TAUBERIAN THEOREMS FOR NÖRLUND-CESÁRO MEAN VIA STATISTICALLY CONVERGENCE IN m-NORMED SPACES

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Abstract. In this paper, we have proved certain kinds of Tauberian theorems for Nörlund-Cesáro summability methods in m-normed space X in statistically sense. We give necessary and sufficient Tauberian condition for this method of summability, in statistically sense. Also, we prove that Tauberian theorems under these summability methods are valid under oscillating slow condition of Schmidt-type and Hardy-type conditions.

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1. Introduction

The notion of statistical convergence was introduced by Fast [10] and Schoenberg [23] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory, topological groups, topological spaces, function spaces, locally convex spaces, measure theory, fuzzy mathematics and many other areas.

Definition 1. A sequence \((x_n)\) is statistically convergent to \(L\), if for every \(\varepsilon > 0\), we have

\[
\lim_{n \to \infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n} = 0,
\]

where \(|A|\) stands for cardinality of the set.

Let \(X\) be a real vector space of dimension \(d \geq m\) (Here \(d\) can be infinite). A mapping \(\|\cdot, \ldots, \cdot\| : X^m \to \mathbb{R}\) satisfying the following four properties

1. \(\|x_1, x_2, \ldots, x_m\| = 0\) if and only if \(x_1, x_2, \ldots, x_m\) are linearly dependent;
2. \(\|x_1, x_2, \ldots, x_m\|\) is invariant under permutation for every \(x_1, x_2, \ldots, x_m \in X\);
3. \(\|\alpha x_1, x_2, \ldots, x_m\| = |\alpha| \|x_1, x_2, \ldots, x_m\|\) for all \(x_1, x_2, \ldots, x_m \in X\) and \(\alpha \in \mathbb{R}\);
is called a $m$-norm on $X$, and the pair $(X, ||\cdot||,\ldots, ||\cdot||)$ is called an $m$-normed space. A trivial example of an $m$-normed space is $X = \mathbb{R}^m$ equipped with the following $m$-norm (Gunawan [11]):

$||x_1, x_2, \ldots, x_m||_E = \text{abs} \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mm} \end{pmatrix}$

where $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}^m$ for each $i = 1, \ldots, m$. (The subscript $E$ is for Euclidean.)

A sequence $(x_k)$ in an $m$-normed space $(X, ||\cdot||,\ldots, ||\cdot||)$ is said to be convergent to $c \in X$ with respect to the $m$-norm if for each $\varepsilon > 0$ there exists an positive integer $m_0$ such that

$||x_k - c, z_1, z_2, \ldots, z_{m-1}|| < \varepsilon,$

for all $k \geq m_0$ and for every $z_1, z_2, \ldots, z_{m-1} \in X$.

**Definition 2.** A sequence $(x_k)$ in an $m$-normed space $(X, ||\cdot||,\ldots, ||\cdot||)$ is said to be statistically-convergent to some $c \in X$ with respect to the $m$-norm, if for each $\varepsilon > 0$

$\lim_{k \to \infty} \frac{1}{k} \left| \left\{n \leq k: ||x_n - c, z_1, z_2, \ldots, z_{m-1}|| \geq \varepsilon \right\} \right| = 0,$

for each $z_1, z_2, \ldots, z_{m-1} \in X$.

This kind of the convergence we denote by $st^m - (x_k)$ and with $st^m - S(X)$ we will denote the set of all statistically convergent sequences in $m$-normed space $X$.

In what follows we give the concept of the summability method known as the generalized Nörlund summability method $(N, p, q)$ (see Borwein [1], Stadtmüller and Tali [30]). In the recent years the Nörlund summability methods, are studied intensively, for more information see([15, 24, 25, 27, 28]). Given two non-negative sequences $(p_n)$ and $(q_n)$, the convolution $(p \ast q)$ is defined by

$R_n: = (p \ast q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$

In this paper, we suppose $R_n \to \infty$ as $n \to \infty$. With $(C, 1)$— we will denote the Cesáro summability method. Let $(x_n)$ be a sequence. When $(p \ast q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund-Cesáro transform of the sequence $(x_n)$ is the sequence $N^n_{p,q}^{C^1}_n$ obtained by putting

$N^n_{p,q}^{C^1}_n = \frac{1}{(p \ast q)_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v.$
We say that the sequence \( (x_n) \) is generalized Nörlund-Cesàro summable to \( L \) determined by the sequences \( (p_n) \) and \( (q_n) \) or briefly summable \( N_{p,q}^n C_1^n \) to \( L \) if

\[
\lim_{n \to \infty} N_{p,q}^n C_1^n = L. \tag{1.1}
\]

**Definition 3.** \( N_{p,q}^n C_1^n \) is said to be statistically-convergent to some \( c \in X \) with respect to the \( m \)-norm if for each \( \varepsilon > 0 \)

\[
\lim_{n \to \infty} \frac{1}{(p \ast q)_n} \left\{ k \leq (p \ast q)_n : \left\| \frac{1}{(p \ast q)_n} \sum_{k=0}^{n} p_k q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - L, z_1, z_2, \ldots, z_{m-1} \right\| \geq \varepsilon \right\} = 0,
\]

for each \( z_1, z_2, \ldots, z_{m-1} \in X \).

This convergence we will denote by \( \text{st}_m \lim_{n \to \infty} N_{p,q}^n C_1^n = L \). Suppose throughout the paper we assume that the sequences \( (q_n) \) and \( (p_n) \) are satisfying the following conditions:

\[
q_n \geq 1, \quad \sum_{k=0}^{n} p_k \sim n, \quad n \in \mathbb{N},
\]

\[
q_{\lambda_n-k} \leq 2q_{n-k}, \quad k = 0, 1, 2, 3, \ldots; n; \lambda > 1,
\]

\[
q_{n-k} \leq 2q_{\lambda_n-k}, \quad k = 0, 1, 2, 3, \ldots; \lambda_n; \quad 0 < \lambda < 1,
\]

where \( \lambda_n = \lfloor \lambda \cdot n \rfloor \), \( a_n \sim b_n \), means that there are constants \( C, C_1 \) such that \( a_n \leq C b_n \leq C_1 a_n \).

If

\[
\lim_{n \to \infty} x_n = L \tag{1.2}
\]

implies (1.1), then the method \( N_{p,q}^n C_1^n \) is called to be regular. However, the converse is not always true.

Notice that (1.1) may imply (1.2) under a certain condition, which is called a Tauberian condition. For further results of Tauberian type theorems, a reader is referred to the following references: Braha [2–5, 7], Canak, Braha and Totur [8], Canak, Erikli, Sezer and Yarasgil [9], Kiesel [16], Kiesel, Stadtmüller [17], Loku, Braha, Et, Tato [18], Loku, Braha [19]. Very recently, Savas, Sezer [21] and Braha, Loku [6], have studied the Tauberian Theorems in the 2-normed spaces. It is also worth mentioning that the papers Srivastava, Jena, Paikray and Mishra [29], Srivastava, Jena, Paikray [26], Jena, Paikray, Mishra [13], Jena, Pairay, Parida and Dutta [14], and Parida, Paikray, and Jena [20] are relevant in the context of our research.

In this paper, we present necessary and sufficient conditions under which the existence of the limit \( \text{st}_m \lim_{n \to \infty} x_n = L \) follows from that of \( \text{st}_m \lim_{n \to \infty} N_{p,q}^n C_1^n = L \),
in $m$-normed spaces. These conditions are one-sided or two-sided if $(x_n)$ is a sequence of real or complex numbers, respectively. In what follows we will mention some concepts and definitions which we need in the sequel.

**Theorem 1.** If $(x_k)$ converges statistically to $L \in X$, in $m$-normed space $X$, such that $\|x_k - L, z_1, z_2, \ldots, z_{m-1}\| \leq M$, for every $k \in \mathbb{N}$, $z_1, z_2, \ldots, z_{m-1} \in X$. Then the sequence $(N_{p,q}C_k^1)$ is statistically convergent to $L$ in $m$-normed space $X$. Conversely is not true.

**Proof.** Let us suppose that $\lim_{k \to \infty} x_k = L$, in $m$-normed space $X$. For every $\varepsilon > 0$, we get

$$\lim_{k \to \infty} \left| \left\{ n \leq k : \|x_n - L, z_1, z_2, \ldots, z_{m-1}\| \geq \varepsilon \right\} \right| = 0,$$

for each $z_1, z_2, \ldots, z_{m-1} \in X$. Let us denote by

$$B_\varepsilon = \{ k \in \mathbb{N} : \|x_k - L, z_1, z_2, \ldots, z_{m-1}\| \geq \varepsilon \}$$

and

$$\overline{B}_\varepsilon = \{ k \in \mathbb{N} : \|x_k - L, z_1, z_2, \ldots, z_{m-1}\| < \varepsilon \}.$$

Then

$$\left\| \frac{1}{R_k} \sum_{i=0}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - L, z_1, z_2, \ldots, z_{m-1} \right\| \leq \frac{1}{R_k} \sum_{i=0}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} \|x_v - L, z_1, z_2, \ldots, z_{m-1}\|$$

$$\leq \frac{1}{R_k} \sum_{i=0}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} \|x_v - L, z_1, z_2, \ldots, z_{m-1}\|$$

$$+ \frac{1}{R_k} \sum_{i \in \overline{B}_\varepsilon} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} \|x_v - L, z_1, z_2, \ldots, z_{m-1}\|$$

$$\leq \frac{M}{R_k} \sum_{i=0}^{k} p_i q_{k-i} + \frac{\varepsilon}{R_k} \sum_{i \in \overline{B}_\varepsilon} p_i q_{k-i}$$

$$\leq \frac{M |B_\varepsilon|}{R_k} \cdot \max_{0 \leq i \leq k} \{ p_i q_{k-i} \} + \varepsilon \leq \frac{M |B_\varepsilon|}{k} \cdot \max_{0 \leq i \leq k} \{ p_i q_{k-i} \} + \varepsilon \to 0 + \varepsilon, \quad \text{as} \quad k \to \infty.$$ 

To show that converse is not true we will use into consideration this

**Proposition 1** (Gunawan [11]). A sequence in a standard $m$-normed space is convergent in the $m$-norm if and only if it is convergent in the standard $(m - 1)$-norm and,
by induction, in the standard \((m-r)\)-norm for all \(r = 1, \ldots, m-1\). In particular, a sequence in a standard \(m\)-normed space is convergent in the \(m\)-norm if and only if it is convergent in the usual norm \(|\cdot|_S = \langle \cdot, \cdot \rangle^1\).

Based on this fact we will give example which proves that converse does not holds, in the 2-norm.

**Example 1.** Let us consider that \(X = \mathbb{R}^3\) and \(|x, y| = \max \{|x_1y_2 - x_2y_1|, |x_1y_3 - x_3y_1|, |x_2y_3 - y_2x_3|\}\), where \(x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)\). Define the sequence \((x_n) \in X\) as follows:

\[
x_n = \left(2 + (-1)^n, \frac{2}{3} + \frac{(-1)^n}{3}, \frac{2}{3} + \frac{2(-1)^n}{3}\right),
\]

and \(y = (y_1, y_2, y_3) \in X\).

The sequence \((N_{p,q}^n C_1^1)\) is

\[
N_{p,q}^n C_1^1 = 2 + \frac{1}{2(n+1)} \left[ \ln 2 + \frac{(-1)^n}{2} \left( \Psi \left( \frac{n + 3}{2} \right) - \Psi \left( \frac{n + 1}{2} \right) \right) \right] + \ln n + C,
\]

\[
= \frac{2}{3} + \frac{1}{6(n+1)} \left[ \ln 2 + \frac{(-1)^n}{2} \left( \Psi \left( \frac{n + 3}{2} \right) - \Psi \left( \frac{n + 1}{2} \right) \right) \right] + \ln n + C,
\]

where \(\Psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, C\)-Lebesgue constant and it is valid for all \(y = (y_1, y_2, y_3) \in \mathbb{R}^3\). Let us denote by \(L = \left(2, \frac{2}{3}, \frac{2}{3}\right)\), then

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \left| N_{p,q}^n C_1^1 - L, y \right| = \lim_{n \to \infty} \max_{y \in \mathbb{R}^3} \left| \frac{1}{2(n+1)} \left[ \ln 2 + \frac{(-1)^n}{2} \left( \Psi \left( \frac{n + 3}{2} \right) - \Psi \left( \frac{n + 1}{2} \right) \right) \right] + \ln n + C \right| y_2
\]

\[
- \frac{1}{6(n+1)} \left[ \ln 2 + \frac{(-1)^n}{2} \left( \Psi \left( \frac{n + 3}{2} \right) - \Psi \left( \frac{n + 1}{2} \right) \right) \right] + \ln n + C \right| y_1,
\]

\[
\frac{1}{2(n+1)} \left[ \ln 2 + \frac{(-1)^n}{2} \left( \Psi \left( \frac{n + 3}{2} \right) - \Psi \left( \frac{n + 1}{2} \right) \right) \right] + \ln n + C \right| y_3
\]

\[
- \frac{1}{3(n+1)} \left[ \ln 2 + \frac{(-1)^n}{2} \left( \Psi \left( \frac{n + 3}{2} \right) - \Psi \left( \frac{n + 1}{2} \right) \right) \right] + \ln n + C \right| y_1,
\]

\[
\frac{1}{6(n+1)} \left[ \ln 2 + \frac{(-1)^n}{2} \left( \Psi \left( \frac{n + 3}{2} \right) - \Psi \left( \frac{n + 1}{2} \right) \right) \right] + \ln n + C \right| y_3.
\]
\[
- \frac{1}{3(n+1)} \left\{ \ln 2 + \frac{(-1)^n}{2} \left( \Psi \left( \frac{n}{2} + \frac{3}{2} \right) - \Psi \left( \frac{n}{2} + 1 \right) \right) + \ln n + C \right\} y_2 \right\} = 0.
\]

So \((x_n)\) is \(N_{p,q}^m C_n^1\) summable to \((2, \frac{2}{3}, \frac{2}{3})\) in 2-normed space \(X\). Now we will prove that \((x_n)\) does not converge to \((2, \frac{2}{3}, \frac{2}{3})\) in 2-normed space \(X\). Let \(y = (0, 1, 1) \in \mathbb{R}^3\) then

\[
\lim_{n \to \infty} ||x_n - L, y|| = \lim_{n \to \infty} \left\| \left( (-1)^n, \frac{(-1)^n}{3}, \frac{2(-1)^n}{3} \right), (y_1, y_2, y_3) \right\|
\]

\[
= \lim_{n \to \infty} \max\left\{ \left| (-1)^n \cdot y_2 - \frac{(-1)^n}{3} \cdot y_1 \right|, \left| (-1)^n \cdot y_3 - \frac{2(-1)^n}{3} \cdot y_2 \right| \right\} \neq 0,
\]

sequence \((x_n)\) is not convergence.

\[\square\]

2. TAUBERIAN THEOREM, FOR \(st^m - N_{p,q}^m C_n^1\)-SUMMABILITY METHOD

In this section, we will show the Tauberian Theorem for \(st^m - N_{p,q}^m C_n^1\)-summability method.

**Theorem 2.** Let \((p_n)\) and \((q_n)\) be two sequences of real numbers defined as above and

\[
st^m - \liminf_{n} \frac{R_n}{R_{\lambda n}} > 1, \quad \lambda > 1
\]  

(2.1)

where \(\lambda_n = [\lambda n]\), denotes the integral parts of the \([\lambda n]\) for every \(n \in \mathbb{N}\), and let \((N_{p,q}^m C_n^1)\) be a sequence of real numbers such that \(\lim_n N_{p,q}^m C_n^1 = L\), in \(m\)-normed space \(X\). Then \((x_n)\) is convergent to the same number \(L\) in \(m\)-normed space \(X\), if and only if the following two conditions holds:

\[
\inf_{\lambda>1} \limsup_{n} \frac{1}{n} \frac{1}{R_{\lambda n} - R_n} \left\| \left\{ k \leq (p \ast q)_n : \left\| \frac{1}{(p \ast q)_k} \sum_{i=k+1}^{\lambda_k} p_i q_{\lambda_k-i-1} \sum_{i=0}^{i} x_i - x_k, z_1, z_2, \ldots, z_{m-1} \right\| \leq -\varepsilon \right\} \right\| = 0,
\]  

(2.2)
and
\[ \inf_{0 < \lambda < 1} \lim_{n \to \infty} \sup_n \frac{1}{R_n - R_{\lambda n}} \left\{ k \leq (p \ast q)_n : \left\| \frac{1}{(p \ast q)_k} \sum_{i=k+1}^{k} \sum_{y=0}^{i} x_k - x_y, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon \right\} = 0, \]

for any \( z_1, z_2, z_3, \ldots, z_{m-1} \in X \), and \( \varepsilon > 0 \).

**Remark 1.** The symmetric counterparts of conditions (2.2) and (2.3) are the following
\[ \sup_{\lambda > 1} \lim_{n \to \infty} \inf_n \frac{1}{R_{\lambda n} - R_n} \left\{ k \leq (p \ast q)_n : \left\| \frac{1}{(p \ast q)_k} \sum_{i=k+1}^{\lambda n} \sum_{y=0}^{i} x_k - x_y, z_1, z_2, \ldots, z_{m-1} \right\| \leq -\varepsilon \right\} = 0, \]

and
\[ \sup_{0 < \lambda < 1} \lim_{n \to \infty} \inf_n \frac{1}{R_n - R_{\lambda n}} \left\{ k \leq (p \ast q)_n : \left\| \frac{1}{(p \ast q)_k} \sum_{i=k+1}^{\lambda n} \sum_{v=0}^{i} x_v - x_k, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon \right\} = 0, \]

for any \( z_1, z_2, z_3, \ldots, z_{m-1} \in X \), and \( \varepsilon > 0 \).

**Remark 2.** The sequence \((x_n) \in X\) is slowly oscillating (see Schmidt [22]) in \( m \)-normed space if:
\[ \inf_{\lambda > 1} \lim_{n \to \infty} \sup_{0 < \lambda < 1} \max_{n \leq k \leq \lambda n} \left\| x_k - x_n, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0, \]

for all \( z_1, z_2, z_3, \ldots, z_{m-1} \in X \), or equivalently
\[ \inf_{0 < \lambda < 1} \lim_{n \to \infty} \sup_{\lambda n \leq k \leq n} \max_{n \leq k \leq \lambda n} \left\| x_n - x_k, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0, \]

for all \( z_1, z_2, z_3, \ldots, z_{m-1} \in X \).

If we denote by \( \Delta x_n = x_n - x_{n-1} \), then we can obtain the following form of the above conditions
\[ \inf_{\lambda > 1} \lim_{n \to \infty} \sup_{0 < \lambda < 1} \max_{n \leq k \leq \lambda n} \left\| - \sum_{i=k+1}^{n} \Delta x_i, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0. \]
and

\[
\inf_{0 < \lambda < 1} \lim_{n \to \infty} \sup_{\lambda_n \leq k \leq n} \left\| \sum_{i=k+1}^{n} \Delta x_i, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0,
\]

for all \( z_1, z_2, z_3, \ldots, z_{m-1} \in X \).

In what follows we will show some auxiliary lemmas which are needful in the sequel.

**Lemma 1.** For the sequences of real numbers \( (p_n) \) and \( (q_n) \), condition given by relation (2.1) is equivalent to this one:

\[
st^m - \lim_{n} \inf \frac{R_n}{R_{\lambda_n}} > 1, \quad 0 < \lambda < 1.
\]

**Proof.** It is similar to the Lemma 2.4 in Braha [5], and we omit it. \( \square \)

**Lemma 2.** Let \( (p_n) \) and \( (q_n) \) be the sequences defined as above and relation (2.1) is satisfied. If \( x = (x_n) \) is \( st^m - N_{p,q} C_{\lambda_n}^1 \)-convergent to \( L \), in \( m \)-normed space \( X \). Then for every \( \lambda > 0 \),

\[
st^m - \lim_{n} \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0, \quad (2.4)
\]

for \( \lambda > 1 \) and

\[
st^m - \lim_{n} \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{\lambda_n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0, \quad (2.5)
\]

for \( 0 < \lambda < 1 \), for every \( z_1, z_2, z_3, \ldots, z_{m-1} \in X \).

**Proof.** Let us suppose that \( \lambda > 1 \). After some calculations we obtain:

\[
\left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
\leq \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{\lambda_n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
+ \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{\lambda_n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{\lambda_n-k} - q_{n-k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\|
\]
Knowing that

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Case where $0 < \lambda < 1$. In this case we have:

Knowing that

\[
\limsup_n \frac{R_n}{R_{\lambda_n} - R_n} = \left( \liminf_n \frac{R_n}{R_{\lambda_n}} - 1 \right)^{-1} < \infty. \tag{2.7}
\]
From equations (2.10) and (2.11), we get relation (2.5).

In what follows we will prove the Theorem 2.

**Proof of Theorem 2.** Necessity. Let us suppose that \( st^m - \lim_{n} x_n = L \), and \( st^m - \lim_{n} N_{p,q} C_{n}^1 = L \), in \( m \)-normed space \( X \). For every \( \lambda > 1 \) following Lemma 2 we get relation (2.2) and in case where \( 0 < \lambda < 1 \), again applying Lemma 2 we obtain relation (2.3).

Sufficient: Assume that \( st^m - \lim_{n} N_{p,q} C_{n}^1 = L \), in \( m \)-normed space \( X \). Or equivalently, \( st^m - \lim_{n} (N_{p,q} C_{n}^1 - x_n) = 0 \), in \( m \)-normed space \( X \).

Given any \( \varepsilon > 0 \), by relation (2.2) we can choose some \( \lambda_1 > 1 \) such that

\[
\liminf_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_n} p_q \lambda_n - \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right\| \geq -\varepsilon,
\]

where \( \lambda_n = \lambda_1 \cdot n \). From above relation and Lemma 2, we get:

\[
\left\| \liminf_n \left( \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_n} p_q \lambda_n - \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right) \right\| \geq -\varepsilon,
\]

respectively

\[
\left\| L - \limsup_n x_n, z_1, z_2, \ldots, z_{m-1} \right\| \geq -\varepsilon \Rightarrow \liminf_n \left\| L - x_n, z_1, z_2, \ldots, z_{m-1} \right\| \geq -\varepsilon.
\]

(2.10)

From relation (2.3), there exists a \( 0 < \lambda_2 < 1 \), such that

\[
\limsup_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_n} p_q \lambda_n - \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon,
\]

where \( \lambda_n = \lambda_2 \cdot n \). From above relation and Lemma 2, we get:

\[
\left\| \limsup_n \left( \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_n} p_q \lambda_n - \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right) \right\| \leq \varepsilon,
\]

respectively

\[
\left\| \limsup_n x_n - L, z_1, z_2, \ldots, z_{m-1} \right\| \geq \varepsilon \Rightarrow \limsup_n \left\| x_n - L, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon.
\]

(2.11)

From equations (2.10) and (2.11), we get

\[-\varepsilon \leq \liminf_n \left\| x_n - L, z_1, z_2, \ldots, z_{m-1} \right\| \leq \limsup_n \left\| x_n - L, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon.\]
Hence, we have obtained that
\[ \lim_{n} \| x_n - L, z_1, z_2, \ldots, z_{m-1} \| = 0. \]

In what follows we will show that under conditions that \((x_n)\) is slowly oscillation sequence, and \(st^m - N_{p,q}^n C^1\) summability, it converges \(st^m\) to the same limit.

**Theorem 3.** Let \((x_n) \in X\) be \(st^m - N_{p,q}^n C^1\)-summable to \(L\). If \((x_n)\) slowly oscillating in m-normed space \(X\), then \((x_n)\) converges \(st^m\) to \(L\).

**Proof.** First case, \(\lambda > 1\).

Let us suppose that \(st^m - N_{p,q}^n C^1\) converges to \(L\) in m-normed space \(X\). To prove that \(st^m - (x_n) \to L\) in m-normed space \(X\), it is enough to prove that
\[ st^m - \lim_{n} \| N_{p,q}^n C^1 - x_n, z_1, z_2, \ldots, z_{m-1} \| = 0, \]
for every \(z_1, z_2, \ldots, z_{m-1} \in X\). Let us start from
\[
\| N_{p,q}^n C^1 - x_n, z_1, z_2, \ldots, z_{m-1} \| = \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right\|
\leq \max_{0 \leq v \leq n} \left\| \sum_{j=v+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \right\|.
\]

Taking limit superior in both sides of the above relation and then infimum, we get:
\[
\inf_{\lambda > 1} \lim_{n \to \infty} \sup \| N_{p,q}^n C^1 - x_n, z_1, z_2, \ldots, z_{m-1} \| = 0.
\]

Hence, it is proved that \((x_n)\) converges to \(L\) in m-normed space \(X\). And from it follows it \(st^m\)-convergence.

Second case, \(0 < \lambda < 1\). This case is similar to the previous one and for this reason we omit it.

Next result show that if \((x_n)\) satisfies Hardy[12] conditions, and is \(st^m - N_{p,q}^n C^1\) summable, then \((x_n)\) converges \(st^m\) to same limit.

**Theorem 4.** Let \((x_n) \in X\) be \(st^m - N_{p,q}^n C^1\)-summable to \(L\). If \((x_n)\) satisfies the following relation:
\[ n \Delta x_n = 0(1), \]
then \((x_n)\) \(st^m\)-convergent to \(L\).
Proof. It is enough to prove that
\[ st^m - \lim_n \| N_{p,q}^n C_n \| = 0 \]
for every \( z_1, z_2, \ldots, z_{m-1} \in X \). Let us consider that \( \lambda > 1 \). From
\[ n \Delta x_n = O(1), \]
it follows that for every \( \varepsilon > 0 \), there exists a \( n_0 \) such that for every \( n > n_0 \) we have
\[ n \Delta x_n < \varepsilon. \]

Let us start from
\[ \| N_{p,q}^n C_n - x_n, y \| = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_n - k \frac{1}{k+1} \sum_{i=0}^{k} (x_i - x_n), z_1, z_2, \ldots, z_{m-1} \|
\]
\[ = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_n - k \frac{1}{k+1} \sum_{i=0}^{k} \sum_{j=i+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \|
\]
\[ \leq \max_{0 \leq k \leq n} \sum_{j=k+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \|. \]

From above relations, we get:
\[ \| N_{p,q}^n C_n - x_n, z_1, z_2, \ldots, z_{m-1} \| \leq \max_{0 \leq k \leq n} \sum_{j=k+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \| \leq \varepsilon. \]

Hence, it is proved that \( (x_n) \) converges \( st^m \) to \( L \) in \( m \)-normed space \( X \).

The second case where \( 0 < \lambda < 1 \), can be proved in similar way. \( \square \)

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