 6 show ratios between the absolute error of MN-iteration and those of other methods and graphs of Table 4 and Table 6 are represented on Figure 3 and Figure 4. Clearly, the graphs on both figures converge to constants less than 1. It indicates that the sequences of absolute error of MN-iteration are less than those sequences of other methods. By Definition 1, we can conclude that MN-iteration converges to the fixed point faster than other method. These results verify the proof on the section 3 which show that MN-iteration converge faster than Mann, Ishikawa, Noor, NSP, and SP iterations.

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