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EXISTENCE AND ESTIMATION OF THE FIXED POINTS OF ENRICHED BERINDE NONEXPANSIVE MAPPINGS

JAVID ALI AND MOHD JUBAIR

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Abstract. In this article, inspired by Berinde (Approximating fixed points of enriched nonexpansive mappings by Krasnoselskii iteration in Hilbert spaces, Carpathian J. Math. 35(3), 2019, 293-304), we define and study a new enriched class of mappings which includes many other contractive type mappings. We prove an existence result for the fixed point of newly introduced mapping. We also estimate fixed points of the proposed mapping via newly modified Mann iteration. In the process, some convergence results are also obtained for the proposed class of mappings in Uniformly convex Banach space.

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1. INTRODUCTION

Fixed point theory is a fascinating subject, with an enormous number of applications in various fields of mathematics, especially in the existence and uniqueness theory. It includes classical results to form the existence and uniqueness theorems in ordinary differential equations, random differential equations, integral equations, functional equation, partial differential equations, variational inequalities, etc. In metric fixed point theory, Banach contraction [3] is one of the crucial results, which ensures that in a complete metric space every contraction mapping always have a unique fixed point. Banach theorem has been advanced and extended in many directions. This result is a very operative and common tool for guaranteeing the existence and uniqueness of the solution of certain nonlinear problems arising within and outside mathematics. Since the applications of Banach contraction principle are restricted to contraction mappings. Because of its limitation, we need other appropriate and nice mappings to study more nonlinear problems.

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{Z}_+$ denotes the set of all nonnegative integers. Let U be a nonempty subset of a Banach space $(X, ||\cdot||)$, $T : U \to U$ a mapping and $F(T) = \{t \in U : \ Tt = t\}$. A self map T on U is called

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nonexpansive if $\|Tp-Tu\| \leq \|p-u\|, \forall p,u \in U$. It is said to be a quasi nonexpansive if $F(T) \neq \varnothing$ and $\|Tp-t\| \leq \|p-t\|, \forall\ p \in U$ and $\forall\ t \in F(T)$.

In 2009, Chumpungam [7] studied Berinde's nonexpansive mappings and proved existence result for their fixed points. Furthermore, many authors approximated common fixed points and proved existence and convergence results for multivalued Berinde nonexpansive mappings (1.1) in linear spaces, e.g. [5, 6].

Let X be a Banach space and U a nonempty closed convex subset of X. A self mapping T on U is said to be Berinde nonexpansive if there exists $\lambda \geq 0$ such that for all $p,u \in U$,

$$\|Tp-Tu\| \leq \|p-u\| + \lambda \|u-Tp\|. \tag{1.1}$$

Recently, Berinde [4] coined the notion of enriched nonexpansive mappings which is also a generalization of nonexpansive mapping and is defined as follows:

Definition 1. [4] Let X be a normed linear space. A mapping $T : X \to X$ is said to be an enriched nonexpansive mapping if there exists $\eta \in [0,\infty)$ such that

$$\|\eta(p-u)+Tp-Tu\| \leq (\eta+1)\|p-u\|,\ \forall\ p,u \in X. \tag{1.2}$$

It can easily be seen in [4] that every nonexpansive mapping is enriched nonexpansive, but the reverse is not true in general. Moreover, if T has at least one fixed point, then T need not be quasi nonexpansive mapping. The following interesting example is available in [4].

Example 1. Let $X = [\frac{1}{2}, 2]$ be a normed space endowed with usual norm and $T : X \to X$ a mapping determine as $T(p) = \frac{1}{p}$ for all $p \in X$. Then

(i) T is $\frac{3}{2}$-enriched nonexpansive mapping,
(ii) T is not nonexpansive,
(iii) $F(T) = \{1\}$ but T is not quasi nonexpansive.

Remark 1. It can be easily seen that any enriched nonexpansive mapping becomes nonexpansive when $\eta = 0$.

Inspired by the above, one can raise the following question:

Question. Does there exist a class of mappings which contains the mappings defined in (1.1) and (1.2)?

Here, we have a partial answer to this question, affirmatively. In fact, we propose a new class of mapping called enriched Berinde nonexpansive mapping. Further, we prove existence and convergence results for such mappings. We have also provided some numerical examples to substantiate the facts and results.

It is well known that by using Picard iteration process [15] we can approximate the fixed points of contraction mappings where the iteration $\{\xi_n\}$ is developed by an arbitrary point $\xi_0 \in U$ as follows:

$$\xi_{n+1} = T\xi_n, \qquad n \in \mathbb{Z}_+. \tag{1.3}$$

But in case of nonexpansive mappings, if we take initial guess different from fixed points of mapping then Picard sequence may fail to converge to the fixed points of such mappings.

Therefore, in 1953, Mann [11] introduced an iteration process to approximate fixed point of nonexpansive mappings which is defined by an initial guess $\xi_0 \in U$ as follows:

$$\xi_{n+1} = (1-\theta_n)\xi_n + \theta_n T\xi_n, \quad n \in \mathbb{Z}_+, \tag{1.4}$$

where $\{\theta_n\}$ is a control sequence in $(0,1)$. In case of Pseudocontractive mappings Mann iteration need not converge to the fixed points of such mappings. Then, to approximate the fixed points of different non-linear mappings several authors introduced and studied remarkable iteration processes. However to approximate the fixed points of enriched class of mappings by Picard and Mann iterations or any other iteration present in the literature is a very difficult task.

2. Preliminaries

Definition 2. A Banach space X is said to have Opial's property [12] if for each weakly convergent sequence $\{\xi_n\}$ to $p \in X$, the following inequality

$$\liminf_{n\to\infty} \|\xi_n - p\| < \liminf_{n\to\infty} \|\xi_n - u\|$$

holds, for all $u \in X$ with $u \neq p$.

Lemma 1. *[16] Let X be a uniformly convex Banach space and $0 < a \leq \omega_n \leq b < 1$ for all $n \in \mathbb{N}$. Assume that $\{\xi_n\}$ and $\{\sigma_n\}$ are two sequences in X such that $\limsup_{n\to\infty} \|\xi_n\| \leq \omega$, $\limsup_{n\to\infty} \|\sigma_n\| \leq \omega$ and $\limsup_{n\to\infty} \|\omega_n\xi_n + (1-\omega_n)\sigma_n\| = \omega$ holds, for some $\omega \geq 0$. Then $\lim_{n\to\infty} \|\xi_n - \sigma_n\| = 0$.*

Definition 3. [10] (Condition$(*)$) A self map T on a nonempty subset U of a Banach space X is said to satisfy *Condition*$(*)$ if there exists a constant $\lambda \geq 0$ such that

$$\|Tp - Tu\| \leq \|p-u\| + \lambda\|p - Tp\| \qquad \forall\, p,u \in U. \tag{2.1}$$

Lemma 2. *[10] Assume that $T : U \to U$ is a Berinde nonexpansive mapping satisfying condition $(*)$, where U is a nonempty closed convex subset of Banach space X that satisfies the Opial's condition. Then, $I - T$ is demiclosed at zero.*

3. Enriched Berinde Non-Expansive Mapping and New Iterative Scheme

In this section, we define enriched Berinde nonexpansive mapping and modified Mann iteration, and also prove some basic results for such mapping.

Definition 4. Let U be a nonempty subset of Banach space X. A mapping $T: U \to U$ is said to be an enriched Berinde nonexpansive or η-enriched Berinde nonexpansive if there exist constants $\eta \in [0,\infty)$ and $\lambda \geq 0$ such that

$$\|\eta(p-u)+Tp-Tu\| \leq (\eta+1)\|p-u\|+\lambda\|(\eta+1)(u-p)+p-Tp\| \quad \forall\, p,u \in U. \tag{3.1}$$

Now we mention some facts and basic properties of enriched Berinde nonexpansive mapping.

Remark 2.

(1) If $\lambda = 0$ in inequality (3.1), then it reduces to enriched nonexpansive mapping (1.2).
(2) If $\lambda = 0$ and $\eta = 0$ in inequality (3.1), then it reduces to nonexpansive mapping.
(3) If $\eta = 0$ in inequality (3.1), then it reduces to Berinde nonexpansive mapping (1.1).

It can be easily seen that every nonexpansive mapping, Berinde nonexpansive mapping and enriched nonexpansive mapping are enriched Berinde nonexpansive mapping. The following examples show the converse is not true in general. An enriched Berinde nonexpansive mapping need not be quasi-nonexpansive.

Example 2. Define a self mapping T on $[0,3]$ by

$$T(p) = \begin{cases} 0, & \text{if } p \neq 3 \\ 1.9, & \text{if } p = 3. \end{cases}$$

Here T is enriched Berinde nonexpansive mapping for $\lambda > 0$ but not Berinde nonexpansive.

Verification.
Case I. If $p \in [0,1.1)$ and $u \in [0,3)$, then condition (4) reduces to

$$\begin{aligned} |\eta(p-u)+Tp-Tu| &= |\eta(p-u)| \\ &\leq (\eta+1)|p-u|+\lambda|(\eta+1)(u-p)+p-Tp|. \end{aligned}$$

Case II. If $p \in [1.1,\ 3]$, $u = 3$ and consider $p < u$, then condition (4) reduces to

$$\begin{aligned} |\eta(p-u)+Tp-Tu| &\leq \eta|p-u|+|Tp-Tu| = \eta|p-u|+1.9 \\ &\leq (\eta+1)|p-u|+\lambda|(\eta+1)(u-p)+p-Tp| \\ &\leq (\eta+1)|p-u|+\lambda\Big((\eta+1)|u-p|+|p|\Big) \\ &\leq (\eta+1)|p-u|+\lambda\Big((\eta+1)|p-u|+|p|\Big). \end{aligned}$$

Thus T is enriched Berinde nonexpansive mapping for $\lambda > 0$. Now take $p = 3, u = 1.9$, then condition (1.1) reduces to

$$\|Tp - Tu\| = 1.9 > 1.1 = \|p - u\|.$$

Thus T is not Berinde nonexpansive mapping, hence not nonexpansive.

Example 3. Let $U = [0,1] \subset \mathbb{R} = X$ and a map $T : U \to U$ defined by

$$T(p) = \begin{cases} p^2 & \text{if } p \in [0, \frac{1}{2}), \\ 1 & \text{if } p \in [\frac{1}{2}, 1]. \end{cases}$$

(a) T is enriched Berinde nonexpansive mapping (3.1) with $\lambda = 4$.
(b) T is not enriched nonexpansive mapping (1.2), hence not nonexpansive.
(c) T is not quasi nonexpansive.

Verification.

(a) To show that T is enriched Berinde nonexpansive mapping, we have the following cases.

Case I. If $p, u \in [0, \frac{1}{2})$, then we have,

$$|\eta(p-u) + (p^2 - u^2)| \leq |\eta(p-u)| + |(p^2 - u^2)| \leq (\eta+1)|p-u|,$$

which is true for all $\eta \geq 0$ and $\lambda \geq 0$.

Case II. If $p \in [0, \frac{1}{2})$ and $u \in [\frac{1}{2}, 1]$, then we have,

$$\begin{aligned} |\eta(p-u) + p^2 - 1| &\leq \eta|p-u| + |p^2 - 1| \leq (\eta+1)|p-u| + 1 \\ &= (\eta+1)|p-u| + 4.\frac{1}{4} \\ &\leq (\eta+1)|p-u| + 4|(\eta+1)(u-p) + p - Tp|. \end{aligned}$$

Case III. If $p, u \in [\frac{1}{2}, 1]$, then we have,

$$|\eta(p-u) + 0| \leq (\eta+1)|p-u| + 4|(\eta+1)(u-p) + p - Tp|.$$

Case IV. If $p \in [\frac{1}{2}, 1]$ and $u \in [0, \frac{1}{2})$, then we have,

$$\begin{aligned} |\eta(p-u) + 1 - u^2| &\leq \eta|p-u| + |1 - u^2| \leq (\eta+1)|p-u| + 1 \\ &= (\eta+1)|p-u| + 4.\frac{1}{4} \leq (\eta+1)|p-u| + 4.\frac{1}{2} \\ &\leq (\eta+1)|p-u| + 4|(\eta+1)(u-p) + p - Tp|. \end{aligned}$$

Hence T is enriched Berinde nonexpansive mapping with $\lambda = 4$.

(b) Assume T is enriched nonexpansive mapping. Then

$$|\eta(p-u) + Tp - Tu| \leq (\eta+1)|p-u|, \qquad \forall\, p, u \in U.$$

For $p = 0.50$ and $u = 0.49$, leads to a contradiction

$$\frac{\eta}{100} + 0.7599 \leq \frac{\eta}{100} + 0.01.$$

Hence T is not enriched nonexpansive mapping.

(c) Assume that T is quasi nonexpansive mapping and $F(T) = \{0,1\}$. Then

$$|Tp - t| \leq |p - t|, \qquad \forall\, p \in U,\ t \in F(T).$$

Now take $p = \frac{1}{3}$ and $t = 1$, then

$$|Tp - t| = \Big|\frac{1}{9} - 1\Big| = \frac{8}{9} > \frac{2}{3} = \Big|\frac{1}{3} - 1\Big|.$$

Which is a contradiction. Hence T is not quasi nonexpansive mapping.

Definition 5. [2] Given a map $T : X \to X$, where X is a Banach space, the κ-Krasnoselskii map or averaged mapping is defined for $\kappa \in (0,1]$ as $T_\kappa : X \to X$,

$$T_\kappa(p) = (1-\kappa)p + \kappa T p \qquad \forall\, p \in X.$$

Remark 3. For a self mapping T on a convex subset U of a Banach space X and for any $\kappa \in (0,1]$, we have

$$F(T_\kappa) = F(T).$$

Theorem 1. *Let X be a Banach space and $T : X \to X$ an enriched Berinde nonexpansive mapping. Then, κ-Krasnoselskii map $T_\kappa : X \to X$ is a Berinde nonexpansive mapping.*

Proof. Since T is an enriched Berinde nonexpansive mapping, so for all $p,u \in X$, we have

$$\|\eta(p-u) + Tp - Tu\| \leq (\eta+1)\|p-u\| + \lambda\|(\eta+1)(u-p) + p - Tp\|.$$

Using $\kappa = \frac{1}{\eta+1}$, we have

$$\begin{aligned}
&\|\Big(\frac{1}{\kappa} - 1\Big)(p-u) + Tp - Tu\| \leq \frac{1}{\kappa}\|p-u\| + \lambda\|\frac{1}{\kappa}(u-p) + p - Tp\| \\
&\|(1-\kappa)(p-u) + \kappa Tp - \kappa Tu\| \leq \|p-u\| + \lambda\|u - T_\kappa p\|.
\end{aligned}$$

This gives

$$\|T_\kappa(p) - T_\kappa(u) \leq \|p-u\| + \lambda\|u - T_\kappa p\|.$$

Hence T_κ is a Berinde nonexpansive mapping with $\lambda \geq 0$. □

Theorem 2. *Let $T : X \to X$ be a Berinde nonexpansive mapping satisfying condition $(*)$, where X is a Banach space. Then T is quasi nonexpansive, i.e. for each $p \in X$ and $t \in F(T)$, we have*

$$\|Tp - t\| \leq \|p - t\|.$$

Proof. Since T is a Berinde nonexpansive map satisfying condition $(*)$, therefore there exists $\lambda \geq 0$ such that

$$\|Tp - Tu\| \leq \|p-u\| + \lambda\|p - Tp\|, \qquad \forall\, p,u \in X.$$

Now, for a fixed point t of T, we have

$$\|Tp-t\| = \|Tp-Tt\| \leq \|p-t\| + \lambda\|t-Tt\| \leq \|p-t\|.$$

□

Lemma 3. *Let $T : U \to U$ be an enriched Berinde nonexpansive map and $U \subset X$ be a nonempty closed and convex satisfying Opial's condition. Then, $I - T$ is demiclosed at zero.*

Proof. From Theorem 1, we know that T_κ is a Berinde nonexpansive map for $\kappa = \frac{1}{\eta+1}$. Now, let $\{\xi_n\}$ be a sequence that converges weakly to $p \in U$ and let $(I - T)\xi_n \to 0$ (strongly). This would mean $\lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0$. However

$$\|\xi_n - T_\kappa(\xi_n)\| = \kappa\|\xi_n - T(\xi_n)\|,$$

so that

$$\lim_{n\to\infty} \|\xi_n - T_\kappa(\xi_n)\| = \kappa \lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0.$$

By Lemma 2, $I - T_\kappa$ is demiclosed at zero. So we have

$$T_\kappa(p) = p.$$

Using the definition of T_κ and simplifying, we get

$$\kappa(p - T(p)) = 0.$$

However $\kappa \neq 0$, therefore $T(p) = p$. In other words, $I - T$ is demiclosed at zero. □

In some past years, many authors gave the generalizations and comparison of nonexpansive mappings such as Hardy and Rogers, Suzuki [18], Karapinar and Taş [9], Pant and Shukla [14], Pandey et al. [13], Ali et al. [1] etc. As Berinde [4] enriched the class of nonexpansive mappings and proved existence results on fixed points. But approximation of the fixed points of enriched class of nonexpansive mappings is very complicated by using the iteration schemes available in the literature, namely Picard, Mann, Ishikawa [8] and some others. Therefore, one can modify or define a new iteration to approximate the fixed points of such enriched mappings. To overcome this problem, we modify Mann iteration to approximate fixed points of enriched class of mappings. For any initial guess $\xi_0 \in U$ and $\eta \geq 0$ the modified Mann iteration can be determine as follows:

$$\xi_{n+1} = \frac{1}{\eta+1}[\eta\xi_n + T(\xi_n)], \qquad n \in \mathbb{Z}_+. \tag{3.2}$$

4. Existence Result

Theorem 3. *Let U be a bounded convex closed subset of a uniformly convex Banach space X and $T : U \to U$ be a η-enriched Berinde nonexpansive mapping. Then the set $F(T)$ is nonempty.*

Proof. Since T is a η-enriched Berinde nonexpansive mapping, by Definition 4, it follows that there exist constants $\eta \in [0,\infty)$ and $\lambda \geq 0$, such that

$$\|\eta(p-u)+Tp-Tu\| \leq (\eta+1)\|p-u\|+\lambda\|(\eta+1)(u-p)+p-Tp\| \qquad \forall\, p,u \in U.$$

By putting $\eta = \frac{1}{\kappa}-1$ for $\eta > 0$, it follows that $\kappa \in (0,1]$ and the previous inequality is equivalent to

$$\|(1-\kappa)(p-u)+\kappa Tp-\kappa Tu\| \leq \|p-u\|+\lambda\|(u-p)+\kappa p-\kappa Tp\| \qquad \forall\, p,u \in U. \tag{4.1}$$

Denote $T_\kappa(p) = (1-\kappa)p+\kappa Tp$. Then inequality (4.1) expresses the fact that

$$\|T_\kappa(p)-T_\kappa(u)\| \leq \|p-u\|+\lambda\|u-T_\kappa p\| \qquad \forall\, p,u \in U$$

i.e. T_κ is Berinde nonexpansive. So, by Chumpungam [7], it follows that $F(T_\kappa) \neq \varnothing$.

By Remark 3, we can see that $F(T)$=$F(T_\kappa)$$\neq$$\varnothing$. Hence T has at least one fixed point. □

In support of Theorem 3, one can see Example 3.

5. Convergence Results

Throughout this section, we presume that $T : U \to U$ is a η-enriched Berinde nonexpansive mapping, where U is a nonempty convex and closed subset of a uniformly convex Banach space X. Now, we prove the following useful lemma which will be used to prove the next results of this section.

Lemma 4. *Let $\{\xi_n\}$ be a sequence developed by the iteration process (3.2). Assume that $F(T) \neq \varnothing$. Then,*

(i) $\lim_{n\to\infty} \|\xi_n - t\|$ *exists for all $t \in F(T)$.*

(ii) $\lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0.$

Proof. (i) Let $t \in F(T)$. Form Theorem 1, we know that for $\kappa = \frac{1}{\eta+1}$, T_κ is a Berinde nonexpansive map. Also from Theorem 2, we have $\|T_\kappa(p)-t\| \leq \|p-t\|$, for all $p \in U$. Therefore we have

$$\begin{aligned}\|\xi_{n+1}-t\| &= \|\frac{1}{\eta+1}[\eta\xi_n+T(\xi_n)]-t\| = \|\kappa[(\frac{1}{\kappa}-1)\xi_n+T(\xi_n)]-t\| \\ &= \|(1-\kappa)\xi_n+\kappa T(\xi_n)-t\| = \|T_\kappa(\xi_n)-t\| \leq \|\xi_n-t\|. \end{aligned}\tag{5.1}$$

Thus the sequence $\{\|\xi_n - t\|\}$ is bounded below and decreasing for all $t \in F(T)$. Hence $\lim_{n\to\infty} \|\xi_n - t\|$ exists.

(ii) Define $T_\kappa : U \to U$, for $\kappa = \frac{1}{\eta+1}$. Then from Theorem 1, we know that T_κ is Berinde nonexpansive. Also from Remark 3, we know that $F(T) = F(T_\kappa) \neq \varnothing$. Moreover, for the same initial guess $\xi_0 \in U$, the sequence generated by the modified Mann iteration process using T is the same as that generated by the Mann iteration process using T_κ. Hence by Theorem 3.2 in [10], we have $\lim_{n\to\infty}\|\xi_n - T_\kappa(\xi_n)\| = 0$. By using the definition of T_κ, we get

$$\kappa\|\xi_n - T(\xi_n)\| = 0.$$

Since $\kappa \neq 0$, then $\lim_{n\to\infty}\|\xi_n - T(\xi_n)\| = 0$. □

Now, we prove that iteration process (3.2) converges weakly for the proposed mapping.

Theorem 4. *Let $\{\xi_n\}$ be defined by (3.2) and X satisfying Opial's property. Then, $\{\xi_n\}$ converges weakly to a fixed point of the mapping T.*

Proof. Let s and q be two weak sub-sequential limits of $\{\xi_{n_j}\}$ and $\{\xi_{n_k}\}$, respectively, where $\{\xi_{n_j}\}$ and $\{\xi_{n_k}\}$ are two subsequences of $\{\xi_n\}$. From Lemma 4(i), we get $\lim_{n\to\infty}\|\xi_n - t\|$ exists. Now our aim is to show that $s = q$ in $F(T)$. From Lemma 4(ii), $\lim_{n\to\infty}\|\xi_n - T(\xi_n)\| = 0$ and by Lemma 3, $I - T$ is demiclosed at zero. Thus $(I-T)s = 0$, that is, $s = Ts$. Similarly, $q = Tq$.

Now it is enough to show that $s = q$. If $s \neq q$, then applying Opial's property, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|\xi_n - s\| &= \lim_{n_j\to\infty}\|\xi_{n_j} - s\| < \lim_{n_j\to\infty}\|\xi_{n_j} - q\| = \lim_{n\to\infty}\|\xi_n - q\| \\ &= \lim_{n_k\to\infty}\|\xi_{n_k} - q\| < \lim_{n_k\to\infty}\|\xi_{n_k} - s\| = \lim_{n\to\infty}\|\xi_n - s\|,\end{aligned}$$

which is a contradiction, Hence $s = q$. It concludes that $\{\xi_n\}$ converges weakly to a fixed point of T. □

Theorem 5. *The sequence $\{\xi_n\}$ developed by the iteration process (3.2) converges strongly to a fixed point of T if and only if $\lim_{n\to\infty}\inf d(\xi_n, F(T)) = 0$, where $d(\xi_n, F(T)) = \inf\{\|\xi_n - t\| : t \in F(T)\}$.*

Proof. First part is trivial. Now, we prove the converse part. Presume that $\lim_{n\to\infty}\inf d(\xi_n, F(T)) = 0$. From Lemma 4(i), $\lim_{n\to\infty}\|\xi_n - t\|$ exists, for all $t \in F(T)$ and by hypothesis $\lim_{n\to\infty} d(\xi_n, F(T)) = 0$. Now our assertion is that $\{\xi_n\}$ a Cauchy sequence in U. As $\lim_{n\to\infty} d(\xi_n, F(T)) = 0$, for a given $\rho > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(\xi_n, F(T)) < \frac{\rho}{2} \qquad \Longrightarrow \inf\{\|\xi_n - t\| : t \in F(T)\} < \frac{\rho}{2}.$$

Specifically, $\inf\{\|\xi_N - t\| : t \in F(T)\} < \frac{\rho}{2}$. So, there exists $t \in F(T)$ such that

$$\|\xi_N - t\| < \frac{\rho}{2}.$$

Now, for $m,n \geq N$,

$$\begin{aligned}\|\xi_{n+m}-\xi_n\| &\leq \|\xi_{n+m}-t\|+\|\xi_n-t\| \\ &\leq \|\xi_N-t\|+\|\xi_N-t\| = 2\|\xi_N-t\| < \rho.\end{aligned}$$

Thus, $\{\xi_n\}$ is a Cauchy sequence in U, so that there exists an element $\ell \in U$ such that $\lim_{n\to\infty} \xi_n = \ell$. Now, $\lim_{n\to\infty} d(\xi_n, F(T)) = 0$ implies $d(\ell, F(T)) = 0$, hence we get $\ell \in F(T)$. □

In 1974, Senter and Dotson [17] defined a condition on mappings called condition (I), using which we can get strong convergence.

Definition 6. [17] Let $\psi : [0,\infty) \to [0,\infty)$ be a nondecreasing function with $\psi(0) = 0$ and $\psi(z) > 0$, $\forall\, z > 0$. A self map T on U is said to satisfy condition (I), if $d(p, Tp) \geq \psi(d(p, F(T)))$, $\forall\, p \in U$.

Theorem 6. *Let condition (I) be satisfied by mapping T. Then strong convergence is achieved by iteration process (3.2) to a fixed point of T.*

Proof. We already proved in Lemma 4(ii) that

$$\lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0. \tag{5.2}$$

By equation (5.2) and condition (I), we obtain

$$\begin{aligned}0 \leq \lim_{n\to\infty} \psi(d(\xi_n, F(T))) &\leq \lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0 \\ \implies \lim_{n\to\infty} \psi(d(\xi_n, F(T))) &= 0. \\ \implies \lim_{n\to\infty} d(\xi_n, F(T)) &= 0.\end{aligned}$$

By Theorem 5, the sequence $\{\xi_n\}$ converges strongly to a fixed point of T. □

6. Conclusion

In this paper, we enriched the class of Berinde nonexpansive mapping and approximated the fixed points of proposed mapping by using modified Mann iteration process in uniformly convex Banach space. Our results are new and generalized several results in literature, in particular the results of Chumpungam [7].

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REFERENCES

[1] F. Ali, J. Ali, and J. J. Nieto, “Some observations on generalized non-expansive mappings with an application,” *Comput. Appl. Math.*, vol. 39, no. 2, p. 20, 2020, id/No 74, doi: 10.1007/s40314-020-1101-4.

[2] J. B. Baillon, R. E. Bruck, and S. Reich, “On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces,” *Houston J. Math.*, vol. 4, pp. 1–9, 1978.

[3] S. Banach, “Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales.” *Fundam. Math.*, vol. 3, pp. 133–181, 1922, doi: 10.4064/fm-3-1-133-181.

[4] V. Berinde, “Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces,” *Carpathian J. Math.*, vol. 35, no. 3, pp. 293–304, 2019.

[5] N. Bunlue and S. Suantai, “Existence and convergence theorems for Berinde nonexpansive multivalued mapping on Banach spaces,” *Afr. Mat.*, vol. 30, no. 3-4, pp. 483–494, 2019, doi: 10.1007/s13370-019-00661-8.

[6] L. Bussaban and A. Kettapun, “Common fixed points of an iterative method for Berinde nonexpansive mappings,” *Thai J. Math.*, vol. 16, no. 1, pp. 49–60, 2018.

[7] D. Chumpungam, “Strong convergence theorems and rate of convergence of some iterative methods for common fixed points of weak contractions in banach spaces,” Masters thesis, Chiang Mai University, Thailand, 2009.

[8] S. Ishikawa, “Fixed points by a new iteration method,” *Proc. Am. Math. Soc.*, vol. 44, pp. 147–150, 1974, doi: 10.2307/2039245.

[9] E. Karapınar and K. Taş, “Generalized C-conditions and related fixed point theorems,” *Comput. Math. Appl.*, vol. 61, no. 11, pp. 3370–3380, 2011, doi: 10.1016/j.camwa.2011.04.035.

[10] S. Kosol, “Weak and strong convergence theorems of some iterative methods for common fixed points of Berinde nonexpansive mappings in Banach spaces,” *Thai J. Math.*, vol. 15, no. 3, pp. 629–639, 2017.

[11] W. R. Mann, “Mean value methods in iteration,” *Proc. Am. Math. Soc.*, vol. 4, pp. 506–510, 1953, doi: 10.2307/2032162.

[12] Z. Opial, “Weak convergence of the sequence of successive approximations for nonexpansive mappings,” *Bull. Am. Math. Soc.*, vol. 73, pp. 591–597, 1967, doi: 10.1090/S0002-9904-1967-11761-0.

[13] R. Pandey, R. Pant, V. Rakočević, and R. Shukla, “Approximating fixed points of a general class of nonexpansive mappings in Banach spaces with applications,” *Result. Math.*, vol. 74, no. 1, p. 24, 2019, id/No 7, doi: 10.1007/s00025-018-0930-6.

[14] R. Pant and R. Shukla, “Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces,” *Numer. Funct. Anal. Optim.*, vol. 38, no. 2, pp. 248–266, 2017, doi: 10.1080/01630563.2016.1276075.

[15] E. Picard, “Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives.” *Journ. de Math. (4)*, vol. 6, pp. 145–210, 1890.

[16] J. Schu, “Weak and strong convergence to fixed points of asymptotically nonexpansive mappings,” *Bull. Aust. Math. Soc.*, vol. 43, no. 1, pp. 153–159, 1991, doi: 10.1017/S0004972700028884.

[17] H. F. Senter and W. G. j. Dotson, “Approximating fixed points of nonexpansive mappings,” *Proc. Am. Math. Soc.*, vol. 44, pp. 375–380, 1974, doi: 10.2307/2040440.

[18] T. Suzuki, “Fixed point theorems and convergence theorems for some generalized nonexpansive mappings,” *J. Math. Anal. Appl.*, vol. 340, no. 2, pp. 1088–1095, 2008, doi: 10.1016/j.jmaa.2007.09.023.

Authors' addresses

Javid Ali
(**Corresponding author**) Department of Mathematics, Aligarh Muslim University, Aligarh- 202002, India
E-mail address: javid.mm@amu.ac.in

Mohd Jubair
Department of Mathematics, Aligarh Muslim University, Aligarh- 202002, India
E-mail address: jubairnnn@gmail.com