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INTERVAL-VALUED Iq^b-CALCULUS AND APPLICATIONS

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Abstract. This study investigates the Iq^b-differentiability and Iq^b-integrability for interval-valued functions defined on the q-geometric set. We also establish some Iq^b-Hermite-Hadamard type inequalities. Furthermore, some examples are presented to illustrate our results.

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1. INTRODUCTION

Quantum calculus was proposed by the famous mathematician Euler in the 18th century. It uses the difference operator instead of the classical derivative to study the set of non-differentiable functions. In 1910, Jackson introduced the concept of quantum definite integral and extended the concept of quantum calculus. Quantum calculus is a bridge between mathematics and physics. It has been widely used in number theory, quantum theory, mechanics and other fields. In recent years, with the higher and higher requirements of analog quantum computing on mathematics, it has attracting considerable scholarly attention and triggered a huge amount of innovative scientific research [6, 7, 9, 14].

As a branch of mathematics, interval analysis is an effective tool for dealing with data inaccurate models caused by some types of measurements. Interval analysis has been attracting considerable interest since it was firstly applied to automatic error analysis by Moore. We now see applications in fuzzy set and possibility theory [5], uncertain quantification and propagation procedure in the case of the small sample measurement data [17], stochastic analysis of structures with uncertain-but-bounded parameters [13], the terminal error bound of the Stewart platform [18], for more profound results and applications, we refer to the papers [3, 10]. Particularly, Younus [19] investigated the fractional q-differentiability and fractional q-integrability for

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interval-valued functions (IVFs) defined on the q-geometric set of real numbers. Lou [11] proved the Iq-Hermite-Hadamard inequalities for IVFs. Recently, several studies have focused on the quantum differentials, integrals and inequalities of IVFs.

Motivated by the works mentioned, in this paper, we discuss the interval-valued quantum calculus for (shortly, Iq^b-calculus). Firstly, we give the concepts of Iq^b-calculus and define the Iq^b-derivative and Iq^b-integral. We also give some basic properties and present some examples to illustrate our theorems. Moreover, using the notion of Iq^b-derivative and Iq^b-integral, some new inequalities like Hermite-Hadamard are offered. The results of this paper can serve as a base for future studies. These results of this paper can be used as a powerful tool in fuzzy analysis, interval optimization, and interval-valued differential equations.

2. Preliminaries

2.1. *Calculus for IVFs*

Let $\mathbb{R}_I$ be the set of all non-empty compact intervals on the real line $\mathbb{R}$. For all $[\underline{a},\, \overline{a}],\ [\underline{b},\, \overline{b}] \in \mathbb{R}_I,\ \lambda \in \mathbb{R}$ we have

$$[\underline{a},\, \overline{a}] + [\underline{b},\, \overline{b}] = [\underline{a} + \underline{b},\, \overline{a} + \overline{b}],$$

and

$$\lambda\,[\underline{a}, \overline{a}] = \begin{cases} [\lambda\,\underline{a},\, \lambda\,\overline{a}], & \lambda > 0, \\ 0, & \lambda = 0, \\ [\lambda\,\overline{a},\, \lambda\,\underline{a}], & \lambda < 0, \end{cases}$$

respectively.

The generalized Hukuhara (gH for brevity) difference of two intervals $[\underline{a},\, \overline{a}],\ [\underline{b},\, \overline{b}] \in \mathbb{R}_I$ is defined by Stefanini [15]:

$$[\underline{a},\, \overline{a}] \ominus_g [\underline{b},\, \overline{b}] = \left[\min\{\underline{a} - \underline{b},\, \overline{a} - \overline{b}\},\ \max\{\underline{a} - \underline{b},\, \overline{a} - \overline{b}\}\right].$$

Let $A = [\underline{a},\, \overline{a}] \in \mathbb{R}_I,\ w(A) = \overline{a} - \underline{a}$ is called the length of the interval A. Then, for all $A = [\underline{a},\, \overline{a}],\ B = [\underline{b},\, \overline{b}] \in \mathbb{R}_I$, we have

$$[\underline{a},\, \overline{a}] \ominus_g [\underline{b},\, \overline{b}] = \begin{cases} \left[\underline{a} - \underline{b},\, \overline{a} - \overline{b}\right], & \text{if } w(A) \geq w(B), \\ \left[\overline{a} - \overline{b},\, \underline{a} - \underline{b}\right], & \text{if } w(A) \leq w(B). \end{cases}$$

The gH-difference $\ominus_g$ have the following properties [12]:

(1) $A \ominus_g A = \{0\},\ A \ominus_g \{0\} = A,\ \{0\} \ominus_g A = -A$,
(2) $A \ominus_g B = (-B) \ominus_g (-A) = -(B \ominus_g A)$,
(3) $A \ominus_g (-B) = B \ominus_g (-A) = -(B \ominus_g A)$,
(4) $(A + B) \ominus_g B = A$,
(5) $\lambda A \ominus_g \lambda B = \lambda\,(A \ominus_g B)$.

For $A,\ B,\ C,\ D \in \mathbb{R}_I$, consider $t_1 = w(A) - w(C),\ t_2 = w(B) - w(D),\ t_3 = w(A) - w(B),\ t_4 = w(C) - w(D)$. The following properties are satisfied [12]:

(1)

$$(A+B)\ominus_g(C+D)=\begin{cases}(A\ominus_g C)+(B\ominus_g D), & t_1t_2\geq 0,\\ (A\ominus_g C)\ominus_g(-B\ominus_g D), & t_1t_2<0.\end{cases}$$

(2)

$$(A\ominus_g B)+(C\ominus_g D)=\begin{cases}(A\ominus_g(-C))\ominus_g(B\ominus_g(-D)), & t_1t_2\geq 0,\ t_3t_4<0,\\ (A\ominus_g(-C))+(-B\ominus_g D), & t_1t_2<0,\ t_3t_4<0,\\ (A+C)\ominus_g(B+D), & t_3t_4\geq 0.\end{cases}$$

(3)

$$(A\ominus_g B)\ominus_g(C\ominus_g D)=\begin{cases}(A\ominus_g C)\ominus_g(B\ominus_g D), & t_1t_2\geq 0,\ t_3t_4\geq 0,\\ (A\ominus_g C)+(-(B\ominus_g D)), & t_1t_2<0,\ t_3t_4\geq 0,\\ (A+(-C))\ominus_g(B+(-D)), & t_3t_4<0.\end{cases}$$

A function $f\colon [a,b]\to\mathbb{R}_I$ is said to be IVF, if $f(x)=\left[\underline{f}(x),\ \overline{f}(x)\right]$ such that $\underline{f}(x)\leq\overline{f}(x)$ for all $x\in[a,b]$. It is well known that $\lim\limits_{x\to x_0} f(x)$ exist if and only if $\lim\limits_{x\to x_0}\underline{f}(x)$ and $\lim\limits_{x\to x_0}\overline{f}(x)$ exist, and is given by

$$\lim_{x\to x_0} f(x)=\left[\lim_{t\to x_0}\underline{f}(x),\ \lim_{x\to x_0}\overline{f}(x)\right].$$

Particularly, an IVF f is continuous if and only if $\underline{f},\ \overline{f}$ are continuous. For two IVFs $f,\ g\colon [a,b]\to\mathbb{R}_I$, we define the IVF $f\ominus_g g\colon [a,b]\to\mathbb{R}_I$ by

$$(f\ominus_g g)(x)=f(x)\ominus_g g(x).$$

If $\lim\limits_{x\to x_0} f(x)=A$ and $\lim\limits_{x\to t_0} g(x)=B$, then

$$\lim_{x\to x_0}(f\ominus_g g)(x)=A\ominus_g B.$$

Further more, if $f,g\colon [a,b]\to\mathbb{R}_I$ are both continuous, then $f\ominus_g g$ is continuous.

Definition 1 ([1, Definition 20])**.** Let $f\colon [a,b]\to\mathbb{R}_I$ be an IVF and let $x_0\in[a,b]$. We define $f'(x_0)$ if it exist by

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)\ominus_g f(x_0)}{h},$$

and call it the gH-derivative of f at x_0. We say that f is gH-differentiable on (a,b) if it is differentiable at every point of (a,b). The IVF $f'\colon [a,b]\to\mathbb{R}_I$ is called a gH-derivative of f on $[a,b]$.

Theorem 1 ([16, Theorem 17]). *An IVF $f\colon [a,b] \to \mathbb{R}_I$ such that $f(x) = \left[\underline{f}(x), \overline{f}(x)\right]$ is gH-differentiable at $x \in [a,b]$ if $\underline{f}, \overline{f}$ are differentiable at $x \in [a,b]$ and*

$$f'(x) = \left[\min\left\{\underline{f}'(x), \overline{f}'(x)\right\}, \max\left\{\underline{f}'(x), \overline{f}'(x)\right\}\right].$$

Let's recall the definition of μ-monotone IVFs given by Markov in [12]:

An IVF $f\colon [a,b] \to \mathbb{R}_I$ is μ-increasing (μ-decreasing) on $[a,b]$ if the function $x \mapsto w(f(x))$ is increasing (decreasing) on $[a,b]$. Therefore, f is called μ-monotone on $[a,b]$ if f is μ-decreasing or μ-increasing on $[a,b]$.

Proposition 1 ([4, Proposition 2]). *Let $f\colon [a,b] \to \mathbb{R}_I$ such that $f(x) = \left[\underline{f}(x), \overline{f}(x)\right]$. If f is μ-monotone and gH-differentiable on $[a,b]$, then $\frac{d}{dx}\underline{f}(x)$ and $\frac{d}{dx}\overline{f}(x)$ exist for all $x \in [a,b]$. Moreover, we have*

(i) $f'(x) = \left[\underline{f}'(x), \overline{f}'(x)\right]$ *for all $t \in [a,b]$, if f is μ-increasing;*

(ii) $f'(x) = \left[\overline{f}'(x), \underline{f}'(x)\right]$ *for all $t \in [a,b]$, if f is μ-decreasing.*

Definition 2 ([12, Definition 4]). The integral of an IVF $f\colon [a,b] \to \mathbb{R}_I$ such that $f(x) = \left[\underline{f}(x), \overline{f}(x)\right]$ is defined by

$$\int_a^b f(x)dt = \left[\int_a^b \underline{f}(x)dx, \int_a^b \overline{f}(x)dx\right].$$

2.2. *q-calculus for real-valued functions*

We first present some known Definitions and related inequalities in q-calculus. Set the following notation [8]:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Let A be a subset of $\mathbb{R}$ and $0 < q < 1$ be a fixed number. The set A is said to be q-geometric if $qz \in A$ whenever $z \in A$.

For a real-valued function f defined on a q-geometric set A, the q-difference operator D_q is defined by Bermudo in [2]:

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \text{ for } x \in A \setminus \{0\}.$$

Definition 3 ([2, Definition 4]). For a continuous $f\colon [a,b] \to \mathbb{R}$ and $q \in (0,1)$, the q^b-derivative of f at $x \in [a,b]$ is characterized by the expression

$${}^bD_q f(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}, \quad x \neq b.$$

For $x = b$, we define ${}^bD_q f(b) = \lim\limits_{x \to b} D_q f(x)$ if it exist and it is finite.

Definition 4 ([2, Definition 6])**.** For a continuous $f\colon [a,b]\to\mathbb{R}$ and $q\in(0,1)$, the q^b-integral of f at $x\in[a,b]$ is characterized by the expression

$$^{b}I_q f(t)=\int_t^b f(x)\ ^{b}d_q x=(1-q)(b-t)\sum_{n=0}^{\infty}q^n f(q^n t+(1-q^n)b).$$

Theorem 2 ([2, Theorem 12])**.** *Let $f\colon [a,b]\to\mathbb{R}$ be a convex function and $q\in(0,1)$. Then we have*

$$f\left(\frac{a+qb}{[2]_q}\right)\le\frac{1}{b-a}\int_a^b f(x)\ ^{b}d_q x\le\frac{f(a)+qf(b)}{[2]_q}. \tag{2.1}$$

3. Iq^b-DERIVATIVE FOR IVFS

Now we introduce Iq^b-derivative and corresponding properties.

Definition 5 (Iq^b-derivative)**.** For a continuous IVF $f\colon [a,b]\to\mathbb{R}_I^+$, the Iq^b-derivative of f at $x\in[a,b]$ is given as:

$$^{b}D_q f(x)=\begin{cases}\frac{f(qx+(1-q)b)\ominus_g f(x)}{(1-q)(b-x)}, & x\neq b,\\ \lim\limits_{x\to b^-}D_q f(x), & x=b,\end{cases}$$

where $q\in(0,1)$ and $D_q f(x)=\frac{f(x)\ominus_g f(qx)}{(1-q)x}$.

Example 1. Consider $f\colon [0,1]\to\mathbb{R}_I$ given by $f(x)=[-x,x]$. It follows that f is Iq^b-differentiable. By Definition 5, for all $x\in[0,1)$, we have

$$\begin{aligned}^{b}D_q f(x)&=\frac{f(qx+(1-q)b)\ominus_g f(x)}{(1-q)(b-x)}\\&=\frac{[-qx-(1-q),qx+(1-q)]\ominus_g[-x,x]}{(1-q)(1-x)}\\&=\frac{[(1-q)(x-1),(1-q)(1-x)]\ominus_g[-x,x]}{(1-q)(1-x)}=[-1,1],\end{aligned}$$

and for $x=1$, we get

$$\begin{aligned}^{b}D_q f(1)=\lim_{x\to1^-}D_q f(x)&=\lim_{x\to1^-}\frac{f(x)\ominus_g f(qx)}{(1-q)x}\\&=\lim_{x\to1^-}\frac{[-x,x]\ominus_g[-qx,qx]}{(1-q)x}=[-1,1].\end{aligned}$$

Theorem 3. *An IVF $f\colon [a,b]\to\mathbb{R}_I^+$ such that $f(x)=\left[\underline{f}(x),\overline{f}(x)\right]$. Then f is Iq^b-differentiable function if and only if $\underline{f},\overline{f}$ are q^b-differentiable function, and*

$$^{b}D_q f(x)=\left[\min\left\{^{b}D_q\underline{f}(x),{}^{b}D_q\overline{f}(x)\right\},\max\left\{^{b}D_q\underline{f}(x),{}^{b}D_q\overline{f}(x)\right\}\right]. \tag{3.1}$$

Proof. Suppose f is a Iq^b-differentiable function, then there exist $\underline{g}, \overline{g}$ such that

$$^bD_q f(x) = \left[\underline{g}(x),\ \overline{g}(x)\right].$$

According to Definition 5,

$$\underline{g}(x) = \min\left\{\frac{\underline{f}(qx+(1-q)b)-\underline{f}(x)}{(1-q)(b-x)},\ \frac{\overline{f}(qx+(1-q)b)-\overline{f}(x)}{(1-q)(b-x)}\right\}$$

and

$$\overline{g}(x) = \max\left\{\frac{\underline{f}(qx+(1-q)b)-\underline{f}(x)}{(1-q)(b-x)},\ \frac{\overline{f}(qx+(1-q)b)-\overline{f}(x)}{(1-q)(b-x)}\right\}$$

exist. Then $^bD_q\underline{f}(x)$ and $^bD_q\overline{f}(x)$ exist, and (3.1) is feasible.

Conversely, if $\underline{f},\ \overline{f}$ are q^b-differentiable at x and suppose $^bD_q\underline{f}(x) \leq\ ^bD_q\overline{f}(x)$, then

$$\begin{aligned}\left[^bD_q\underline{f}(x),\ ^bD_q\overline{f}(x)\right] &= \left[\frac{\underline{f}(qx+(1-q)b)-\underline{f}(x)}{(1-q)(b-x)},\ \frac{\overline{f}(qx+(1-q)b)-\overline{f}(x)}{(1-q)(b-x)}\right]\\ &= \frac{f(qx+(1-q)b)\ominus_g f(x)}{(1-q)(b-x)} = {}^bD_q f(x).\end{aligned}$$

So, f is a Iq^b-differentiable IVF.

Similarly, If $^bD_q\underline{f}(x) \geq {}^bD_q\overline{f}(x)$, then $^bD_q f(x) = \left[^bD_q\overline{f}(x),\ ^bD_q\underline{f}(x)\right]$. □

To illustrate the nature of the derivatives more clears, we give the following results.

Theorem 4. *Let $f\colon [a,b] \to \mathbb{R}_I^+$. If f is Iq^b-differentiable on $[a,b]$, then we have*

(1) $^bD_q f(x) = \left[^bD_q\underline{f}(x),\ ^bD_q\overline{f}(x)\right]$ for all $x \in [a,b]$ if f is μ-increasing;

(2) $^bD_q f(x) = \left[^bD_q\overline{f}(x),\ ^bD_q\underline{f}(x)\right]$ for all $x \in [a,b]$ if f is μ-decreasing.

Proof. First, we supposed f is μ-increasing and Iq^b-differentiable on $[a,b]$. For all $x \in [a,b]$, we have

$$\begin{aligned}&\left[\overline{f}(qx+(1-q)b)-\underline{f}(qx+(1-q)b)\right]-\left[\overline{f}(x)-\underline{f}(x)\right] > 0,\\ &\qquad\qquad\overline{f}(qx+(1-q)b)-\overline{f}(x) > \underline{f}(qx+(1-q)b)-\underline{f}(x).\end{aligned}$$

Therefore,

$$\begin{aligned}^bD_q f(x) &= \frac{\left[\overline{f}(qx+(1-q)b)-\underline{f}(qx+(1-q)b)\right]\ominus_g\left[\overline{f}(x)-\underline{f}(x)\right]}{(1-q)(b-x)}\\ &= \left[\frac{\underline{f}(qx+(1-q)b)-\underline{f}(x)}{(1-q)(b-x)},\ \frac{\overline{f}(qx+(1-q)b)-\overline{f}(x)}{(1-q)(b-x)}\right]\\ &= \left[^bD_q\underline{f}(x),\ ^bD_q\overline{f}(x)\right].\end{aligned}$$

The other condition can be similarly proved. □

Example 2. Furthermore, by Example 1, we have

$$w(f(x)) = 2x, \quad x \in [0,1].$$

So, f is a μ-increasing IVF.

On the other hand,

$$
\begin{aligned}
{}^bD_q\underline{f}(x) &= \frac{\underline{f}(qx+(1-q)b)-\underline{f}(x)}{(1-q)(b-x)} = \frac{(1-q)(x-1)}{(1-q)(1-x)} = -1,\\
{}^bD_q\overline{f}(x) &= \frac{\overline{f}(qx+(1-q)b)-\overline{f}(x)}{(1-q)(b-x)} = \frac{(1-q)(1-x)}{(1-q)(1-x)} = 1.
\end{aligned}
$$

Then, we obtained

$${}^bD_qf(x) = [-1,1] = [{}^bD_q\underline{f}(x), {}^bD_q\overline{f}(x)].$$

Theorem 5. *Let $f,g\colon [a,b] \to \mathbb{R}_I^+$ be Iq^b-differentiable IVFs. For all $x \in [a,b]$, $q \in (0,1)$, consider*

$$
\begin{aligned}
t_1 &= w(f(qx+(1-q)b)) - w(f(x)),\\
t_2 &= w(g(qx+(1-q)b)) - w(g(x)),\\
t_3 &= w(f(qx+(1-q)b)) - w(g(qx+(1-q)b)),\\
t_4 &= w(f(x)) - w(g(x)).
\end{aligned}
$$

(1) The sum $f+g\colon [a,b] \to \mathbb{R}_I^+$ is a Iq^b-differentiable IVF with

$${}^bD_q(f+g)(x) = \begin{cases} {}^bD_qf(x) + {}^bD_qg(x), & t_1t_2 \geq 0,\\ {}^bD_qf(x) \ominus_g \left(-{}^bD_qg(x)\right), & t_1t_2 < 0. \end{cases}$$

(2) For any constant $\lambda \in \mathbb{R}$, $\lambda f\colon [a,b] \to \mathbb{R}_I^+$ is a Iq^b-differentiable IVF with

$${}^bD_q\,\lambda f(x) = \lambda\, {}^bD_qf(x).$$

(3) The difference $f \ominus_g g\colon [a,b] \to \mathbb{R}_I^+$ is a Iq^b-differentiable IVF with

$${}^bD_q(f \ominus_g g)(x) = \begin{cases} {}^bD_qf(x) \ominus_g {}^bD_qg(x), & t_1t_2 \geq 0,\ t_3t_4 \geq 0,\\ {}^bD_qf(x) + \left(-{}^bD_qg(x)\right), & t_1t_2 < 0,\ t_3t_4 \geq 0. \end{cases}$$

Proof.

(1) By Definition 5, we have

$$
\begin{aligned}
{}^bD_q(f+g)(x) &= \frac{(f+g)(qx+(1-q)b) \ominus_g (f+g)(x)}{(1-q)(b-x)}\\
&= \frac{(f(qx+(1-q)b)+g(qx+(1-q)b)) \ominus_g (f(x)+g(x))}{(1-q)(b-x)}
\end{aligned}
$$

$$= \begin{cases} \frac{(f(qx+(1-q)b)\ominus_g f(x))+(g(qx+(1-q)b)\ominus_g g(x))}{(1-q)(b-x)}, & t_1t_2 \geq 0, \\ \\ \frac{(f(qx+(1-q)b)\ominus_g f(x))\ominus_g(-g(qx+(1-q)b)\ominus_g g(x))}{(1-q)(b-x)}, & t_1t_2 < 0, \end{cases}$$

$$= \begin{cases} \frac{(f(qx+(1-q)b)\ominus_g f(x))}{(1-q)(b-x)} + \frac{(g(qx+(1-q)b)\ominus_g g(x))}{(1-q)(b-x)}, & t_1t_2 \geq 0, \\ \\ \frac{(f(qx+(1-q)b)\ominus_g f(x))}{(1-q)(b-x)} \ominus_g \frac{(-g(qx+(1-q)b)\ominus_g g(x))}{(1-q)(b-x)}, & t_1t_2 < 0, \end{cases}$$

$$= \begin{cases} {}^bD_q f(x) + {}^b D_q g(x), & t_1t_2 \geq 0, \\ {}^bD_q f(x) \ominus_g \left(-{}^bD_q g(x)\right), & t_1t_2 < 0. \end{cases}$$

(2) By Definition 5, we have

$$\begin{aligned} {}^bD_q (\lambda f)(x) &= \frac{(\lambda f)(qx+(1-q)b) \ominus_g (\lambda f)(x)}{(1-q)(b-x)} \\ &= \lambda \, \frac{f(qx+(1-q)b) \ominus_g f(x)}{(1-q)(b-x)} \\ &= \lambda \, {}^bD_q f(x). \end{aligned}$$

(3) By Definition 5, we have

$$\begin{aligned} {}^bD_q (f \ominus_g g)(x) &= \frac{(f \ominus_g g)(qx+(1-q)b) \ominus_g (f \ominus_g g)(x)}{(1-q)(b-x)} \\ &= \begin{cases} \frac{(f(qx+(1-q)b)\ominus_g f(x))\ominus_g(g(qx+(1-q)b)\ominus_g g(x))}{(1-q)(b-x)}, & t_1t_2 \geq 0,\ t_3t_4 \geq 0, \\ \\ \frac{(f(qx+(1-q)b)\ominus_g f(x))+(-g(qx+(1-q)b)\ominus_g g(x))}{(1-q)(b-x)}, & t_1t_2 < 0,\ t_3t_4 \geq 0, \end{cases} \\ &= \begin{cases} {}^bD_q f(x) \ominus_g^b D_q g(x), & t_1t_2 \geq 0,\ t_3t_4 \geq 0, \\ {}^bD_q f(x) + \left(-{}^bD_q g(x)\right), & t_1t_2 < 0,\ t_3t_4 \geq 0. \end{cases} \end{aligned}$$

□

4. Iq^b-INTEGRAL FOR INTERVAL-VALUED FUNCTIONS

In this section, we present the concepts of Iq^b-integral for IVFs and give some properties.

Definition 6 (Iq^b-integral)**.** Let $f\colon [a,b] \to \mathbb{R}_I$ be a continuous IVF, the definite integral Iq^b-integral of f on $[a,b]$ is given by

$${}^b I_q f(x) = \int_x^b f(t) \, {}^b d_q^I t = (1-q)(b-x) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b).$$

Theorem 6. *Let $f,\ g\colon [a,b] \to \mathbb{R}_I$ are continuous IVFs and $\lambda \in \mathbb{R}$, we have the following properties:*

(1) ${}^{b}I_q(f+g)(x) = {}^{b}I_q f(x) + {}^{b}I_q g(x)$,

(2) ${}^{b}I_q(\lambda f)(x) = \lambda\, {}^{b}I_q f(x)$.

Proof. By Definition 6, we have

$$\begin{aligned}
{}^{b}I_q(f+g)(x) &= (1-q)(b-x)\sum_{n=0}^{\infty} q^n (f+g)(q^n x + (1-q^n)b) \\
&= (1-q)(b-x)\sum_{n=0}^{\infty} q^n \left(f(q^n x + (1-q^n)b) + g(q^n x + (1-q^n)b) \right) \\
&= (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b) \\
&\quad + (1-q)(b-x)\sum_{n=0}^{\infty} q^n g(q^n x + (1-q^n)b) \\
&= {}^{b}I_q f(x) + {}^{b}I_q g(x),
\end{aligned}$$

and

$$\begin{aligned}
{}^{b}I_q(\lambda f)(x) &= (1-q)(b-x)\sum_{n=0}^{\infty} q^n (\lambda f)(q^n x + (1-q^n)b) \\
&= \lambda\,(1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b) = \lambda\, {}^{b}I_q f(x).
\end{aligned}$$

□

Theorem 7. *Let $f\colon [a,b] \to \mathbb{R}_I$ be a continuous IVF, then f is Iq^b-integral on $[a,b]$ if and only if $\underline{f}$ and $\overline{f}$ is q^b-integral on $[a,b]$, and*

$$ {}^{b}I_q f(x) = \left[{}^{b}I_q \underline{f}(x),\ {}^{b}I_q \overline{f}(x) \right]. $$

Proof. The proof can be obtained by combining Definitions 4 and 6 and hence is omitted. □

Example 3. Let $f\colon [0,1] \to \mathbb{R}_I$ be given by

$$ f(x) = \left[x^2,\ x\right]. $$

For $0 < q < 1$, we have

$$\begin{aligned}
\int_0^1 f(x)\ {}^{1}d_q^I x &= \left[\int_0^1 x^2\ {}^{1}d_q^I x,\ \int_0^1 x\ {}^{1}d_q^I x \right] \\
&= \left[(1-q)\sum_{n=0}^{\infty} q^n (1-q^n)^2,\ (1-q)\sum_{n=0}^{\infty} q^n (1-q^n) \right] = \left[\frac{q(1+q^2)}{[3]_q},\ \frac{q}{[2]_q} \right].
\end{aligned}$$

Theorem 8. *Let $f, g\colon [a,b] \to \mathbb{R}_I$ are continuous IVFs, then*

$$\int_a^b f(x)\ {}^b d_q^I x \ominus_g \int_a^b g(x)\ {}^b d_q^I x \subseteq \int_a^b (f(x) \ominus_g g(x))\ {}^b d_q^I x.$$

Moreover, if $w(f(x)) - w(g(x))$ has a constant sign on $[a,b]$, then

$$\int_a^b f(x)\ {}^b d_q^I x \ominus_g \int_a^b g(x)\ {}^b d_q^I x = \int_a^b (f(x) \ominus_g g(x))\ {}^b d_q^I x.$$

Proof. First, we have

$$\begin{aligned}
&\int_a^b \min\left[\underline{f}(x) - \underline{g}(x),\ \overline{f}(x) - \overline{g}(x)\right]\ {}^b d_q^I x \\
&\leq \min\left\{\int_a^b \left(\underline{f}(x) - \underline{g}(x)\right)\ {}^b d_q^I x,\ \int_a^b \left(\overline{f}(x) - \overline{g}(x)\right)\ {}^b d_q^I x\right\} \\
&\leq \max\left\{\int_a^b \left(\underline{f}(x) - \underline{g}(x)\right)\ {}^b d_q^I x,\ \int_a^b \left(\overline{f}(x) - \overline{g}(x)\right)\ {}^b d_q^I x\right\} \\
&\leq \int_a^b \max\left[\underline{f}(x) - \underline{g}(x),\ \overline{f}(x) - \overline{g}(x)\right]\ {}^b d_q^I x.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_a^b f(x)\ {}^b d_q^I x \ominus_g \int_a^b g(x)\ {}^b d_q^I x \\
&= \left[\min\left\{\int_a^b \left(\underline{f}(x) - \underline{g}(x)\right)\ {}^b d_q^I x,\ \int_a^b \left(\overline{f}(x) - \overline{g}(x)\right)\ {}^b d_q^I x\right\},\right. \\
&\qquad \left.\max\left\{\int_a^b \left(\underline{f}(x) - \underline{g}(x)\right)\ {}^b d_q^I x,\ \int_a^b \left(\overline{f}(x) - \overline{g}(x)\right)\ {}^b d_q^I x\right\}\right] \\
&\subseteq \left[\int_a^b \min\left\{\underline{f}(x) - \underline{g}(x), \overline{f}(x) - \overline{g}(x)\right\}\ {}^b d_q^I x,\right. \\
&\qquad \left.\int_a^b \max\left\{\underline{f}(x) - \underline{g}(x), \overline{f}(x) - \overline{g}(x)\right\}\ {}^b d_q^I x\right] \\
&= \int_a^b (f(x) \ominus_g g(x))\ {}^b d_q^I x.
\end{aligned}$$

Moreover, we assume that $w(f(x)) - w(g(x)) \geq 0,\ x \in [a,b]$, then

$$f(x) \ominus_g g(x) = \left[\underline{f}(x) - \underline{g}(x), \overline{f}(x) - \overline{g}(x)\right].$$

So, we have

$$\int_a^b \left(\underline{f}(x) - \underline{g}(x)\right)\ {}^b d_q^I x \leq \int_a^b \left(\overline{f}(x) - \overline{g}(x)\right)\ {}^b d_q^I x.$$

This implies that

$$\int_a^b f(x)\ {}^b d_q^I x \ominus_g \int_a^b g(x)\ {}^b d_q^I x$$

$$
\begin{aligned}
&= \left[\min\left\{\int_a^b \left(\underline{f}(x)-\underline{g}(x)\right)\ {}^b d_q^I x,\ \int_a^b \left(\overline{f}(x)-\overline{g}(x)\right)\ {}^b d_q^I x\right\},\right. \\
&\qquad \left.\max\left\{\int_a^b \left(\underline{f}(x)-\underline{g}(x)\right)\ {}^b d_q^I x,\ \int_a^b \left(\overline{f}(x)-\overline{g}(x)\right)\ {}^b d_q^I x\right\}\right] \\
&= \left[\int_a^b \underline{f}(x)\ {}^b d_q^I x, \int_a^b \overline{f}(x)\ {}^b d_q^I x\right] \ominus_g \left[\int_a^b \underline{g}(x)\ {}^b d_q^I x, \int_a^b \overline{g}(x)\ {}^b d_q^I x\right] \\
&= \int_a^b \left(f(x)\ominus_g g(x)\right)\ {}^b d_q^I x.
\end{aligned}
$$

□

Example 4. Let $f,g,h\colon [0,1]\to\mathbb{R}_I$ are given by

$$f(x)=[-x,x],\ g(x)=[-x^2,x^2],\ h(x)=[-2x^2,2x^2].$$

We have $w(f(x))-w(g(x))=2x(1-x)\ge 0,\ x\in[0,1]$ and

$$
\begin{aligned}
f(x)\ominus_g g(x) &= [-x+x^2, x-x^2], \\
f(x)\ominus_g h(x) &= \begin{cases} \left[2x^2-x, x-2x^2\right], & x\in\left[0,\frac{1}{2}\right], \\ \left[x-2x^2, 2x^2-x\right], & x\in\left[\frac{1}{2},1\right]. \end{cases}
\end{aligned}
$$

Then, we obtained that

$$
\begin{aligned}
&\int_0^1 f(x)\ {}^b d_q^I x \ominus_g \int_0^1 g(x)\ {}^b d_q^I x = \left[-\frac{q^2}{[2]_q[3]_q}, \frac{q^2}{[2]_q[3]_q}\right], \\
&\int_0^1 f(x)\ {}^b d_q^I x \ominus_g \int_0^1 h(x)\ {}^b d_q^I x = \left[-\frac{q(1-q+q^2)}{[2]_q[3]_q}, \frac{q(1-q+q^2)}{[2]_q[3]_q}\right], \\
&\int_0^1 \left(f(x)\ominus_g g(x)\right)\ {}^b d_q^I x = \left[-\frac{q^2}{[2]_q[3]_q}, \frac{q^2}{[2]_q[3]_q}\right], \\
&\int_0^1 \left(f(x)\ominus_g h(x)\right)\ {}^b d_q^I x = \frac{1}{2}\left[-\frac{q}{[2]_q}, \frac{q}{[2]_q}\right].
\end{aligned}
$$

Hence,

$$\int_0^1 f(x)\ {}^b d_q^I x \ominus_g \int_0^1 g(x)\ {}^b d_q^I x = \int_0^1 \left(f(x)\ominus_g g(x)\right)\ {}^b d_q^I x$$

and

$$\int_0^1 f(x)\ {}^b d_q^I x \ominus_g \int_0^1 h(x)\ {}^b d_q^I x \subseteq \int_0^1 \left(f(x)\ominus_g h(x)\right)\ {}^b d_q^I x.$$

Theorem 9. *Let $f\colon [a,b]\to\mathbb{R}_I$. If f is Iq^b-differentiable on $[a,b]$, then ${}^bD_q f(x)$ is Iq^b-integrable. Moreover, if f is μ-monotone on $[a,b]$, then*

$$f(c)\ominus_g f(x) = \int_x^c {}^bD_q f(s)\ {}^b d_q^I s \quad \textit{for all } s\in[x,b]. \tag{4.1}$$

Proof. If f is Iq^b-integrable on $[a,b]$, then from Theorem 7 it follows that $\underline{f}$ and $\overline{f}$ are q^b-integrable. Hence, ${}^bD_q\underline{f}(x)$ and ${}^bD_q\overline{f}(x)$ are q^b-integrable. Therefore, Theorem 3 imply that ${}^bD_qf(x)$ is Iq^b-integrable.

If f is μ-increasing on $[a,b]$, then

$$ {}^bD_qf(x) = \left[{}^bD_q\underline{f}(x),\ {}^bD_q\overline{f}(x)\right], $$

for all $x \in [a,b]$. Then we have that

$$ \underline{f}(c) - \underline{f}(x) = \int_x^c {}^bD_q\underline{f}(s)\ {}^bd_qs, \qquad \overline{f}(c) - \overline{f}(x) = \int_x^c {}^bD_q\overline{f}(s)\ {}^bd_qs. $$

It follows that

$$ f(c) = f(x) + \int_x^c {}^bD_qf(s)\ {}^bd_q^Is. $$

Since f is μ-increasing on $[a,b]$, we have

$$ f(c) \ominus_g f(x) = \int_x^c {}^bD_qf(s)\ {}^bd_q^Is. $$

If f is μ-decreasing on $[a,b]$, then

$$ {}^bD_qf(x) = \left[{}^bD_q\overline{f}(x),\ {}^bD_q\underline{f}(x)\right], $$

for all $x \in [a,b]$. Then we get that

$$ \begin{aligned} \int_x^c {}^bD_qf(s)\ {}^bd_q^Is &= \left[\int_x^c {}^bD_q\overline{f}(s)\ {}^bd_qs,\ \int_x^c {}^bD_q\underline{f}(s)\ {}^bd_qs\right] \\ &= \left[\underline{f}(c) - \underline{f}(x),\ \overline{f}(c) - \overline{f}(x)\right] \\ &= \left[\underline{f}(c),\ \overline{f}(c)\right] \ominus_g \left[\underline{f}(x),\ \overline{f}(x)\right] = f(c) \ominus_g f(x). \end{aligned} $$

□

Remark 1. If f is μ-increasing on $[a,b]$, then (4.1) is equivalent with

$$ f(c) = f(x) + \int_x^c {}^bD_qf(s)\ {}^bd_q^Is, $$

and if f is μ-decreasing on $[a,b]$, then (4.1) is equivalent with

$$ f(x) = f(c) + \left(-\int_x^c {}^bD_qf(s)\ {}^bd_q^Is\right). $$

5. Iq^b-HERMITE-HADAMARD INEQUALITIES

Now we review the definition of interval convex function.

Definition 7. [20, Definition 5] Let $f\colon [a,b] \to \mathbb{R}_I^+$ be an IVF. We say that f is interval convex function or that $f \in SX([a,b],\mathbb{R}_I^+)$, if for all $x,y \in [a,b]$ and $t \in [0,1]$, we have

$$f(tx+(1-t)y) \supseteq tf(x)+(1-t)f(y).$$

Next, we prove the Hermite-Hadamard type inequalities for Iq^b-integrable IVFs.

Theorem 10. *Let $f\colon [a,b] \to \mathbb{R}_I^+$ be a gH-differentiable convex IVF over $[a,b]$, then the following inequalities hold for Iq^b-integral:*

$$f\left(\frac{a+qb}{[2]_q}\right) \supseteq \frac{1}{b-a}\int_a^b f(x)\ {}^b d_q^I x \supseteq \frac{f(a)+qf(b)}{[2]_q}, \tag{5.1}$$

where $q \in (0,1)$.

Proof. We observe that

$$\frac{a+qb}{[2]_q} = \sum_{n=0}^{\infty}(1-q)q^n\left(q^n a+(1-q^n)b\right),$$

where $\sum\limits_{n=0}^{\infty}(1-q)q^n = 1$. Thus, by the convexity of f, Jensen's inequality implies

$$f\left(\frac{a+qb}{[2]_q}\right) \supseteq \sum_{n=0}^{\infty}(1-q)q^n f\left(q^n a+(1-q^n)b\right) = \frac{1}{b-a}\int_a^b f(x)\ {}^b d_q^I x$$

and the first inequality in (5.1) is proved.

Now, for the second inequality in (5.1), we suppose that

$$s(x) = f(b)+\frac{f(b)-f(a)}{b-a}(x-b) = \frac{f(b)-f(a)}{b-a}x+\frac{bf(a)-af(b)}{b-a}$$

and for the convexity of f, we have

$$f(x) \supseteq s(x).$$

Hence by using Iq^b-integration, we have

$$\int_a^b f(x)\ {}^b d_q^I x \supseteq \int_a^b s(x)\ {}^b d_q^I x = (b-a)\left[\frac{f(a)+qf(b)}{[2]_q}\right]$$

and the second inequality in (5.1) is also proved. The proof is completed. □

Lemma 1. *If $f(x) = \left[\underline{f}(x),\overline{f}(x)\right]$ and $\underline{f}(x) = \overline{f}(x)$ for all $x \in [a,b]$, then we get the (2.1). If in (5.1), we get the classical Hermite-Hadamard inequality.*

Example 5. Let $f\colon [0,1] \to \mathbb{R}_I^+$ is given by $f(x) = [x^2, x]$, then f is interval convex function, and

$$f\left(\frac{a+qb}{[2]_q}\right) = \left[\frac{q^2}{[2]_q^2}, \frac{q}{[2]_q}\right],$$
$$\frac{1}{b-a}\int_a^b f(x)\ {}^b d_q^I x = \left[\frac{q(1+q^2)}{[2]_q[3]_q}, \frac{q}{[2]_q}\right],$$
$$\frac{f(a)+qf(b)}{[2]_q} = \left[\frac{q}{[2]_q}, \frac{q}{[2]_q}\right].$$

Then, we obtain that

$$\left[\frac{q^2}{[2]_q^2}, \frac{q}{[2]_q}\right] \supseteq \left[\frac{q(1+q^2)}{[2]_q[3]_q}, \frac{q}{[2]_q}\right] \supseteq \left[\frac{q}{[2]_q}, \frac{q}{[2]_q}\right].$$

Consequently, Theorem 10 is verified.

Summing up the results in Theorem 10 and Theorem 5.3 of [11] yields the next corollary.

Corollary 1. *Let $f\colon [a,b] \to \mathbb{R}_I^+$ be a gH-differentiable convex IVF over $[a,b]$, then we have*

$$\begin{aligned} f\left(\frac{qa+b}{[2]_q}\right) + f\left(\frac{a+qb}{[2]_q}\right) &\supseteq \frac{1}{b-a}\int_a^b f(x)\ {}^b d_q^I x + \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q^I x \\ &\supseteq f(a)+f(b) \end{aligned} \tag{5.2}$$

where $q \in (0,1)$.

Corollary 2. *Let $f\colon [a,b] \to \mathbb{R}_I^+$ be a gH-differentiable convex IVF over $[a,b]$, then we have*

$$2f\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\left\{\int_a^b f(x)\ {}^b d_q^I x + \int_a^b f(x)\ {}_a d_q^I x\right\} \supseteq f(a)+f(b), \tag{5.3}$$

where $q \in (0,1)$.

Proof. By Corollary 1, it is enough to see that by the convexity of f,

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}\frac{qa+b}{[2]_q} + \frac{1}{2}\frac{a+qb}{[2]_q}\right) \supseteq \frac{1}{2}f\left(\frac{qa+b}{[2]_q}\right) + \frac{1}{2}f\left(\frac{aq+b}{[2]_q}\right).$$

□

Remark 2. If in (5.2) or (5.3), we make $q \to 1$, we get the classical Hermite-Hadamard inequality.

Theorem 11. *Let $f\colon [a,b] \to \mathbb{R}_I^+$ be a gH-differentiable convex IVF over $[a,b]$, then the following inequalities hold for Iq^b-integral:*

$$f\left(\frac{qa+b}{[2]_q}\right)+\frac{(1-q)(b-a)}{[2]_q}\ {}^bD_qf\left(\frac{qa+b}{[2]_q}\right) \supseteq \frac{1}{b-a}\int_a^b f(x)\ {}^bd_q^I x \supseteq \frac{f(a)+qf(b)}{[2]_q},$$

where $q \in (0,1)$.

Proof. According to the Iq^b-differentiability of f on $[a,b]$, there are two tangents at the point $\frac{qa+b}{1+q} \in (a,b)$, and their equations are

$$\underline{h}_2(x)=\underline{f}\left(\frac{qa+b}{[2]_q}\right)+\frac{(1-q)(b-a)}{1+q}\ {}^bD_q\underline{f}\left(\frac{qa+b}{[2]_q}\right)$$

and

$$\overline{h}_2(x)=\overline{f}\left(\frac{qa+b}{[2]_q}\right)+\frac{(1-q)(b-a)}{1+q}\ {}^bD_q\overline{f}\left(\frac{qa+b}{[2]_q}\right).$$

Since $f \in SX([a,b],\mathbb{R}_I)$, we have

$$h_2(x) \supseteq f(x),$$

for all $x \in [a,b]$. By Iq^b-integrating this inequality with respect to x on $[a,b]$ we have

$$\begin{aligned}
&\int_a^b h_2(x)\ {}^bd_q^I x\\
&= \int_a^b \left[f\left(\frac{qa+b}{[2]_q}\right)+\ {}^bD_qf\left(\frac{qa+b}{[2]_q}\right)\left(x-\frac{qa+b}{[2]_q}\right)\right] {}^bd_q^I x\\
&= (b-a)f\left(\frac{qa+b}{[2]_q}\right)+\ {}^bD_qf\left(\frac{qa+b}{[2]_q}\right)\left(\int_a^b x\ \ {}^bd_q^I x-(b-a)\frac{qa+b}{[2]_q}\right)\\
&= (b-a)f\left(\frac{qa+b}{[2]_q}\right)+\ {}^bD_qf\left(\frac{qa+b}{[2]_q}\right)\\
&\quad\times\left((1-q)(b-a)\sum_{n=0}^{\infty} q^n\left(q^n a+(1-q^n)b\right)-(b-a)\frac{qa+b}{[2]_q}\right)\\
&= (b-a)f\left(\frac{qa+b}{[2]_q}\right)+\ {}^bD_qf\left(\frac{qa+b}{[2]_q}\right)\\
&\quad\times\left((1-q)(b-a)\left[\left(\frac{1}{1-q}-\frac{1}{1-q^2}\right)a+\frac{1}{1-q^2}b\right]-(b-a)\frac{a+qb}{[2]_q}\right)\\
&= (b-a)f\left(\frac{qa+b}{[2]_q}\right)+\ {}^bD_qf\left(\frac{qa+b}{[2]_q}\right)\left((b-a)\frac{a+qb}{[2]_q}-(b-a)\frac{qa+b}{[2]_q}\right)
\end{aligned}$$

$$= (b-a)f\left(\frac{a+qb}{[2]_q}\right) + {}^bD_qf\left(\frac{qa+b}{[2]_q}\right)\frac{(b-a)^2(1-q)}{1+q}$$
$$\supseteq \int_a^b f(x)\ {}^bd_q^Ix.$$

Combining the above formula with (5), we come to the conclusion. □

Conclusions

We introduced the concept of Iq^b-calculus of interval-valued functions and studied their important properties. Furthermore, we gave Hermite-Hadamard-type inequalities by using these results. Our results generalized some existing theories of quantum calculus. Next, we intend to further study some applications of quantum calculus and quantum calculus of fuzzy interval-valued functions.

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