Miskolc Mathematical Notes
Vol. 24 (2023), No. 3, pp. 1439–1452
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3976

TAUBERIAN THEOREMS FOR NÖRLUND-CESÁRO MEAN VIA STATISTICALLY CONVERGENCE IN m-NORMED SPACES

NAIM BRAHA, VALDETE LOKU, AND RAM MOHAPATRA

Received 27 October, 2021

Abstract. In this paper, we have proved certain kinds of Tauberian theorems for Nörlund-Cesáro summability methods in m-normed space X in statistically sense. We give necessary and sufficient Tauberian condition for this method of summability, in statistically sense. Also, we prove that Tauberian theorems under these summability methods are valid under oscillating slow condition of Schmidt-type and Hardy-type conditions.

2010 *Mathematics Subject Classification:* 40E05; 40G05; 40A05; 46B15

Keywords: Nörlund-Cesáro summability method, statistical convergent, Schmidt-type oscillating slowly sequences, Hardy-type condition, m-normed spaces

1. INTRODUCTION

The notion of statistical convergence was introduced by Fast [10] and Schoenberg [23] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory, topological groups, topological spaces, function spaces, locally convex spaces, measure theory, fuzzy mathematics and many other areas.

Definition 1. A sequence (x_n) is statistically convergent to L, if for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{|\{k \le n\colon |x_k - L| \ge \varepsilon\}|}{n} = 0,$$

where $|A|$, stands for cardinality of the set.

Let X be a real vector space of dimension $d \ge m$ (Here d can be infinite). A mapping $||\cdot,\dots,\cdot||\colon X^m \to \mathbb{R}$ satisfying the following four properties

(1) $||x_1, x_2, \dots, x_m|| = 0$ if and only if $x_1, x_2, \dots, x_m$ are linearly dependent;
(2) $||x_1, x_2, \dots, x_m||$ is invariant under permutation for every $x_1, x_2, \dots, x_m \in X$;
(3) $||\alpha x_1, x_2, \dots, x_m|| = |\alpha|||x_1, x_2, \dots, x_m||$ for all $x_1, x_2, \dots, x_m \in X$ and $\alpha \in \mathbb{R}$;

(4) $||x_, x_2, \dots, x_{m-1}, y+z|| \leq ||x_, x_2, \dots, x_{m-1}, y|| + ||x_, x_2, \dots, x_{m-1}, z||$
for all $x_1, x_2, \dots, x_{m-1}, y, z \in X$,

is called a m-norm on X, and the pair $(X, ||\cdot, \dots, \cdot||)$ is called an m-normed space. A trivial example of an m-normed space is $X = R^m$ equipped with the following m-norm (Gunawan [11]):

$$||x_1, x_2, \dots, x_m||_E = \text{abs}\left(\begin{vmatrix} x_{11} & \dots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{m1} & \dots & x_{mm} \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \dots, x_{im}) \in \mathbb{R}^m$ for each $i = 1, \dots, m$. (The subscript E is for Euclidean.)

A sequence (x_k) in an m-normed space $(X, ||\cdot, \dots, \dots||)$ is said to be convergent to $c \in X$ with respect to the m-norm if for each $\varepsilon > 0$ there exists an positive integer m_0 such that

$$||x_k - c, z_1, z_2, \dots, z_{m-1}|| < \varepsilon,$$

for all $k \geq m_0$ and for every $z_1, z_2, \dots, z_{m-1} \in X$.

Definition 2. A sequence (x_k) in an m-normed space $(X, ||\cdot, \dots, \dots||)$ is said to be statistically-convergent to some $c \in X$ with respect to the m-norm, if for each $\varepsilon > 0$

$$\lim_{k\to\infty} \frac{1}{k} |\{n \leq k \colon ||x_n - c, z_1, z_2, \dots, z_{m-1}|| \geq \varepsilon\}| = 0,$$

for each $z_1, z_2, \dots, z_{m-1} \in X$.

This kind of the convergence we denote by $st^m - (x_k)$ and with $st^m - S(X)$ we will denote the set of all statistically convergent sequences in m-normed space X.

In what follows we give the concept of the summability method known as the generalized Nörlund summability method (N, p, q) (see Borwein [1], Stadtmüller and Tali [30]). In the recent years the Nörlund summability methods, are studied intensively, for more information see([15, 24, 25, 27, 28]). Given two non-negative sequences (p_n) and (q_n), the convolution $(p \star q)$ is defined by

$$R_n \colon = (p \star q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$$

In this paper, we suppose $R_n \to \infty$ as $n \to \infty$. With $(C;1)-$ we will denote the Cesáro summability method. Let (x_n) be a sequence. When $(p \star q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund-Cesáro transform of the sequence (x_n) is the sequence $N^n_{p,q} C^1_n$ obtained by putting

$$N^n_{p,q} C^1_n = \frac{1}{(p \star q)_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v.$$

We say that the sequence (x_n) is generalized Nörlund-Cesáro summable to L determined by the sequences (p_n) and (q_n) or briefly summable $N^n_{p,q}C^1_n$ to L if

$$\lim_{n\to\infty} N^n_{p,q}C^1_n = L. \tag{1.1}$$

Definition 3. $N^n_{p,q}C^1_n$, is said to be statistically-convergent to some $c \in X$ with respect to the m-norm if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{(p\star q)_n} \left| \left\{ k \leq (p\star q)_n : \left\| \frac{1}{(p\star q)_k} \sum_{i=0}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - L, z_1, z_2, \ldots, z_{m-1} \right\| \geq \varepsilon \right\} \right| = 0,$$

for each $z_1, z_2, \ldots, z_{m-1} \in X$.

This convergence we will denote by $st^m - lim_n N^n_{p,q}C^1_n = L$. Suppose throughout the paper we assume that the sequences (q_n) and (p_n) are satisfying the following conditions:

$$q_n \geq 1, \qquad \sum_{k=0}^{n} p_k \sim n, \qquad n \in \mathbb{N},$$
$$q_{\lambda_n - k} \leq 2q_{n-k}, \qquad k = 0,1,2,3,\ldots, \qquad n; \lambda > 1,$$
$$q_{n-k} \leq 2q_{\lambda_n - k}, \qquad k = 0,1,2,3,\ldots,\lambda_n; \qquad 0 < \lambda < 1,$$

where $\lambda_n = [\lambda \cdot n]$, $a_n \sim b_n$, means that there are constants C, C_1 such that $a_n \leq Cb_n \leq C_1 a_n$.

If

$$\lim_{n\to\infty} x_n = L \tag{1.2}$$

implies (1.1), then the method $N^n_{p,q}C^1_n$ is called to be regular. However, the converse is not always true.

Notice that (1.1) may imply (1.2) under a certain condition, which is called a Tauberian condition. For further results of Tauberian type theorems, a reader is referred to the following references: Braha [2–5,7], Canak, Braha and Totur [8], Canak, Erikli, Sezer and Yarasgil [9], Kiesel [16], Kiesel, Stadtmüller [17], Loku, Braha, Et, Tato [18], Loku, Braha [19]. Very recently, Savas, Sezer [21] and Braha, Loku [6], have studied the Tauberian Theorems in the 2-normed spaces. It is also worth mentioning that the papers Srivastava, Jena, Paikray and Mishra [29], Srivastava, Jena, Paikray [26], Jena, Paikray, Mishra [13], Jena, Pairay, Parida and Dutta [14], and Parida, Paikray, and Jena [20] are relevant in the context of our research.

In this paper, we present necessary and sufficient conditions under which the existence of the limit $st^m - \lim_{n\to\infty} x_n = L$ follows from that of $st^m - \lim_{n\to\infty} N^n_{p,q}C^1_n = L$, in m-normed spaces. These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively. In what follows we will mention some concepts and definitions which we need in the sequel.

Theorem 1. *If (x_k) converges statistically to $L \in X$, in m-normed space X, such that $\|x_k - L, z_1, z_2, \ldots, z_{m-1}\| \leq M$, for every $k \in \mathbb{N}$, $z_1, z_2, \ldots, z_{m-1} \in X$. Then the sequence $(N^k_{p,q} C^1_k)$ is statistically convergent to L in m-normed space X. Conversely is not true.*

Proof. Let us suppose that $st - \lim_k x_k = L$, in m-normed space X. For every $\varepsilon > 0$, we get

$$\lim_{k\to\infty} \frac{|\{n \leq k\colon \|x_n - L, z_1, z_2, \ldots, z_{m-1}\| \geq \varepsilon\}|}{k} = 0,$$

for each $z_1, z_2, \ldots, z_{m-1} \in X$. Let us denote by

$$B_\varepsilon = \{k \in \mathbb{N}\colon \|x_k - L, z_1, z_2, \ldots, z_{m-1}\| \geq \varepsilon\}$$

and

$$\overline{B_\varepsilon} = \{k \in \mathbb{N}\colon |x_k - L, z_1, z_2, \ldots, z_{m-1}\| < \varepsilon\}.$$

Then

$$\begin{aligned}
&\left\| \frac{1}{R_k} \sum_{i=0}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - L, z_1, z_2, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_k} \sum_{i=0}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} (x_v - L), z_1, z_2, \ldots, z_{m-1} \right\| \\
&\leq \frac{1}{R_k} \sum_{\substack{i=0 \\ i\in B_\varepsilon}}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} \|x_v - L, z_1, z_2, \ldots, z_{m-1}\| \\
&\quad + \frac{1}{R_k} \sum_{\substack{i=0 \\ i\in \overline{B}_\varepsilon}}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} \|x_v - L, z_1, z_2, \ldots, z_{m-1}\| \\
&\leq \frac{M}{R_k} \sum_{\substack{i=0 \\ i\in B_\varepsilon}}^{k} p_i q_{k-i} + \frac{\varepsilon}{R_k} \sum_{\substack{i=0 \\ i\in \overline{B}_\varepsilon}}^{k} p_i q_{k-i} \\
&\leq \frac{M|B_\varepsilon|}{R_k} \cdot \max_{0\leq i\leq k} \{p_i q_{k-i}\} + \varepsilon \leq \frac{M|B_\varepsilon|}{k} \cdot \max_{0\leq i\leq k} \{p_i q_{k-i}\} + \varepsilon \to 0 + \varepsilon, \quad \text{as} \quad k \to \infty.
\end{aligned}$$

To show that converse is not true we will use into consideration this

Proposition 1 (Gunawan [11]). *A sequence in a standard m-normed space is convergent in the m-norm if and only if it is convergent in the standard $(m-1)$-norm and,* *by induction, in the standard $(m-r)$-norm for all $r = 1,\dots,m-1$. In particular, a sequence in a standard m-normed space is convergent in the m-norm if and only if it is convergent in the usual norm $||\cdot||_S = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.*

Based on this fact we will give example which proves that converse does not holds, in the 2-norm.

Example 1. Let us consider that $X = \mathbb{R}^3$ and $||x,y|| = \max\{|x_1y_2 - x_2y_1|, |x_1y_3 - x_3y_1|, |x_2y_3 - y_2x_3|\}$, where $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3)$. Define the sequence $(x_n) \in X$ as follows:

$$x_n = \left(2+(-1)^n, \frac{2}{3}+\frac{(-1)^n}{3}, \frac{2}{3}+\frac{2(-1)^n}{3}\right),$$

and $y = (y_1,y_2,y_3) \in X$.
The sequence $(N^n_{p,q}C^1_n)$ is

$$\begin{aligned} N^n_{p,q}C^1_n = \Bigg(& 2+\frac{1}{2(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right], \\ & \frac{2}{3}+\frac{1}{6(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right], \\ & \frac{2}{3}+\frac{1}{3(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]\Bigg), \end{aligned}$$

where $\Psi(z) = \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}$, C-Lebesgue constant and it is valid for all $y = (y_1,y_2,y_3) \in \mathbb{R}^3$. Let us denote by $L = \left(2,\frac{2}{3},\frac{2}{3}\right)$, then

$$\begin{aligned} & \lim_{n\to\infty} ||N^n_{p,q}C^1_n - L, y|| \\ & = \lim_{n\to\infty} \max\Bigg\{\Bigg|\frac{1}{2(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_2 \\ & \qquad -\frac{1}{6(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_1\Bigg|, \\ & \qquad \Bigg|\frac{1}{2(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_3 \\ & \qquad -\frac{1}{3(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_1\Bigg|, \\ & \qquad \Bigg|\frac{1}{6(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_3 \end{aligned}$$

$$-\frac{1}{3(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_2\Bigg|\Bigg\}$$
$$=0.$$

So (x_n) is $N_{p,q}^n C_n^1$-summable to $\left(2,\frac{2}{3},\frac{2}{3}\right)$ in 2-normed space X. Now we will prove that (x_n) does not converges to $\left(2,\frac{2}{3},\frac{2}{3}\right)$ in 2-normed space X. Let $y=(0,1,1)\in\mathbb{R}^3$ then

$$\begin{aligned}\lim_{n\to\infty}||x_n-L,y|| &= \lim_{n\to\infty}\left\|\left((-1)^n,\frac{(-1)^n}{3},\frac{2(-1)^n}{3}\right),(y_1,y_2,y_3)\right\| \\ &= \lim_{n\to\infty}\max\Bigg\{\left|(-1)^n\cdot y_2-\frac{(-1)^n\cdot y_1}{3}\right|,\left|(-1)^n\cdot y_3-\frac{2(-1)^n y_1}{3}\right|, \\ &\qquad \left|\frac{(-1)^n\cdot y_3}{3}-\frac{2(-1)^n\cdot y_2}{3}\right|\Bigg\}\neq 0,\end{aligned}$$

sequence (x_n) is not convergence.

□

2. TAUBERIAN THEOREM, FOR $st^m - N_{p,q}^n C_n^1$-SUMMABILITY METHOD

In this section, we will show the Tauberian Theorem for $st^m - N_{p,q}^n C_n^1$-summability method.

Theorem 2. *Let (p_n) and (q_n) be two sequences of real numbers defined as above and*

$$st^m-\liminf_n \frac{R_{\lambda_n}}{R_n}>1,\quad \lambda>1 \tag{2.1}$$

where $\lambda_n=[\lambda n]$, denotes the integral parts of the $[\lambda n]$ for every $n\in\mathbb{N}$, and let $(N_{p,q}^n C_n^1)$ be a sequence of real numbers such that $\lim_n N_{p,q}^n C_n^1 = L$, in m-normed space X. Then (x_n) is convergent to the same number L in m-normed space X, if and only if the following two conditions holds:

$$\begin{aligned}&\inf_{\lambda>1}\limsup_n \frac{1}{R_{\lambda_n}-R_n} \\ &\left|\left\{k\leq (p\star q)_n:\left\|\frac{1}{(p\star q)_k}\sum_{i=k+1}^{\lambda_k} p_i q_{\lambda_k-i}\frac{1}{i+1}\sum_{v=0}^{i} x_v - x_k, z_1,z_2,\ldots,z_{m-1}\right\|\leq -\varepsilon\right\}\right| \\ &=0,\end{aligned} \tag{2.2}$$

and

$$\inf_{0<\lambda<1} \limsup_{n} \frac{1}{R_n - R_{\lambda_n}} \left| \left\{ k \leq (p \star q)_n : \left\| \frac{1}{(p \star q)_k} \sum_{i=\lambda_k+1}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_k - x_v, z_1, z_2, \dots, z_{m-1} \right\| \leq \varepsilon \right\} \right| = 0, \tag{2.3}$$

for any $z_1, z_2, z_3, \dots, z_{m-1} \in X$, *and* $\varepsilon > 0$.

Remark 1. The symmetric counterparts of conditions (2.2) and (2.3) are the following

$$\sup_{\lambda>1} \liminf_{n} \frac{1}{R_{\lambda_n} - R_n} \left| \left\{ k \leq (p \star q)_n : \left\| \frac{1}{(p \star q)_k} \sum_{i=k+1}^{\lambda_k} p_i q_{\lambda_k - i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_k, z_1, z_2, \dots, z_{m-1} \right\| \leq -\varepsilon \right\} \right| = 0,$$

and

$$\sup_{0<\lambda<1} \liminf_{n} \frac{1}{R_n - R_{\lambda_n}} \left| \left\{ k \leq (p \star q)_n : \left\| \frac{1}{(p \star q)_k} \sum_{i=\lambda_k+1}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_k - x_v, z_1, z_2, \dots, z_{m-1} \right\| \leq \varepsilon \right\} \right| = 0,$$

for any $z_1, z_2, z_3, \dots, z_{m-1} \in X$, and $\varepsilon > 0$.

Remark 2. The sequence $(x_n) \in X$ is slowly oscillating (see Schmidt [22]) in m-normed space if:

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n \leq k \leq \lambda_n} ||x_k - x_n, z_1, z_2, z_3, \dots, z_{m-1}|| = 0,$$

for all $z_1, z_2, z_3, \dots, z_{m-1} \in X$, or equivalently

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \max_{\lambda_n \leq k \leq n} ||x_n - x_k, z_1, z_2, z_3, \dots, z_{m-1}|| = 0,$$

for all $z_1, z_2, z_3, \dots, z_{m-1} \in X$.
If we denote by $\Delta x_n = x_n - x_{n-1}$, then we can obtain the following form of the above conditions

$$\inf_{\lambda>1} \limsup_{n\to\infty} \max_{n \leq k \leq \lambda_n} \left\| - \sum_{i=k+1}^{n} \Delta x_i, z_1, z_2, z_3, \dots, z_{m-1} \right\| = 0$$

and

$$\inf_{0<\lambda<1} \limsup_{n\to\infty} \max_{\lambda_n \le k \le n} \left\| \sum_{i=k+1}^{n} \Delta x_i, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0,$$

for all $z_1, z_2, z_3, \ldots, z_{m-1} \in X$.

In what follows we will show some auxiliary lemmas which are needful in the sequel.

Lemma 1. *For the sequences of real numbers (p_n) and (q_n), condition given by relation (2.1) is equivalent to this one:*

$$st^m - \liminf_n \frac{R_n}{R_{\lambda_n}} > 1, \quad 0 < \lambda < 1.$$

Proof. It is similar to the Lemma 2.4 in Braha [5], and we omit it. □

Lemma 2. *Let (p_n) and (q_n) be the sequences defined as above and relation (2.1) is satisfied. If $x = (x_n)$ is $st^m - N^n_{p,q}C^1_n$-convergent to L, in m-normed space X. Then for every $\lambda > 0$,*

$$st^m - \lim_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0, \quad (2.4)$$

for $\lambda > 1$ and

$$st^m - \lim_n \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0, \quad (2.5)$$

for $0 < \lambda < 1$, for every $z_1, z_2, z_3, \ldots, z_{m-1} \in X$.

Proof. Let us suppose that $\lambda > 1$. After some calculations we obtain:

$$\begin{aligned}
&\left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\le \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{n-k} + q_{\lambda_n - k} - q_{n-k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{\lambda_n - k} - q_{n-k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\|. \qquad (2.6)
\end{aligned}
$$

We know that

$$
st^m - \limsup_n \frac{R_n}{R_{\lambda_n} - R_n} = st^m - \left(\liminf_n \frac{R_{\lambda_n}}{R_n} - 1 \right)^{-1} < \infty. \qquad (2.7)
$$

Now from relations (2.6) and (2.7), we get relation (2.4).

Case where $0 < \lambda < 1$. In this case we have:

$$
\begin{aligned}
&\left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&\le \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{\lambda_n - k} + q_{n-k} - q_{\lambda_n - k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&\le \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{n-k} - q_{\lambda_n - k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \dots, z_{m-1} \right\|. \qquad (2.8)
\end{aligned}
$$

Knowing that

$$
\limsup_n \frac{R_{\lambda_n}}{R_n - R_{\lambda_n}} = \left(\liminf_n \frac{R_n}{R_{\lambda_n}} - 1 \right)^{-1} < \infty. \qquad (2.9)
$$

Now from relations (2.8) and (2.9), we get relation (2.5). □

In what follows we will prove the Theorem 2.

Proof of Theorem 2. Necessity. Let us suppose that $st^m - \lim_n x_n = L$, and $st^m - \lim_n N^n_{p,q} C^1_n = L$, in m-normed space X. For every $\lambda > 1$ following Lemma 2 we get relation (2.2) and in case where $0 < \lambda < 1$, again applying Lemma 2 we obtain relation (2.3).

Sufficient: Assume that $st^m - \lim_n N^n_{p,q} C^1_n = L$, in m-normed space X and conditions (2.1), (2.2) and (2.3) are satisfied. In what follows we will prove that $st^m - \lim_n x_n = L$, in m-normed space X. Or equivalently, $st^m - \lim_n (N^n_{p,q} C^1_n - x_n) = 0$, in m-normed space X.

Given any $\varepsilon > 0$, by relation (2.2) we can choose some $\lambda_1 > 1$ such that

$$\liminf_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_{n_1}} p_i q_{\lambda_{n_1}-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_n, z_1, z_2, \dots, z_{m-1} \right\| \geq -\varepsilon,$$

where $\lambda_{n_1} = \lambda_1 \cdot n$. From above relation and Lemma 2, we get:

$$\left\| \liminf_n \left(\frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_{n_1}} p_i q_{\lambda_{n_1}-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_n, z_1, z_2, \dots, z_{m-1} \right) \right\| \geq -\varepsilon,$$

respectively

$$\left\| L - \limsup_n x_n, z_1, z_2, \dots, z_{m-1} \right\| \geq -\varepsilon \;\Rightarrow\; \liminf_n \| L - x_n, z_1, z_2, \dots, z_{m-1} \| \geq -\varepsilon. \tag{2.10}$$

From relation (2.3), there exists a $0 < \lambda_2 < 1$, such that

$$\limsup_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_{n_2}} p_i q_{\lambda_{n_1}-i} \frac{1}{i+1} \sum_{v=0}^{i} x_n - x_v, z_1, z_2, \dots, z_{m-1} \right\| \leq \varepsilon,$$

where $\lambda_{n_2} = \lambda_2 \cdot n$. From above relation and Lemma 2, we get:

$$\left\| \limsup_n \left(\frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_{n_1}} p_i q_{\lambda_{n_1}-i} \frac{1}{i+1} \sum_{v=0}^{i} x_n - x_v, z_1, z_2, \dots, z_{m-1} \right) \right\| \leq \varepsilon,$$

respectively

$$\left\| \limsup_n x_n - L, z_1, z_2, \dots, z_{m-1} \right\| \geq \varepsilon \;\Rightarrow\; \limsup_n \| x_n - L, z_1, z_2, \dots, z_{m-1} \| \leq \varepsilon. \tag{2.11}$$

From equations (2.10) and (2.11), we get

$$-\varepsilon \leq \liminf_n \| x_n - L, z_1, z_2, \dots, z_{m-1} \| \leq \limsup_n \| x_n - L, z_1, z_2, \dots, z_{m-1} \| \leq \varepsilon.$$

Hence, we have obtain that

$$\lim_{n} \|x_n - L, z_1, z_2, \ldots, z_{m-1}\| = 0.$$

□

In what follows we will show that under conditions that (x_n) is slowly oscillation sequence, and $st^m - N^n_{p,q}C^1_n$ summability, it converges st^m to the same limit.

Theorem 3. *Let $(x_n) \in X$ be $st^m - N^{\lambda_n}_{p,q}C^1_n$-summable to L. If (x_n) slowly oscillating in m-normed space X, then (x_n) converges st^m- to L.*

Proof. First case, $\lambda > 1$.

Let us suppose that $st^m - N^{\lambda_n}_{p,q}C^1_n$ converges to L in m-normed space X. To prove that $st^m - (x_n) \to L$ in m-normed space X, it is enough to prove that

$$st^m - \lim_{n} ||N^{\lambda_n}_{p,q}C^1_n - x_n, z_1, z_2, \ldots, z_{m-1}|| = 0,$$

for every $z_1, z_2, \ldots, z_{m-1} \in X$. Let us start from

$$\begin{aligned}
\left\|N^n_{p,q}C^1_n - x_n, z_1, z_2, \ldots, z_{m-1}\right\| &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n), z_1, z_2, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} \sum_{j=v+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \right\| \\
&\leq \max_{0 \leq v \leq n} \left\| \sum_{j=v+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \right\|.
\end{aligned}$$

Taking limit superior in both sides of the above relation and then infimum, we get:

$$\inf_{\lambda > 1} \lim_{n\to\infty} \sup \left\|N^n_{p,q}C^1_n - x_n, z_1, z_2, \ldots, z_{m-1}\right\| = 0.$$

Hence, it is proved that (x_n) converges to L in m-normed space X. And from it follows it st^m- convergence.

Second case, $0 < \lambda < 1$. This case is similar to the previous one and for this reason we omit it. □

Next result show that if (x_n) satisfies Hardy[12] conditions, and is $st^m - N^n_{p,q}C^1_n$ summable, then (x_n) converges st^m- to same limit.

Theorem 4. *Let $(x_n) \in X$ be $st^m - N^n_{p,q}C^1_n$-summable to L. If (x_n) satisfies the following relation:*

$$n\Delta x_n = 0(1),$$

then (x_n) st^m- convergent to L.

Proof. It is enough to prove that

$$st^m - \lim_n ||N^n_{p,q}C^1_n - x_n, z_1, z_2, \dots, z_{m-1}|| = 0$$

for every $z_1, z_2, \dots, z_{m-1} \in X$. Let us consider that $\lambda > 1$. From

$$n\Delta x_n = 0(1),$$

it follows that for every $\varepsilon > 0$, there exists a n_0 such that for every $n > n_0$ we have

$$|n\Delta x_n| < \varepsilon.$$

Let us start from

$$\begin{aligned}\left\|N^n_{p,q}C^1_n - x_n, y\right\| &= \left\|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k}(x_v - x_n), z_1, z_2, \dots, z_{m-1}\right\| \\ &= \left\|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{k+1}\sum_{v=0}^{k}\sum_{j=v+1}^{n}\Delta x_j, z_1, z_2, \dots, z_{m-1}\right\| \\ &\leq \max_{0\leq k\leq n}\left\|\sum_{j=k+1}^{n}\Delta x_j, z_1, z_2, \dots, z_{m-1}\right\|.\end{aligned}$$

From above relations, we get:

$$\left\|N^n_{p,q}C^1_n - x_n, z_1, z_2, \dots, z_{m-1}\right\| \leq \max_{0\leq k\leq n}\left\|\sum_{j=k+1}^{n}\Delta x_j, z_1, z_2, \dots, z_{m-1}\right\| \leq \varepsilon. \tag{2.12}$$

Hence, it is proved that (x_n) converges st^m to L in m-normed space X.

The second case where $0 < \lambda < 1$, can be proved in similar way. □

REFERENCES

[1] D. Borwein, "On products of sequences," *J. London Math. Soc.*, vol. 33, pp. 352–357, 1958, doi: 10.1112/jlms/s1-33.3.352.

[2] N. L. Braha, "Tauberian conditions under which λ-statistical convergence follows from statistical summability (V,λ)," *Miskolc Math. Notes.*, vol. 16, no. 2, pp. 695–703, 2015, doi: 10.18514/MMN.2015.1254.

[3] N. L. Braha, *Some applications of summability theory. Dutta, Hemen (ed.) et al., Current topics in summability theory and applications.* New York: Springer, 2016. doi: 10.1007/978-981-10-0913-6.

[4] N. L. Braha, "Tauberian theorems via the generalized de la Vallée-Poussin mean for sequences in 2-Normed spaces," *Acta Univ. Sapientiae Math.*, vol. 11, no. 2, pp. 251–263, 2019, doi: 10.2478/ausm-2019-0019.

[5] N. L. Braha, "Tauberian theorems under statistically Nörlund-Cesáro summability method," *J. Math. Inequal.*, vol. 14, no. 4, pp. 967–975, 2020, doi: 10.7153/jmi-2020-14-63.

[6] N. L. Braha and V. Loku, "Tauberian theorems via the generalized de la Vallée-Poussin mean for sequences in 2-Normed spaces," *Submitted to journal.*

[7] N. L. Braha and I. Temaj, "Tauberian conditions under which statistical convergence follows from statistical summability $(EC)^n_1$," *Bol. Soc. Parana. Mat.*, vol. 37, no. 3, pp. 9–17, 2019, doi: 10.5269/bspm.v37i4.32297.

[8] I. Canak, N. L. Braha, and U. Totur, "A Tauberian theorem for the generalized Nörlund summability method," *Georgian Math. J.*, vol. 27, no. 1, pp. 31–36, 2020, doi: 10.1515/gmj-2017-0062.

[9] I. Çanak, G. Erikli, S. A. Sezer, and E. Yarasgil, "Necessary and sufficient Tauberian conditions for weighted mean methods of summability in two-normed spaces," *Acta Comment. Univ. Tartu. Math.*, vol. 24, no. 1, pp. 49–57, 2020, doi: 10.12697/ACUTM.2020.24.04.

[10] F. Fast, "Sur la convergence statistique," *Colloq. Math.*, vol. 2, pp. 241–244, 1951. [Online]. Available: http://eudml.org/doc/209960

[11] H. Gunawan and M. Mashadi, "On n-normed spaces," *Int. J. Math. Math. Sci.*, vol. 27, no. 10, pp. 631–639, 2001, doi: 10.1155/S0161171201010675.

[12] G. H. Hardy, *Divergent series*. Chelsea, New York, NY., 1991.

[13] B. B. Jena, S. K. Paikray, and U. K. Misra, "A Tauberian theorem for double Cesàro summability method," *Int. J. Math. Math. Sci.*, pp. 1–4, 2016, doi: 10.1155/2016/2431010.

[14] B. B. Jena, S. K. Paikray, P. Parida, and H. Dutta, "Results on Tauberian theorem for Cesàro summable double sequences of fuzzy numbers," *Kragujevac journal of mathematics*, vol. 44, no. 4, pp. 495–508, 2020, doi: 10.46793/KgJMat2004.495J.

[15] U. Kadak, N. L. Braha, and H.M.Srivastava, "Statistical weighted B-summability and its applications to approximation theorems," *Appl. Math. Comput.*, vol. 302, pp. 80–96, 2017, doi: 10.1016/j.amc.2017.01.011.

[16] R. Kiesel, "General Nörlund transforms and power series methods," *Math. Z.*, vol. 214, no. 2, pp. 273–286, 1993, doi: 10.1007/BF02572404.

[17] R. Kiesel and U. Stadtmüller, "Tauberian and convexity theorems for certain (N,p,q)-means," *Canad. J. Math.*, vol. 46, no. 5, pp. 982–994, 1994, doi: 10.4153/CJM-1994-056-6.

[18] V. Loku, N. L. Braha, M. Et, and A. Tato, "Tauberian theorems for the generalized de la Vallée-Poussin mean-convergent sequences of fuzzy numbers," *Bull. Math. Anal. Appl.*, vol. 9, no. 2, pp. 45–56, 2017. [Online]. Available: http://www.bmathaa.org

[19] V. Loku and N. L. Braha, "Tauberian theorems by weighted summability method," *Armen. J. Math.*, vol. 9, no. 1, pp. 35–42, 2017.

[20] P. Parida, S. K. Paikray, and B. B. Jena, "Statistical Tauberian theorems for Cesàro integrability mean based on post-quantum calculus," *Arabian Journal of Mathematics*, vol. 9, no. 3, pp. 653–663, 2020, doi: 10.1007/s40065-020-00284-z.

[21] R. Savas and S. A. Sezer, "Tauberian theorems for sequences in 2-normed spaces," *Results Math.*, vol. 72, no. 4, pp. 1919–1931, 2017, doi: 10.1007/s00025-017-0747-8.

[22] R. Schmidt, "über divergente folgen und lineare mittelbildungen," *Math. Z.*, vol. 22, no. 1, pp. 89–152, 1925, doi: 10.1007/BF01479600.

[23] I. J. Schoenberg, "The integrability of certain functions and related summability methods," *Amer. Math. Monthly*, vol. 66, pp. 361–375, 1959, doi: 10.1080/00029890.1959.11989303.

[24] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "Generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 112, no. 4, pp. 1487–1501, 2018, doi: 10.1007/s13398-017-0442-3.

[25] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "Deferred Cesàro statistical probability convergence and its applications to approximation theorems," *J. Nonlinear Convex Anal.*, vol. 20, no. 9, pp. 1999–1792, 2019.

[26] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "A certain class of statistical probability convergence and its applications to approximation theorems," *Appl. Anal. Discrete Math.*, vol. 14, pp. 579–598, 2020, doi: 10.2298/AADM190220039S.

[27] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "Statistical deferred Nörlund summability and Korovkin-type approximation theorem," *Mathematics*, vol. 8, no. 636, pp. 1–11, 2020, doi: 10.3390/math8040636.

[28] H. M. Srivastava, B. B. Jena, and S. K. Paikray, "Deferred Cesaro statistical convergence of martingale sequence and Korovkin-type approximation theorems," *Miskolc Math. Notes.*, vol. 23, no. 1, pp. 443–456, 2022, doi: 10.18514/MMN.2022.3624.

[29] H. M. Srivastava, B. B. Jena, S. K. Paikray, and U. K. Misra, "Statistically and relatively modular deferred weighted summability and Korovkin-type approximation theorems," *Symmetry*, vol. 11, pp. 1–20, 2019, doi: 10.3390/sym11040448.

[30] U. Stadtmüller and A. Tali, "On certain families of generalized Nörlund methods and power series methods," *J. Math. Anal. Appl.*, vol. 238, no. 1, pp. 44–66, 1999, doi: 10.1006/jmaa.1999.6503.

Authors' addresses

Naim Braha

University of Prishtina, Department of Mathematics and Computer Sciences, Avenue Mother Theresa, No-5, Prishtine, 10000, Kosovo and ILIRIAS Research Institute (www.ilirias.com), Janina No-2, Ferizaj, 70000, Kosovo

E-mail address: nbraha@yahoo.com

Valdete Loku

(**Corresponding author**) University of Applied Sciences, Rr. Universiteti, p.n. 70000 Ferizaj, Kosovo

E-mail address: valdeteloku@gmail.com

Ram Mohapatra

Department of Mathematics, University of Central Florida, P. O. Box 161364 Orlando, FL 32816-1364, U.S.A, URL:http://www.math.ucf.edu/ ramm/

E-mail address: ramm@mail.ucf.edu