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CHARACTERIZATIONS OF Rad-SUPPLEMENTED MODULES

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Abstract. We prove that a commutative ring R is an artinian principal ideal ring if and only if the ring is semilocal and every Rad-supplemented R-module is a direct sum of w-local R-modules. Moreover, we study of extensions of Rad-supplemented modules over commutative noetherian rings, and we show that if $\frac{M}{N}$ is reduced, M is Rad-supplemented if and only if N and $\frac{M}{N}$ are Rad-supplemented. We also prove that over a dedekind domain an indecomposable, amply Rad-supplemented radical module is hollow radical.

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1. INTRODUCTION

In this note R will be an associative ring with identity. Unless otherwise mentioned, all modules will be unital left R-modules. Let R be such a ring and M be an R-module. The notation $N \subseteq M$ means that N is a submodule of M. A submodule S of M is called *small* in M, denoted by $S << M$, if $S + N \neq M$ for every proper submodule N of M. We denote by $\text{Rad}(M)$ the radical of M. A non-zero module M is called *hollow* if every proper submodule of M is small in M, and it is called *local* if it is hollow and $\text{Rad}(M)$ is a maximal submodule of M. Let M be a module. M is called *supplemented* if every submodule N of M has a *supplement*, that is a submodule K of M minimal with respect to $N + K = M$. Equivalently, $N + K = M$ and $N \cap K << K$ ([12]). Following [12], M is called *amply supplemented* if, for any two submodules U and V of M with $U + V = M$, V contains a supplement of U in M. Clearly, hollow modules are amply supplemented and amply supplemented modules are supplemented.

Recall from Lomp [7] that a module M is said to be *semilocal* if $\frac{M}{\text{Rad}(M)}$ is semisimple, and a ring R is said to be *semilocal* if it is semilocal as a left (right) module over itself. It is shown in [7, Teorem 3.5] that a ring R is semilocal if and only if every left R-module is semilocal.

As a proper generalization of supplemented modules, the notion of Rad-supplemented modules, which has been introduced by Xue [13], has been studied recently (see [1, 4, 5]). Let M be a module and N be a submodule of M. A submodule K of M is called a Rad*-supplement* of N in M (according to [13], *generalized supplement*) if $N + K = M$ and $N \cap K \subseteq \text{Rad}(K)$. Since $\text{Rad}(K)$ is the sum of all small submodules of K, every supplement submodule is a Rad-supplement in M. A module M is called Rad*-supplemented* (according to [13], *generalized supplemented*) if every submodule N of M has a Rad-supplement K in M, and it is called *amply* Rad*-supplemented* (according to [13], *generalized amply supplemented*) if every submodule N of M has *ample* Rad*-supplements* in M, i.e., $N + L = M$ implies that N has a Rad-supplement $K \subseteq L$. In [5], the various properties of Rad-supplemented modules are extensively studied. In addition, it is shown in [1, 2.2.(2) and 2.3.(3)] that factor modules of a Rad-supplemented module and finite sums of Rad-supplemented modules are Rad-supplemented. It is of obvious interest to investigate extensions and characterizations of Rad-supplemented modules. This is the focus of our investigations in this paper.

Let Γ be a class of modules and let $0 \to N \to M \to K \to 0$ be any short exact sequence. Here M is an *extension* of N by K and Γ is called *closed under extensions* if $N, K \in \Gamma$ implies $M \in \Gamma$. It is clear that, for modules $N \subseteq M$, M is an extension of N.

In this article, we prove that a commutative ring R is an artinian principal ideal ring if and only if the ring is semilocal and every Rad-supplemented R-module is a direct sum of w-local R-modules if and only if every left R-module is a direct sum of w-local R-modules. We give a characterization of semisimple rings via Rad-supplements. We show that a semilocal ring R is left perfect if and only if every Rad-supplemented module is (generalized) semiperfect. Some examples are given in order to show that the class of Rad-supplemented modules is not generally closed under extensions. Let R be a commutative noetherian ring and M be an R-module with $N \subseteq M$. If $\frac{M}{N}$ is reduced, M is Rad-supplemented if and only if N and $\frac{M}{N}$ are Rad-supplemented. It follows that a ring R is semilocal if and only if every left R-module with Rad-supplemented radical is Rad-supplemented. Over a dedekind domain a radical module is amply Rad-supplemented and indecomposable if and only if the module is hollow radical. Every indecomposable, w-local and amply Rad-supplemented module over a dedekind domain is local.

2. Rad-SUPPLEMENTED MODULES OVER ANY RINGS

Let R be any ring and M be an R-module. A submodule N of M is called *radical* if N has no maximal submodules, i.e. $N = \text{Rad}(N)$. Note that radical modules are Rad-supplemented. This fact plays a key role in our study. By $P(M)$ we denote the sum of all radical submodule of a module M. It is clear that, for any module M, $P(M)$ is the largest radical submodule and so $P(M)$ is Rad-supplemented. Using the mentioned facts, we give examples of a module, which is Rad-supplemented but not supplemented. We see, for example, the left $\mathbb{Z}$-module $M =_{\mathbb{Z}} \mathbb{Q}$.

Firstly we have the following lemma.

Lemma 1. *Let M be a module and $N \subseteq U \subseteq M$. Then U is* Rad*-supplemented if and only if $\frac{U}{P(N)}$ is* Rad*-supplemented.*

Proof. ($\Rightarrow$) Let U be Rad-supplemented. By [1, 2.2 (2)], $\frac{U}{P(N)}$ is Rad-supplemented as a factor module of U.

($\Leftarrow$) Let U' be any submodule of U. By the assumption, there exists a submodule $\frac{V}{P(N)}$ of $\frac{U}{P(N)}$ such that $\frac{U'+P(N)}{P(N)} + \frac{V}{P(N)} = \frac{U}{P(N)}$ and

$$\Big(\frac{U'+P(N)}{P(N)}\Big)\bigcap\Big(\frac{V}{P(N)}\Big) \subseteq \mathrm{Rad}\Big(\frac{V}{P(N)}\Big).$$

Then $(U'+P(N))+V = U$ and hence $U'+V = U$. Since $P(N) = \mathrm{Rad}(P(N)) \subseteq \mathrm{Rad}(V)$, it follows that $\frac{U'\cap V+P(N)}{P(N)} = \frac{(U'+P(N))\cap V}{P(N)} = (\frac{U'+P(N)}{P(N)}) \cap (\frac{V}{P(N)}) \subseteq \mathrm{Rad}(\frac{V}{P(N)}) = \frac{\mathrm{Rad}(V)}{P(N)}$, which means that $U'\cap V \subseteq \mathrm{Rad}(V)$. So V is a Rad-supplement of U' in U. Hence U is Rad-supplemented. □

Corollary 1. *Let M be a module and N be a submodule of M. M is* Rad*-supplemented if and only if $\frac{M}{P(N)}$ is* Rad*-supplemented. In particular, M is* Rad*-supplemented if and only if $\frac{M}{P(M)}$ is* Rad*-supplemented.*

Proof. It follows from Lemma 1. □

Recall from [5, Corollary 4.2] that if a submodule V of a module M is a Rad-supplement in M, then $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M)$.

Now we shall show that the rings whose modules are Rad-supplement submodules in every extension are semisimple in the following theorem.

Theorem 1. *Let R be any ring. Then the following statements are equivalent.*

(1) *R is semisimple.*
(2) *Every left R-module is a* Rad*-supplement in every extension.*
(3) *Every left R-module is a* Rad*-supplement in every injective extension .*
(4) *Every left ideal of R is a* Rad*-supplement in every injective extension.*

Proof. (1) $\Rightarrow$ (2) Let N be an R-module and M be any extension of N. By the hypothesis and [6, Corollary 8.2.2 (a)], M is semisimple, and so N is a direct summand of M. It follows that N is a Rad-supplement in M.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let I be any left ideal of R. By the hypothesis, I is a Rad-supplement in its injective hull $E(I)$. Then we have $I + J = E(I)$ and $I \cap J \subseteq \mathrm{Rad}(I)$ for some submodule $J \subseteq E(I)$. If $m \in I \cap J$, then $Rm \subseteq \mathrm{Rad}(I) \subseteq \mathrm{Rad}(E(I))$. By (4), Rm is a Rad-supplement in $E(I)$ and so $\mathrm{Rad}(Rm) = Rm \cap \mathrm{Rad}(E(I)) = Rm$.

Therefore $m = 0$. This means that $I \oplus J = E(I)$ and so I is injective, and hence a direct summand of R. By [6, Corollary 8.2.2 (a)], R is semisimple. □

A ring R is Rad-supplemented if ${}_RR$ (or R_R) is a Rad-supplemented module. It is clear that semiperfect (i.e., supplemented) rings are Rad-supplemented. Characterizations of semiperfect rings have been studied extensively by many authors recently. Now we shall give a characterization of Rad-supplemented rings. Firstly, we need the following simple lemmas.

Lemma 2. *Let R be any ring with identity. Then R is* Rad*-supplemented if and only if every cyclic R-module is* Rad*-supplemented.*

Proof. Let R be a Rad-supplemented ring. Suppose that M is any cyclic R-module. Then there exists an element m of M such that $M = Rm$. Note that $\frac{R}{\mathrm{Ann}(m)} \cong Rm$, where $\mathrm{Ann}(m)$ is the set of all elements r of R such that $rm = 0$. From [1, 2.2.(2)] the hypothesis implies that $\frac{R}{\mathrm{Ann}(m)}$ is Rad-supplemented and so Rm is Rad-supplemented. The converse is clear. □

Lemma 3. *Let M be a module with $U + V = M$ for submodules U, V of M. If V contains a* Rad*-supplement of U in M, then $U \cap V$ has a* Rad*-supplement in V.*

Proof. Suppose that a submodule K of V is a Rad-supplement of U in M. Then, we have $U + K = M$ and $U \cap K \subseteq \mathrm{Rad}(K)$. From the modular law, $U \cap V + K = V$. Since $K \subseteq V$, then $(U \cap V) \cap K = U \cap K \subseteq \mathrm{Rad}(K)$. So K is a Rad-supplement of $U \cap V$ in V. □

Theorem 2. *The following statements are equivalent for any ring R.*

(1) *R is* Rad*-supplemented.*
(2) *R has ample* Rad*-supplements in every finitely generated extension.*
(3) *Every cyclic R-module has ample* Rad*-supplements in every finitely generated extension.*

Proof. (1) ⇒ (3) Let N be any cyclic R-module and M be any finitely generated extension of N. Since R is Rad-supplemented, by Lemma 2, every cyclic submodule of M is Rad-supplemented and so M is amply Rad-supplemented by [11, Corollary 3.6]. Therefore N has ample Rad-supplements in M.

(3) ⇒ (2) It is obvious.

(2) ⇒ (1) For any left ideal I of R, consider the finitely generated pushout R-module $N = \frac{R \oplus R}{K}$, where K is the set of all elements k of $R \oplus R$ such that $k = (r, -r)$ for all $r \in I$. Then there exist monomorphisms $f, g : R \to N$ such that $N = f(R) + g(R)$. The hypothesis implies that $f(R)$ has a Rad-supplement V in N with $V \subseteq g(R)$. So, by Lemma 3, V is a Rad-supplement of $f(R) \cap g(R)$ in $g(R)$. Note that $I = g^{-1}(f(R) \cap g(R))$. It follows that $R = I + g^{-1}(V)$ and $I \cap g^{-1}(V) \subseteq \mathrm{Rad}(g^{-1}(V))$. Hence R is Rad-supplemented. □

We say that a module M *w-local* if $\mathrm{Rad}(M)$ is a maximal submodule of M as in [4]. Every local module is w-local. It is well known that a commutative ring R has the property that every R-module is a direct sum of local R-modules if and only if R is an artinian principal ideal ring. Now, we prove that if R is a commutative ring and every R-module is a direct sum of w-local R-modules, then R is an artinian principal ideal ring in the following theorem.

Theorem 3. *The following are equivalent for a commutative ring R.*

1. *Every left R-module is a direct sum of w-local R-modules.*
2. *R is semilocal and every* Rad*-supplemented left R-module is a direct sum of w-local R-modules.*
3. *R is an artinian principal ideal ring.*

Proof. (1) ⇒ (2) Write $\frac{R}{\mathrm{Rad}(R)} = \bigoplus_{i\in I} N_i$, where each N_i is w-local. Since $\mathrm{Rad}(\frac{R}{\mathrm{Rad}(R)}) = 0$, for all $i \in I$, $\mathrm{Rad}(N_i) = 0$. So N_i is simple. Thus $\frac{R}{\mathrm{Rad}(R)}$ is semisimple and so R is semilocal. The rest of the proof is clear.

(2) ⇒ (3) Let $F = R^{(\Lambda)}$any index set Λ. Suppose that $\mathrm{Rad}(\frac{F}{N}) = \frac{F}{N}$ for some submodule N of F. By the assumption, we can write $\frac{F}{N} = \bigoplus_{i\in I} M_i$ where M_i is w-local for all $i \in I$. By [12, 21.6.(5)], $\mathrm{Rad}(\frac{F}{N}) = \bigoplus_{i\in I} \mathrm{Rad}(M_i)$ and so each M_i is radical as a direct summand of $\frac{F}{N}$. Since M_i is w-local, we obtain that, for all $i \in I$, $M_i = 0$. Therefore $\frac{F}{N} = 0$. This means that $\mathrm{Rad}(F) << F$. It follows from [12, 43.9] that R is left perfect. Applying [12, 43.9] again, we deduce that every left R-module is Rad-supplemented and so every left R-module is a direct sum of w-local R-modules. If N is aw-local, then N is local because R is left perfect. Hence every left R-module is a direct sum of cyclic R-modules. By [9, Theorem 6.7], R is an artinian principal ideal ring.

(3) ⇒ (1) is clear. □

The following corollary is an immediate consequence of Theorem 3.

Corollary 2. *Let R be a commutative semilocal ring. Then, R is an artinian principal ideal ring if and only if every* Rad*-supplemented left R-module is a direct sum of w-local R-modules.*

Let $f : P \to M$ be an epimorphism. Xue [13] calls f a *(generalized) cover* if $(\mathrm{Ker}(f) \subseteq \mathrm{Rad}(P))$ $\mathrm{Ker}(f) << P$, and calls a (generalized) cover f a *(generalized) projective cover* if P is a projective module. In the spirit of [13], a module M is said to be *(generalized) semiperfect* if every factor module of M has a (generalized) projective cover. He [13, Theorem 2.2] proved that every generalized semiperfect module is Rad-supplemented. Now, we obtain the following result.

Proposition 1. *Let R be a semilocal ring. Every* Rad*-supplemented left R-module is (generalized) semiperfect if and only if R is left perfect.*

Proof. ($\Rightarrow$) Let $M = \text{Rad}(M)$. Since M is Rad-supplemented, it follows from the hypothesis that M is generalized semiperfect. Then, there exists a generalized cover $f : F \to M$ with a projective module F. Since $\text{Ker}(f) \subseteq \text{Rad}(F) \neq F$, it follows that $M = 0$. By [12, 43.9], R is left perfect.

($\Leftarrow$) This is immediate. □

3. Rad-SUPPLEMENTED MODULES OVER COMMUTATIVE NOETHERIAN RINGS

Throughout this section, unless otherwise stated, we shall consider commutative noetherian rings.

An R-module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M, and it is called *reduced* if every submodule of M contains a maximal submodule, that is, $P(M) = 0$. Note that $\text{Rad}(M)$ is small in M for every coatomic R-module M.

Lemma 4. *The following statements are equivalent for a* Rad*-supplemented module M.*

(1) *M is coatomic.*
(2) *M is reduced.*
(3) Rad*(M) is small in M.*

If the module M satisfies one of the equivalent conditions, then M is supplemented.

Proof. (1) $\Rightarrow$ (2) Let M be a coatomic module. By [15, Lemma 1.1], every submodule of M is coatomic and so $P(M) = 0$, which means that M is reduced.

(2) $\Rightarrow$ (3) Suppose that $M = \text{Rad}(M) + N$ for some submodule N of M. Then we can write $\text{Rad}(\frac{M}{N}) = \frac{M}{N}$. Since M is Rad-supplemented, N has a Rad-supplement V in M. From (2) it follows that V has a maximal submodule K. So $\frac{K}{N \cap V}$ is a maximal submodule of $\frac{V}{N \cap V}$. Note that

$$\frac{M}{N} \cong \frac{V}{N \cap V}$$

contains a maximal submodule and thus $\frac{M}{N} = 0$. Therefore $M = N$. This proves (3).

(3) $\Rightarrow$ (1) The assumption implies that, for any proper submodule $U \subseteq M$, there exists a submodule V of M such that $U + V = M$ and $U \cap V \subseteq \text{Rad}(V)$. Since $\text{Rad}(M) << M$, V is not contained in a maximal submodule K of M. Then the submodule $U + V \cap K$ of M is maximal. Thus M is coatomic.

Suppose that Rad-supplemented module M satisfies one of these conditions. Then M is supplemented by [5, Proposition 7.3]. □

The following result follows from [5, Proposition 7.3]. We give this result as a consequence of Lemma 4.

Corollary 3. *For a module M, M is* Rad*-supplemented if and only if $\frac{M}{P(M)}$ is supplemented.*

A submodule of a Rad-supplemented module need not be Rad-supplemented, in general. To see this actuality, we shall consider the left $\mathbb{Z}$-module $M =_{\mathbb{Z}} \mathbb{Q}$. It is well known that M is Rad-supplemented. On the other hand, the submodule ${}_{\mathbb{Z}}\mathbb{Z}$ of M is not semisimple.

Now, we show that a submodule of a Rad-supplemented module is Rad-supplemented under a certain condition.

Proposition 2. *Let M be a module and $N \subseteq M$. Suppose that $\frac{M}{N}$ is reduced. If M is* Rad*-supplemented, then N is* Rad*-supplemented.*

Proof. According to [1, 2.2.(2)], $\frac{M}{N}$ is Rad-supplemented as a factor module of M. Since $\frac{M}{N}$ is reduced, $P(\frac{M}{N}) = 0$. Therefore $\frac{M}{N}$ is supplemented by Lemma 4. Since M is Rad-supplemented, $\frac{M}{P(N)}$ is Rad-supplemented by Corollary 1. Note that

$$\frac{\frac{M}{P(N)}}{\frac{N}{P(N)}} \cong \frac{M}{N}$$

is reduced and thus $\frac{M}{P(N)}$ is reduced by [14, Lemma 1.5 (a)]. It follows from Lemma 4 that $\frac{M}{P(N)}$ is supplemented. Thus $\frac{N}{P(N)}$ is supplemented by [8, Proposition 2.6]. So $\frac{N}{P(N)}$ is Rad-supplemented. Hence N is Rad-supplemented by Lemma 1. □

Using Proposition 2, we obtain the following result.

Corollary 4. *The following statements are equivalent for any module M.*

(1) *M is* Rad*-supplemented.*
(2) *Every maximal submodule of M is* Rad*-supplemented.*
(3) *Every cofinite submodule of M is* Rad*-supplemented.*

Proof. (1) ⇒ (3) If N is a cofinite submodule of M, then $\frac{M}{N}$ is finitely generated and so $\frac{M}{N}$ is reduced. From Proposition 2, the proof follows.

(3) ⇒ (2) is clear.

(2) ⇒ (1) Let $M = M_1 + M_2$, where M_1 and M_2 are maximal submodules of M. Since M_1 and M_2 are Rad-supplemented modules, M is Rad-supplemented according to [1, 2.3.(3)]. If M is w-local, $\text{Rad}(M)$ is maximal and so $M = \text{Rad}(M) + U$ for every proper submodule U of M with $U \nsubseteq \text{Rad}(M)$. By [1, 2.3.(1)], U has a Rad-supplement in M since $\text{Rad}(M)$ is Rad-supplemented. Hence M is Rad-supplemented. □

The following example shows that the class of Rad-supplemented modules is not closed under extensions, in general.

Example 1. Let Λ be a collection of maximal ideals of the noetherian commutative ring $\mathbb{Z}$. Suppose that M is the left $\mathbb{Z}$-module $\prod_{\mathfrak{p}\in\Lambda}(\frac{\mathbb{Z}}{\mathfrak{p}})$. Then $\text{Rad}(M) = 0$. By [3, Lemma 2.9], for some submodule N of M, we have $\frac{N}{T} \cong \mathbb{Q}$, where T is the direct sum of simple $\mathbb{Z}$-modules $\frac{\mathbb{Z}}{\mathfrak{p}}$. Then N is an extension of T by $\mathbb{Q}$. Since T is semisimple, it is Rad-supplemented. On the other hand, the submodule N is not Rad-supplemented.

Later we shall give another example of such modules (see Example 2).

Theorem 4. *Let $0 \to N \to M \to K \to 0$ be a short exact sequence. Suppose that K is reduced. Then M is* Rad*-supplemented if and only if N and K are* Rad*-supplemented.*

Proof. ($\Rightarrow$) It follows from Proposition 2 and [1, 2.2.(2)].

($\Leftarrow$) By Lemma 4, K is supplemented. Since N is Rad-supplemented, $\frac{N}{P(N)}$ is supplemented by Corollary 3. It follows from [8, Proposition 2.6] that $\frac{M}{P(N)}$ is Rad-supplemented. Hence M is Rad-supplemented by Corollary 1. □

Corollary 5. *A module M is* Rad*-supplemented if and only if it is an extension of a* Rad*-supplemented submodule by a reduced supplemented module.*

Proof. If M has no maximal submodules, the result is obvious as $\frac{M}{P(M)} = 0$. Suppose that $M \neq P(M)$. Then this gives the existence of a reduced factor module of M. Therefore the assertion follows from Theorem 4. □

Proposition 3. *Let M be a module. M is* Rad*-supplemented if and only if M is semilocal and* $\text{Rad}(M)$ *is* Rad*-supplemented.*

Proof. If M is Rad-supplemented, then M is semilocal. Thus $\frac{M}{\text{Rad}(M)}$ is reduced. By Proposition 2, $\text{Rad}(M)$ is Rad-supplemented . Conversely, suppose that M is semilocal and $\text{Rad}(M)$ is Rad-supplemented. From Theorem 4 the assumption implies that M is Rad-supplemented. □

Using the above proposition we obtain the following characterization of semilocal rings.

Corollary 6. *The following conditions on a ring R is equivalent:*

(1) *R is semilocal.*
(2) *Every left R-module with* Rad*-supplemented radical is* Rad*-supplemented.*

Proof. (1) $\Rightarrow$ (2) If R is semilocal, then every left R-module is semilocal by [7, Theorem 3.5]. The result follows from Proposition 3.

(2) $\Rightarrow$ (1) Since $\text{Rad}(\frac{R}{\text{Rad}(R)}) = 0$, it follows from the hypothesis that $\frac{R}{\text{Rad}(R)}$ is Rad-supplemented. So $\frac{R}{\text{Rad}(R)}$ is semisimple, i.e. R is semilocal. □

In [5], a module M is said to be *totally* Rad*-supplemented* if every submodule of M is Rad-supplemented. Every semisimple module is totally Rad-supplemented. It is easy to check that the class of totally Rad-supplemented modules is closed under factor modules and submodules. The following fact is a modification of Theorem 4.

Theorem 5. *Let M be a module and $\frac{M}{N}$ be reduced for some submodule N of M. Then M is totally* Rad*-supplemented if and only if N and $\frac{M}{N}$ are totally* Rad*-supplemented.*

Proof. Suppose that N and $\frac{M}{N}$ are totally Rad-supplemented. Let U be any submodule of M. By the hypothesis, $U \cap N$ and $\frac{U+N}{N}$ are Rad-supplemented. Note that

$$\frac{U+N}{N} \cong \frac{U}{U \cap N}$$

is reduced because $\frac{M}{N}$ is reduced. By Theorem 4, U is Rad-supplemented. Hence M is totally Rad-supplemented. □

Corollary 7. *Let M be a* Rad*-supplemented module. Then, M is totally* Rad*-supplemented if and only if $P(M)$ is totally* Rad*-supplemented.*

Proof. Suppose that $P(M)$ is totally Rad-supplemented. By the hypothes is and Corollary 3, $\frac{M}{P(M)}$ is supplemented. Applying [8, Proposition 2.6], we deduce that $\frac{M}{P(M)}$ is totally supplemented. Therefore M is totally Rad-supplemented by Theorem 5. □

4. Rad-SUPPLEMENTED MODULES OVER COMMUTATIVE DOMAINS

In this section a ring R will be a commutative domain. Let R be such a ring and M be an R-module. We denote by $T(M)$ the set of all elements m of M for which there exists a non-zero element r of R such that $rm = 0$, i.,e., $\mathrm{Ann}(m) \neq 0$. Then $T(M)$, which is a submodule of M, called *the torsion submodule* of M. If $M = T(M)$, then M is called a *torsion module* and M is called *torsion-free* provided $T(M) = 0$.

Proposition 4. *Let R be a non-semilocal commutative domain and M be an R-module. If M is totally* Rad*-supplemented, M is a torsion module.*

Proof. Let $0 \neq m \in M$. Suppose that $\mathrm{Ann}(m) = 0$, i.e. $R \cong Rm$. Since M is totally Rad-supplemented, the left R-submodule Rm of M is Rad-suppleme nted. So ${}_RR$ is Rad-supplemented. Therefore $\frac{R}{\mathrm{Rad}(R)}$ is semisimp le, i.e. R is semilocal. This contradicts the assumption. Hence $\mathrm{Ann}(m) \neq 0$, this implies that M is torsion. □

Corollary 8. *Let R be a non-semilocal dedekind domain and M be a totally* Rad*-supplement ed R-module. Then M is torsion.*

Let R be a dedekind domain and M be an R-module. We denote by Ω the set of all maximal (i.e., prime) ideals of R. Suppose that $\mathfrak{p}$ is any element of Ω. We denote by $T_{\mathfrak{p}}(M)$, which is a submodule of M, the set of all elements m of M for which there exists a positive integer n such that $\mathfrak{p}^n m = 0$. Then $T_{\mathfrak{p}}(M)$ is called the $\mathfrak{p}$-*primary part* of M. For a torsion module M over a dedekind domain, we have the decomposition $M = \bigoplus_{\mathfrak{p} \in \Omega} T_{\mathfrak{p}}(M)$.

Lemma 5. *Let R be a non-local dedekind domain and M be an R-module. Then M is* Rad*-supplemented if and only if $\frac{M}{P(M)}$ is torsion and every $\mathfrak{p}$-primary part of $\frac{M}{P(M)}$ is (*Rad*-)supplemented.*

Proof. According to [14, Theorem 3.1] and [5, Theorem 7.4], the proof of the lemma is clear. □

Let R be a dedekind domain and M be an R-module. By [2, Lemma 4.4], $P(M)$ is injective and so there exists a direct summand N of M such that $\frac{M}{P(M)} \cong N$. This fact and Lemma 5 give the following basic result for torsion-free modules.

Corollary 9. *Let M be a torsion-free* Rad*-supplemented module over a non-local dedekind domain. Then M is radical.*

Let M be a radical module. M is called *simply radical* if M has no proper radical submodules.

Proposition 5. *Let R be a noetherian ring and M be a simply radical R-module. If M is amply* Rad*-supplemented, M is hollow radical. In particular, every* Rad*-supplemented proper submodule of M is supplemented.*

Proof. Let U be any proper submodule of M. Suppose that $U + V = M$ for some submodule V of M. By the hypothesis, there exists a submodule $V^{'}$ of V such that $U + V^{'} = M$ and $U \cap V^{'} \subseteq \mathrm{Rad}(V^{'})$. Since M is simply radical, it follows that $\mathrm{Rad}(V^{'}) = V^{'} \cap \mathrm{Rad}(M) = V^{'} \cap M = V^{'}$. So $V^{'}$ is radical. Therefore $V^{'} = M$ and so $V = M$. Then we deduce that U is small in M. Hence M is hollow radical. Suppose that a proper submodule N of M is Rad-supplemented. Since M is simply radical, every submodule of N contains a maximal submodule, i. e., $P(N) = 0$. By Lemma 4, N is supplemented. □

Corollary 10. *Let R be a dedekind domain and M be a radical R-module. Then M is amply* Rad*-supplemented and indecomposable if and only if the module is hollow radical.*

Proof. Since indecomposable radical modules over dedekind domains is simply radical, M is hollow radical by Proposition 5. The converse is clear. □

Proposition 6. *Let M be a module over a Dedekind domain. Then the following statements are equivalent.*

(1) *M is indecomposable, w-local and amply* Rad*-supplemented.*
(2) *M is local.*

Proof. (1) ⇒ (2) Let U be any proper submodule of M. Suppose that U is not contained $\mathrm{Rad}(M)$. Since M is w-local, $\mathrm{Rad}(M)$ is maximal and so $U + \mathrm{Rad}(M) = M$. By the hypothesis, there exists a submodule V of $\mathrm{Rad}(M)$ such that $U + V = M$ and $U \cap V \subseteq \mathrm{Rad}(V)$. It follows that $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M) = V$, i.e. V is radical.

Then, by [2, Lemma 4.4], V is injective and so there exists a submodule L of M such that $M = V \oplus L$. Since M is indecomposable and w-local, we get $V = 0$. Thus, $U = M$, implying that M is local.

(2) $\Rightarrow$ (1) is clear. □

Now, we give an analogous characterization of [14, Theorem 3.1] for totally Rad-supplemented modules.

Theorem 6. *Let M be a non-semilocal dedekind domain and M be an R-module. Then M is totally* Rad*-supplemented if and only if M is torsion and every $\mathfrak{p}$-primary part of M is totally* Rad*-supplemented.*

Proof. The necessity of the condition is obvious by Corollary 8. Conversely, suppose that M is torsion and every $\mathfrak{p}$-primary part of M is totally Rad-supplemented. Let $N \subseteq U \subseteq M$. Since $M = \bigoplus_{\mathfrak{p}\in\Omega} T_{\mathfrak{p}}(M)$, we have $U = \bigoplus_{\mathfrak{p}\in\Omega}(U \cap T_{\mathfrak{p}}(M))$ and $N = \bigoplus_{\mathfrak{p}\in\Omega}(N \cap T_{\mathfrak{p}}(M))$. By the hypothesis, $N \cap T_{\mathfrak{p}}(M)$ has a Rad-supplement $V_{\mathfrak{p}}$ in $U \cap T_{\mathfrak{p}}(M)$. So $U \cap T_{\mathfrak{p}}(M) = N \cap T_{\mathfrak{p}}(M) + V_{\mathfrak{p}}$ and $N \cap V_{\mathfrak{p}} \subseteq \mathrm{Rad}(V_{\mathfrak{p}})$. Let $V = \bigoplus_{\mathfrak{p}\in\Omega} V_{\mathfrak{p}}$. Then $N + V = U$. Since $N \cap V_{\mathfrak{p}} \subseteq \mathrm{Rad}(V_{\mathfrak{p}})$ for every $\mathfrak{p} \in \Omega$, by [6, Corollaries 9.1.5 (c)], $N \cap V = (\bigoplus_{\mathfrak{p}\in\Omega}(N \cap T_{\mathfrak{p}}(M))) \cap (\bigoplus_{\mathfrak{p}\in\Omega} V_{\mathfrak{p}}) \subseteq \mathrm{Rad}(V)$. Hence U is Rad-supplemented. This completes the proof. □

Finally, we give an example showing the class of (totally) Rad-supplemented modules is not closed under extensions, in general. For a module M, $\mathrm{Soc}(M)$ will indicate the sum of all simple submodules of M.

Example 2. (see [10, Example 2.3]) Consider the non-Noetherian commutative ring which is the direct product $\prod_{i\geq 1}^{\infty} F_i$, where $F_i = F$ is any field. Suppose that R is the subring of the ring consisting of all sequences $(r_n)_{n\in\mathbb{N}}$ such that there exist $r \in F, m \in \mathbb{N}$ with $r_n = r$ for all $n \geq m$. Let $M = _R R$. Then M is a regular module which is not semisimple. Therefore $\mathrm{Soc}(M)$ is a maximal submodule of M. This means that $\mathrm{Soc}(M)$ and $\frac{M}{\mathrm{Soc}(M)}$ are Rad-supplemented. On the other hand, M is not Rad-supplemented.

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