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PSEUDO CORE INVERTIBILITY AND DMP INVERTIBILITY IN TWO SEMIGROUPS OF A RING WITH INVOLUTION

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Abstract. In 2004, Patrício and Puystjens characterized the relation between Drazin invertible elements (resp., Moore-Penrose invertible elements) of two semigroups pRp and $pRp+1-p$ of a ring R for some idempotent (resp., projection) $p \in R$. In this paper, we consider the relevant result for pseudo core invertible elements of such two semigroups of a ring for some projection, which is then applied to characterize the relation between pseudo core invertible elements of the matrix semigroup $AA^\dagger R^{m\times m}AA^\dagger + I_m - AA^\dagger$ and the matrix semigroup $A^\dagger AR^{n\times n}A^\dagger A + I_n - A^\dagger A$, where $A \in R^{m\times n}$ with $A^\dagger$ existing. Also, similar equivalence involving DMP invertible elements is investigated.

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1. Introduction

Moore-Penrose inverses [22] and Drazin inverses [5] are well-known classical generalized inverses, which play important roles in many fields. Some characterizations of Moore-Penrose invertibility and Drazin invertibility are given, such as [13, 19–21, 23]. Let R be a ring with identity 1 and $R^{m\times n}$ the set of $m \times n$ matrices over R. In [21], Patrício and Puystjens first characterized the relation between Drazin invertible elements of two semigroups pRp and $pRp+1-p$ of R when p is an idempotent. Specifically, for any $x \in R$, they proved that $pxp+1-p$ is Drazin invertible in R if and only if pxp is Drazin invertible in pRp, in which case, $(pxp)^d = p(pxp+1-p)^d p \in pRp$ and $(pxp+1-p)^d = (pxp)^d + 1 - p \in pRp+1-p$. Similar equivalence appeared in the characterization of Moore-Penrose invertibility when R is a ring with an involution and p is a projection. Using the previous result, they related the Drazin invertible elements between the semigroup $AA^- R^{m\times m}AA^- + I_m - AA^-$ and the semigroup $A^= AR^{n\times n}A^= A + I_n - A^= A$, where A^- and $A^=$ are inner inverses of A, and also related the Moore-Penrose invertible elements between the semigroup $AA^\dagger R^{m\times m}AA^\dagger + I_m - AA^\dagger$ and the semigroup $A^\dagger AR^{n\times n}A^\dagger A + I_n - A^\dagger A$ when $A^\dagger$ exists.

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In addition to Drazin inverses and Moore-Penrose inverses, some new generalized inverses, such as core inverses, pseudo core inverses and DMP inverses, have also been introduced and deeply studied in recent years. The concept of the core inverse of a complex matrix was defined by Baksalary and Trenkler [1] in 2010, which was generalized to a ring with involution by Rakić et al. [24] later. As two extensions of the core inverse, in 2014, Manjunatha Prasad and Mohana [17], Malik and Thome [16] introduced, respectively, namely the core-EP inverse and DMP inverse which both exist for arbitrary square complex matrices. Later, Gao and Chen [7] defined the pseudo core inverse by three equations in a ring with involution, extending the core-EP inverse of a complex matrix. Additionally, Gao et al. [8] discussed existence criteria and formulae of the pseudo core inverse of a companion matrix over a ring with involution. Zhou and Chen [27] showed some characterizations of pseudo core inverses and the relation with other generalized inverses. For more details of pseudo core inverses and DMP inverses, for example, see [18, 28]. Noted that some generalized inverse matrices can provide a method to define pre-orders or partial orders and to analyze binary relations, for example, see [3,9,11,12,15]. Except that, many scholars have studied projections related to generalized inverses and their generalizations, such as [4, 14, 25].

Motivated by the above discussion, it is natural to ask whether similar equivalences hold for pseudo core invertibility and DMP invertibility. The article is organized as follows. In Section 2, we present some necessary definitions mentioned above and lemmas. In Section 3, we characterize the relation between pseudo core invertible elements (resp., DMP invertible elements) of two semigroups pRp and $pRp+1-p$ of R, where R is equipped with an involution and p is a projection. As an application, in Section 4, the relation between pseudo core invertible elements (resp., DMP invertible elements) of the semigroup $AA^{\dagger}R^{m\times m}AA^{\dagger}+I_m-AA^{\dagger}$ and the semigroup $A^{\dagger}AR^{n\times n}A^{\dagger}A+I_n-A^{\dagger}A$ is characterized, assuming that $A\in R^{m\times n}$ with $A^{\dagger}$ existing.

2. PRELIMINARIES

Let R be a ring with involution $*$ and have identity 1, where an involution $*: a\mapsto a^*$ is an anti-isomorphism satisfying $(a^*)^*=a$, $(a+b)^*=a^*+b^*$ and $(ab)^*=b^*a^*$ for all $a,b\in R$.

Definition 1 ([10, 22])**.** An element $a\in R$ is said to be Moore-Penrose invertible if there exists $x\in R$ such that

$$(1)\ axa=a,\quad (2)\ xax=x,\quad (3)\ (ax)^*=ax,\quad (4)\ (xa)^*=xa.$$

Such x is called the Moore-Penrose inverse of a and is unique if it exists, denoted by $a^{\dagger}$. If $axa=a$, then a is called regular and x is called an inner inverse of a. We use a^- to denote an inner inverse of a. If the equations (1) and (3) hold, then x is called a $\{1,3\}$-inverse of a. We use $a^{(1,3)}$ to denote a $\{1,3\}$-inverse of a. The set of all $\{1,3\}$-invertible elements in R will be denoted by $R^{\{1,3\}}$.

Definition 2 ([5])**.** Let $a \in R$. Then a is said to be Drazin invertible if there exist $x \in R$ and a positive integer k such that

$$xa^{k+1} = a^k,\ ax^2 = x,\ ax = xa.$$

Such x is unique if it exists and called the Drazin inverse of a, denoted by a^d. If k is the smallest positive integer such that the above equations hold, then k is called the Drazin index of a and denoted by $\mathrm{ind}(a)$. When $\mathrm{ind}(a) = 1$, the Drazin inverse of a is called the group inverse of a, denoted by $a^{\#}$.

Definition 3 ([7])**.** Let $a \in R$. If there exist $x \in R$ and a positive integer k such that

$$xa^{k+1} = a^k,\ ax^2 = x,\ (ax)^* = ax,$$

then x is called the pseudo core inverse of a. It is unique and denoted by $a^{\textcircled{D}}$ when the pseudo core inverse of a exists. If a is pseudo core invertible, then it must be Drazin invertible. The smallest positive integer k satisfying the above equations is called the pseudo core index of a, which coincides with its Drazin index, and still denoted by $\mathrm{ind}(a)$. When $\mathrm{ind}(a) = 1$, the pseudo core inverse of a reduces to its core inverse $a^{\textcircled{\#}}$.

Lemma 1 ([26])**.** *Let $a \in R$. Then $x = a^{\textcircled{\#}}$ is the core inverse of a if and only if there exists $x \in R$ such that*

$$xa^2 = a,\ ax^2 = x,\ (ax)^* = ax.$$

Let $a \in R$. Gao and Chen [7] proved that a is pseudo core invertible if and only if a^n is core invertible for some positive integer n. Here we give the further result.

Lemma 2. *Let $a \in R$. Then a is pseudo core invertible with $\mathrm{ind}(a) = n$ if and only if n is the smallest positive integer such that a^n is core invertible. In this case, $a^{\textcircled{D}} = a^{n-1}(a^n)^{\textcircled{\#}}$ and $(a^n)^{\textcircled{\#}} = (a^{\textcircled{D}})^n$.*

Proof. Suppose that n is the smallest positive integer such that a^n is core invertible. Then we obtain

$$(a^n)^{\textcircled{\#}}(a^n)^2 = a^n,\ \ a^n\left((a^n)^{\textcircled{\#}}\right)^2 = (a^n)^{\textcircled{\#}}, \left(a^n(a^n)^{\textcircled{\#}}\right)^* = a^n(a^n)^{\textcircled{\#}}.$$

According to the proof of [7, Theorem 2.5], we get that $a^{\textcircled{D}} = a^{n-1}(a^n)^{\textcircled{\#}}$ with $\mathrm{ind}(a) \le n$. Assume $\mathrm{ind}(a) = k < n$, then it is easy to obtain that a^k is core invertible, a contradiction. Hence, $\mathrm{ind}(a) = n$.

Conversely, following the proof of [7, Theorem 2.5], we know that a^n is core invertible with $(a^n)^{\textcircled{\#}} = (a^{\textcircled{D}})^n$. We suppose that there exists a positive integer $k < n$ such that a^k is core invertible. Then according to the sufficiency of the proof, it follows that $k \ge \mathrm{ind}(a) = n$, a contradiction. Therefore, n is the smallest positive integer such that a^n is core invertible. $\square$

Let $R^\dagger$ and R^d denote the sets of all Moore-Penrose invertible elements and Drazin invertible elements in R, respectively.

Definition 4 ([16])**.** Let $a \in R^d \cap R^\dagger$. The DMP inverse of a, denoted by $a^{d,\dagger}$, is the unique solution $x \in R$ of the system

$$xax = x,\ xa = a^d a,\ a^k x = a^k a^{\dagger},$$

where $k = \mathrm{ind}(a)$. In fact, it can be proved that $a^{d,\dagger} = a^d a a^{\dagger}$.

3. Pseudo core invertibility and DMP invertibility in a corner ring

The following lemmas will be useful in the sequel.

Lemma 3 ([2]). *Let $a, b \in R$ with $ab = ba$ and $a^*b = ba^*$. If $a \in R^{\#\!\!\!\bigcirc}$, then $a^{\#\!\!\!\bigcirc} b = b a^{\#\!\!\!\bigcirc}$.*

Lemma 4 ([26]). *Let $a, b \in R$ be core invertible. If $ab = ba = 0$ and $a^*b = 0$, then $a + b$ is core invertible, and $(a+b)^{\#\!\!\!\bigcirc} = a^{\#\!\!\!\bigcirc} + b^{\#\!\!\!\bigcirc}$.*

Lemma 5 ([7]). *Let $a, b \in R$ be pseudo core invertible. If $ab = ba = 0$ and $a^*b = 0$, then $a + b$ is pseudo core invertible, and $(a+b)^{\textcircled{\scriptsize D}} = a^{\textcircled{\scriptsize D}} + b^{\textcircled{\scriptsize D}}$.*

Recall that an element $p \in R$ is a projection if $p^2 = p = p^*$. Let $p \in R$ be a projection. Then $pRp + 1 - p = \{pxp + 1 - p : x \in R\}$ is a (multiplicative) semigroup. The subrings of the form pRp are called corner rings. It should be remarked that for pseudo core inverses (resp., core inverses) and DMP inverses we still keep the usual notation as $(pxp)^{\textcircled{\scriptsize D}}$ (resp., $(pxp)^{\#\!\!\!\bigcirc}$) and $(pxp)^{d,\dagger}$ both belong to pRp if they exist in R. Next, we illustrate the relation between pseudo core invertibility (resp., core invertibility) of the corresponding elements in two semigroups pRp and $pRp + 1 - p$ of R. The next two theorems will play an important role in the forthcoming section.

Theorem 1. *Let $p \in R$ be a projection and $x \in R$. Then the following statements hold.*

(1) *$pxp + 1 - p$ is core invertible in R if and only if pxp is core invertible in pRp. In this case,*
$$(pxp)^{\#\!\!\!\bigcirc} = p(pxp + 1 - p)^{\#\!\!\!\bigcirc} p \in pRp,$$
and
$$(pxp + 1 - p)^{\#\!\!\!\bigcirc} = (pxp)^{\#\!\!\!\bigcirc} + 1 - p \in pRp + 1 - p.$$

(2) *$pxp + 1 - p$ is pseudo core invertible with $\mathrm{ind}(pxp + 1 - p) = k$ in R if and only if pxp is pseudo core invertible with $\mathrm{ind}(pxp) = k$ in pRp. In this case,*
$$(pxp)^{\textcircled{\scriptsize D}} = p(pxp + 1 - p)^{\textcircled{\scriptsize D}} p \in pRp,$$
and
$$(pxp + 1 - p)^{\textcircled{\scriptsize D}} = (pxp)^{\textcircled{\scriptsize D}} + 1 - p \in pRp + 1 - p.$$

Proof. (1) Suppose that $pxp + 1 - p$ is core invertible. Then
$$\big((pxp + 1 - p)(pxp + 1 - p)^{\#\!\!\!\bigcirc}\big)^* = (pxp + 1 - p)(pxp + 1 - p)^{\#\!\!\!\bigcirc}.$$
Multiplying on both sides by p, we have
$$\big((pxp)p(pxp + 1 - p)^{\#\!\!\!\bigcirc} p\big)^* = (pxp)p(pxp + 1 - p)^{\#\!\!\!\bigcirc} p.$$

Moreover,

$$(pxp+1-p)^{\#}(pxp+1-p)^2 = pxp+1-p.$$

Multiplying the both sides by p, it follows that $p(pxp+1-p)^{\#}p(pxp)^2 = pxp$. Also,

$$(pxp+1-p)\left((pxp+1-p)^{\#}\right)^2 = (pxp+1-p)^{\#}.$$

Multiplying on both sides by p, we get

$$(pxp)p\left((pxp+1-p)^{\#}\right)^2 p = p(pxp+1-p)^{\#}p.$$

Since

$$(pxp+1-p)(1-p) = 1-p = (1-p)(pxp+1-p),$$
$$(pxp+1-p)^*(1-p) = 1-p = (1-p)(pxp+1-p)^*,$$

by Lemma 3, we have

$$(pxp+1-p)^{\#}(1-p) = (1-p)(pxp+1-p)^{\#},$$

that is

$$(pxp+1-p)^{\#}p = p(pxp+1-p)^{\#}.$$

Hence,

$$(pxp)\left(p(pxp+1-p)^{\#}p\right)^2 = p(pxp+1-p)^{\#}p.$$

According to Lemma 1, $(pxp)^{\#} = p(pxp+1-p)^{\#}p \in pRp$.

Conversely, if $(pxp)^{\#}$ is the core inverse of pxp in $pRp \subseteq R$, then by Lemma 4,

$$\begin{aligned}(pxp+1-p)^{\#} &= (pxp)^{\#} + (1-p)^{\#}\\ &= (pxp)^{\#} + 1 - p\end{aligned}$$

since $(pxp)(1-p) = 0 = (1-p)(pxp)$ and $(pxp)^*(1-p) = 0$.

(2) According to Lemma 2, if $pxp+1-p$ is pseudo core invertible with $\mathrm{ind}(pxp+1-p) = k$, then k is the smallest positive integer such that $(pxp+1-p)^k = (pxp)^k + 1 - p = p\left(x(px)^{k-1}\right)p + 1 - p$ is core invertible, and therefore $p\left(x(px)^{k-1}\right)p = (pxp)^k$ is core invertible. We remark that k is also the smallest positive integer such that $(pxp)^k$ is core invertible according to (1). In fact, we suppose that there exists a positive integer $m < k$ such that $(pxp)^m$ is core invertible. It follows that $p\left(x(px)^{m-1}\right)p = (pxp)^m$ is core invertible, then $(pxp+1-p)^m = (pxp)^m + 1 - p = p\left(x(px)^{m-1}\right)p + 1 - p$ is core invertible according to (1), and therefore $m \geq \mathrm{ind}(pxp+1-p) = k$, a contradiction. Hence, pxp is pseudo core invertible with $\mathrm{ind}(pxp) = k$. For the expression of $(pxp)^{\mathbb{D}}$, by Lemma 2 we can obtain that

$$(pxp)^{\mathbb{D}} = (pxp)^{k-1}\left((pxp)^k\right)^{\#} = (pxp)^{k-1}p\left((pxp)^k\right)^{\#}$$

$$
\begin{aligned}
&= \left((pxp)^{k-1} + 1 - p\right) p \left((pxp)^{k} + 1 - p\right)^{\#} p \\
&= p(pxp+1-p)^{k-1}\left((pxp+1-p)^{k}\right)^{\#} p \\
&= p(pxp+1-p)^{\textcircled{D}} p.
\end{aligned}
$$

Conversely, if $(pxp)^{\textcircled{D}}$ is the pseudo core inverse of pxp in $pRp \subseteq R$ and $\mathrm{ind}(pxp) = k$, then by Lemma 5 it follows that $pxp+1-p$ is pseudo core invertible since $(pxp)(1-p) = 0 = (1-p)(pxp)$ and $(pxp)^*(1-p) = 0$, and

$$
\begin{aligned}
(pxp+1-p)^{\textcircled{D}} &= (pxp)^{\textcircled{D}} + (1-p)^{\textcircled{D}} \\
&= (pxp)^{\textcircled{D}} + 1 - p.
\end{aligned}
$$

Moreover, it can be derived from the necessity of the proof that $\mathrm{ind}(pxp+1-p) = k$. $\square$

Similar equivalence involving DMP invertibility is given in the following result.

Theorem 2. *Let $p \in R$ be a projection and $x \in R$. Then $pxp+1-p$ is DMP invertible in R if and only if pxp is DMP invertible in pRp. In this case,*

$$(pxp)^{d,\dagger} = p(pxp+1-p)^{d,\dagger} p \in pRp,$$

and

$$(pxp+1-p)^{d,\dagger} = (pxp)^{d,\dagger} + 1 - p \in pRp + 1 - p.$$

Proof. According to [21, Theorem 1] and Definition 4, it is easy to obtain the sufficiency and necessity of the theorem. For the expressions of $(pxp)^{d,\dagger}$ and $(pxp+1-p)^{d,\dagger}$, since $(pxp+1-p)^d \in pRp+1-p$, $(pxp)^d \in pRp$ and $(pxp)^\dagger \in pRp$, it follows that

$$
\begin{aligned}
(pxp)^{d,\dagger} &= (pxp)^d(pxp)(pxp)^\dagger \\
&= p(pxp+1-p)^d p(pxp)p(pxp+1-p)^\dagger p \\
&= p(pxp+1-p)^d (pxp+1-p)(pxp+1-p)^\dagger p \\
&\quad - p(pxp+1-p)^d(1-p)(pxp+1-p)^\dagger p \\
&= p(pxp+1-p)^{d,\dagger} p,
\end{aligned}
$$

and

$$
\begin{aligned}
(pxp+1-p)^{d,\dagger} &= (pxp+1-p)^d(pxp+1-p)(pxp+1-p)^\dagger \\
&= ((pxp)^d + 1 - p)(pxp+1-p)((pxp)^\dagger + 1 - p) \\
&= (pxp)^d(pxp)(pxp)^\dagger + 1 - p \\
&= (pxp)^{d,\dagger} + 1 - p.
\end{aligned}
$$

$\square$

4. Pseudo core invertibility and DMP invertibility in two matrix semigroups

Let $R^{m\times n}$ denote the set of $m\times n$ matrices over R. And $A^*\in R^{n\times m}$ is defined as (a^*_{ji}) for $A=(a_{ij})\in R^{m\times n}$. Let $P\in R^{m\times m}$ be a projection. In Section 3, we relate pseudo core invertible (resp., DMP invertible) elements between the semigroup $PR^{m\times m}P+I_m-P$ and the corner ring $PR^{m\times m}P$. If $A\in R^{m\times n}$ is Moore-Penrose invertible, then $AA^\dagger$ and $A^\dagger A$ are two projections. As an application, in this section, we will relate pseudo core invertible elements between the semigroup $AA^\dagger R^{m\times m}AA^\dagger+I_m-AA^\dagger$ and the semigroup $A^\dagger AR^{n\times n}A^\dagger A+I_n-A^\dagger A$ using Theorem 1. Also, the DMP invertibility case is investigated.

Let R be a ring with involution ι and S a ring with involution τ. Recall from [21] that $\varphi:R\to S$ is a (ι,τ)-invariant homomorphism if φ is a ring homomorphism and $\varphi(x^\iota)=(\varphi(x))^\tau$ for all $x\in R$. If ι and τ coincide, then it is written as ι-invariant for short. Let $A\in R^{m\times n}$ such that $A^\dagger$ exists. Suppose that $\phi_A: AA^\dagger R^{m\times m}AA^\dagger\to A^\dagger AR^{n\times n}A^\dagger A$ is defined by $\phi_A(AA^\dagger XAA^\dagger)=A^\dagger XA$. Then they called that A is $*$-invariant if ϕ_A is $*$-invariant. Furthermore, they showed that ϕ_A is $*$-invariant if and only if $A^\dagger YA=A^*Y(A^\dagger)^*$ for all $Y\in R^{m\times m}$.

Let $A\in R^{m\times n}$ such that $A^\dagger$ exists and $B\in R^{m\times m}$. Denote $\Gamma=AA^\dagger BAA^\dagger+I_m-AA^\dagger$ and $\Omega=A^\dagger BA+I_n-A^\dagger A$. In [21], Patrício and Puystjens proved that Γ is Moore-Penrose invertible if and only if Ω is Moore-Penrose invertible, under the hypothesis that A is $*$-invariant. Furthermore, they gave examples to explain that the $*$-invariance of A is sufficient but not necessary for this equivalence. Analogously, we will give equivalences for pseudo core inverses and DMP inverses providing that A is $*$-invariant, respectively. Moreover, the examples in [21] can also show that $*$-invariance of A is sufficient but not necessary for the pseudo core inverse case. For completeness of the conclusion, we employ examples again as follows.

Let $a\in R$. We first recall that a is $\{1,3\}$-invertible if and only if $a\in Ra^*a$ [10]. Also, an element a is core invertible if and only if $a\in R^\#\cap R^{\{1,3\}}$ [26].

Let $R=\mathbb{C}^{2\times 2}$ be the set of 2×2 matrices over the complex field $\mathbb{C}$.

Example 1. Take $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $B=\begin{bmatrix}1&0\\i&0\end{bmatrix}\in\mathbb{C}^{2\times 2}$ and transposition as the involution. Then $A^\dagger=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$. By calculation,

$$\Gamma=\begin{bmatrix}1&0\\i&0\end{bmatrix},\ \Omega=\begin{bmatrix}1&0\\-1+i&0\end{bmatrix},$$

and we get that Γ and Ω are two idempotents, and hence are group invertible. In addition, it is easy to check that $\Omega\in R^{\{1,3\}}$ however $\Gamma\notin R^{\{1,3\}}$. Therefore, Ω is core invertible, but Γ is not core invertible.

The example above is to say that the equivalence that Γ is core invertible if and only if Ω is core invertible does not hold. Therefore, we consider to give a sufficient condition that A is $*$-invariant for this equivalence concerning core invertibility analogously. Furthermore, the example as follow is to show that the $*$-invariance of A is not necessary for the equivalence that Γ is pseudo core invertible if and only if Ω is pseudo core invertible.

Example 2. Consider $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \in \mathbb{C}^{2\times 2}$ and let $*$ be the involution defined as the transposed conjugate. Then $A^\dagger = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Clearly Γ and Ω are pseudo core invertible since every complex matrix has a pseudo core inverse. We next prove that ϕ_A is not $*$-invariant. In fact, if we take $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$, then $A^\dagger XA = \begin{bmatrix} 0 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$, $AXA^\dagger = \begin{bmatrix} 0 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$. Hence, $A^\dagger XA = A^*X(A^\dagger)^*$ does not hold.

Theorem 3. *Let $A \in R^{m\times n}$ be Moore-Penrose invertible and $B \in R^{m\times m}$. Consider the following conditions:*

(1) $\Gamma = AA^\dagger BAA^\dagger + I_m - AA^\dagger$ *is pseudo core invertible with* $\mathrm{ind}(\Gamma) = k$ *(core invertible if $k = 1$).*

(2) $\Omega = A^\dagger BA + I_n - A^\dagger A$ *is pseudo core invertible with* $\mathrm{ind}(\Omega) = k$ *(core invertible if $k = 1$).*

If A is $*$-invariant then (1)$\Leftrightarrow$(2), in which case

$$\Gamma^{\textcircled{D}} = A\Omega^{\textcircled{D}}A^\dagger + I_m - AA^\dagger$$

and

$$\Omega^{\textcircled{D}} = A^\dagger\Gamma^{\textcircled{D}}A + I_n - A^\dagger A.$$

Proof. Let us first consider the case $k = 1$, i.e., the core invertibility case.

Suppose that Γ is core invertible. Then by Theorem 1, we have that $AA^\dagger BAA^\dagger$ has a core inverse $\Gamma_0^{\textcircled{\#}}$ in $AA^\dagger R^{m\times m}AA^\dagger$. As $\Gamma_0^{\textcircled{\#}}(AA^\dagger BAA^\dagger)^2 = AA^\dagger BAA^\dagger$ then we have

$$A^\dagger\Gamma_0^{\textcircled{\#}}A(A^\dagger BA)^2 = A^\dagger BA.$$

Moreover, $AA^\dagger BAA^\dagger(\Gamma_0^{\textcircled{\#}})^2 = \Gamma_0^{\textcircled{\#}}$. Since

$$AA^\dagger(AA^\dagger BAA^\dagger) = (AA^\dagger BAA^\dagger)AA^\dagger,$$
$$AA^\dagger(AA^\dagger BAA^\dagger)^* = (AA^\dagger BAA^\dagger)^*AA^\dagger,$$

it follows that $AA^\dagger\Gamma_0^{\textcircled{\#}} = \Gamma_0^{\textcircled{\#}}AA^\dagger$ by Lemma 3, and therefore

$$A^\dagger BA(A^\dagger\Gamma_0^{\textcircled{\#}}A)^2 = A^\dagger\Gamma_0^{\textcircled{\#}}A.$$

Since ϕ_A is $*$-invariant, for all $Y \in R^{m\times m}$, we have

$$A^\dagger YA = A^*Y(A^\dagger)^*.$$

Also, $(AA^{\dagger}BAA^{\dagger}\Gamma_0^{\#})^* = AA^{\dagger}BAA^{\dagger}\Gamma_0^{\#}$. Multiplying on left side by A^*, it follows that

$$\begin{aligned}(AA^{\dagger}BAA^{\dagger}\Gamma_0^{\#}A)^* &= A^*AA^{\dagger}BAA^{\dagger}\Gamma_0^{\#}\\ &= A^*B(A^{\dagger})^*A^*\Gamma_0^{\#}\\ &= A^{\dagger}BAA^*\Gamma_0^{\#},\end{aligned}$$

then multiplying on right side by $(A^{\dagger})^*$, we get

$$\begin{aligned}(A^{\dagger}BAA^{\dagger}\Gamma_0^{\#}A)^* &= A^{\dagger}BAA^*\Gamma_0^{\#}(A^{\dagger})^*\\ &= A^{\dagger}BAA^{\dagger}\Gamma_0^{\#}A.\end{aligned}$$

Therefore, by Lemma 1, $A^{\dagger}\Gamma_0^{\#}A$ is the core inverse of $A^{\dagger}A\Omega A^{\dagger}A$ in $A^{\dagger}AR^{n\times n}A^{\dagger}A$. Then by Theorem 1, it follows that $\Omega^{\#} = A^{\dagger}\Gamma_0^{\#}A + I_n - A^{\dagger}A$. Since $\Gamma_0^{\#} = AA^{\dagger}\Gamma^{\#}AA^{\dagger}$, we get

$$\Omega^{\#} = A^{\dagger}\Gamma^{\#}A + I_n - A^{\dagger}A.$$

The converse is analogous. If $\Omega^{\#}$ exists, then $\Omega_0^{\#} = A^{\dagger}A\Omega^{\#}A^{\dagger}A$ is the core inverse of $A^{\dagger}BA = A^{\dagger}A\Omega A^{\dagger}A$ in the ring $A^{\dagger}AR^{n\times n}A^{\dagger}A$. Analogous to the necessity of the proof, we can check that $A\Omega_0^{\#}A^{\dagger}$ is the core inverse of $AA^{\dagger}\Gamma AA^{\dagger}$ in $AA^{\dagger}R^{m\times m}AA^{\dagger}$. Therefore,

$$\begin{aligned}\Gamma^{\#} &= A\Omega_0^{\#}A^{\dagger} + I_m - AA^{\dagger}\\ &= A\Omega^{\#}A^{\dagger} + I_m - AA^{\dagger}.\end{aligned}$$

For the general case, suppose $\Gamma^{\mathbb{D}}$ exists with $\mathrm{ind}(\Gamma) = k$. Then by Lemma 2,

$$(\Gamma^k)^{\#} = \left(AA^{\dagger}(BAA^{\dagger})^kAA^{\dagger} + I_m - AA^{\dagger}\right)^{\#}$$

exists. Using the first part of the proof and keeping in mind that B is arbitrary, we can obtain that $\Omega^k = A^{\dagger}(BAA^{\dagger})^kA + I_n - A^{\dagger}A$ is core invertible. Thus, $\Omega^{\mathbb{D}}$ exists with $\mathrm{ind}(\Omega) \le k$. Moreover,

$$\begin{aligned}\Omega^{\mathbb{D}} &= \Omega^{k-1}(\Omega^k)^{\#}\\ &= \Omega^{k-1}\left(A^{\dagger}(BAA^{\dagger})^kA + I_n - A^{\dagger}A\right)^{\#}\\ &= \Omega^{k-1}\left(A^{\dagger}(\Gamma^k)^{\#}A + I_n - A^{\dagger}A\right)\\ &= \left(A^{\dagger}(BAA^{\dagger})^{k-1}A + I_n - A^{\dagger}A\right)\left(A^{\dagger}(\Gamma^k)^{\#}A + I_n - A^{\dagger}A\right)\\ &= A^{\dagger}(BAA^{\dagger})^{k-1}AA^{\dagger}(\Gamma^k)^{\#}A + I_n - A^{\dagger}A\\ &= A^{\dagger}\Gamma^{k-1}(\Gamma^k)^{\#}A + I_n - A^{\dagger}A\\ &= A^{\dagger}\Gamma^{\mathbb{D}}A + I_n - A^{\dagger}A.\end{aligned}$$

The converse is analogous and $\mathrm{ind}(\Gamma) \leq k$. For the expression of $\Gamma^{Ⓓ}$, we have

$$\begin{aligned}
\Gamma^{Ⓓ} &= \Gamma^{k-1}(\Gamma^k)^{\#} \\
&= \Gamma^{k-1}\left(AA^\dagger(BAA^\dagger)^k AA^\dagger + I_m - AA^\dagger\right)^{\#} \\
&= \Gamma^{k-1}\left(A(\Omega^k)^{\#}A^\dagger + I_m - AA^\dagger\right) \\
&= \left(AA^\dagger(BAA^\dagger)^{k-1}AA^\dagger + I_m - AA^\dagger\right)\left(A(\Omega^k)^{\#}A^\dagger + I_m - AA^\dagger\right) \\
&= AA^\dagger(BAA^\dagger)^{k-1}A(\Omega^k)^{\#}A^\dagger + I_m - AA^\dagger \\
&= A\Omega^{k-1}(\Omega^k)^{\#}A^\dagger + I_m - AA^\dagger \\
&= A\Omega^{Ⓓ}A^\dagger + I_m - AA^\dagger.
\end{aligned}$$

□

Finally, we will show that a similar equivalence holds for DMP inverses.

Lemma 6 ([6])**.** *Let $a, b \in R$ with $ab = ba$. If $a \in R^d$, then $a^d b = ba^d$.*

Theorem 4. *Let $A \in R^{m\times n}$ be Moore-Penrose invertible and $B \in R^{m\times m}$. Consider the following conditions:*

(1) $\Gamma = AA^\dagger BAA^\dagger + I_m - AA^\dagger$ *is DMP invertible.*
(2) $\Omega = A^\dagger BA + I_n - A^\dagger A$ *is DMP invertible.*

If A is $*$-invariant then (1)⇔(2), in which case

$$\Gamma^{d,\dagger} = A\Omega^{d,\dagger}A^\dagger + I_m - AA^\dagger$$

and

$$\Omega^{d,\dagger} = A^\dagger\Gamma^{d,\dagger}A + I_n - A^\dagger A.$$

Proof. According to [21, Propositions 5 and 6] and Definition 4, the sufficiency and necessity of the theorem are obtained. Since $A^\dagger A\Omega = \Omega A^\dagger A$ and $AA^\dagger\Gamma = \Gamma AA^\dagger$, it follows that $A^\dagger A\Omega^d = \Omega^d A^\dagger A$ and $AA^\dagger\Gamma^d = \Gamma^d AA^\dagger$ by Lemma 6. For the expressions of $\Gamma^{d,\dagger}$ and $\Omega^{d,\dagger}$, we can obtain

$$\begin{aligned}
\Gamma^{d,\dagger} &= \Gamma^d\Gamma\Gamma^\dagger \\
&= (A\Omega^d A^\dagger + I_m - AA^\dagger)(AA^\dagger BAA^\dagger + I_m - AA^\dagger)(A\Omega^\dagger A^\dagger + I_m - AA^\dagger) \\
&= A\Omega^d A^\dagger BA\Omega^\dagger A^\dagger + I_m - AA^\dagger \\
&= A\Omega^d A^\dagger A\Omega\Omega^\dagger A^\dagger + I_m - AA^\dagger \\
&= A\Omega^d\Omega\Omega^\dagger A^\dagger + I_m - AA^\dagger \\
&= A\Omega^{d,\dagger}A^\dagger + I_m - AA^\dagger,
\end{aligned}$$

and
$$\begin{aligned}\Omega^{d,\dagger} &= \Omega^d\Omega\Omega^\dagger \\ &= (A^\dagger\Gamma^d A + I_n - A^\dagger A)(A^\dagger BA + I_n - A^\dagger A)(A^\dagger\Gamma^\dagger A + I_n - A^\dagger A) \\ &= A^\dagger\Gamma^d AA^\dagger BAA^\dagger\Gamma^\dagger A + I_n - A^\dagger A = A^\dagger\Gamma^d AA^\dagger\Gamma\Gamma^\dagger A + I_n - A^\dagger A \\ &= A^\dagger\Gamma^d\Gamma\Gamma^\dagger A + I_n - A^\dagger A = A^\dagger\Gamma^{d,\dagger}A + I_n - A^\dagger A.\end{aligned}$$

□

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