Characterizations of Rad-supplemented modules

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Abstract. We prove that a commutative ring $R$ is an artinian principal ideal ring if and only if the ring is semilocal and every Rad-supplemented $R$-module is a direct sum of w-local $R$-modules. Moreover, we study of extensions of Rad-supplemented modules over commutative noetherian rings, and we show that if $\frac{M}{N}$ is reduced, $M$ is Rad-supplemented if and only if $N$ and $\frac{M}{N}$ are Rad-supplemented. We also prove that over a dedekind domain an indecomposable, amply Rad-supplemented radical module is hollow radical.

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1. INTRODUCTION

In this note $R$ will be an associative ring with identity. Unless otherwise mentioned, all modules will be unital left $R$-modules. Let $R$ be such a ring and $M$ be an $R$-module. The notation $N \subseteq M$ means that $N$ is a submodule of $M$. A submodule $S$ of $M$ is called small in $M$, denoted by $S \ll M$, if $S + N \neq M$ for every proper submodule $N$ of $M$. We denote by $\text{Rad}(M)$ the radical of $M$. A non-zero module $M$ is called hollow if every proper submodule of $M$ is small in $M$, and it is called local if it is hollow and $\text{Rad}(M)$ is a maximal submodule of $M$. Let $M$ be a module. $M$ is called supplemented if every submodule $N$ of $M$ has a supplement, that is a submodule $K$ of $M$ minimal with respect to $N \subseteq K \subseteq M$. Equivalently, $N + K = M$ and $N \cap K \ll K$ ([12]). Following [12], $M$ is called amply supplemented if, for any two submodules $U$ and $V$ of $M$ with $U + V = M$, $V$ contains a supplement of $U$ in $M$. Clearly, hollow modules are amply supplemented and amply supplemented modules are supplemented.

Recall from Lomp [7] that a module $M$ is said to be semilocal if $\frac{M}{\text{Rad}(M)}$ is semisimple, and a ring $R$ is said to be semilocal if it is semilocal as a left (right) module over itself. It is shown in [7, Theorem 3.5] that a ring $R$ is semilocal if and only if every left $R$-module is semilocal.

As a proper generalization of supplemented modules, the notion of Rad-supplemented modules, which has been introduced by Xue [13], has been studied recently.
Let $M$ be a module and $N$ be a submodule of $M$. A submodule $K$ of $M$ is called a Rad-supplement of $N$ in $M$ (according to [13], generalized supplement) if $N + K = M$ and $N \cap K \subseteq \text{Rad}(K)$. Since $\text{Rad}(K)$ is the sum of all small submodules of $K$, every supplement submodule is a Rad-supplement in $M$. A module $M$ is called Rad-supplemented (according to [13], generalized supplemented) if every submodule $N$ of $M$ has a Rad-supplement $K$ in $M$, and it is called amply Rad-supplemented (according to [13], generalized amply supplemented) if every submodule $N$ of $M$ has ample Rad-supplements in $M$, i.e., $N + L = M$ implies that $N$ has a Rad-supplement $K \subseteq L$. In [5], the various properties of Rad-supplemented modules are extensively studied. In addition, it is shown in [1, 2.2.(2) and 2.3.(3)] that factor modules of a Rad-supplemented module and finite sums of Rad-supplemented modules are Rad-supplemented. It is of obvious interest to investigate extensions and characterizations of Rad-supplemented modules. This is the focus of our investigations in this paper.

Let $\Gamma$ be a class of modules and let $0 \rightarrow N \rightarrow M \rightarrow K \rightarrow 0$ be any short exact sequence. Here $M$ is an extension of $N$ by $K$ and $\Gamma$ is called closed under extensions if $N, K \in \Gamma$ implies $M \in \Gamma$. It is clear that, for modules $N \subseteq M$, $M$ is an extension of $N$.

In this article, we prove that a commutative ring $R$ is an artinian principal ideal ring if and only if the ring is semilocal and every Rad-supplemented $R$-module is a direct sum of w-local $R$-modules if and only if every left $R$-module is a direct sum of w-local $R$-modules. We give a characterization of semisimple rings via Rad-supplements. We show that a semilocal ring $R$ is left perfect if and only if every Rad-supplemented module is (generalized) semiperfect. Some examples are given in order to show that the class of Rad-supplemented modules is not generally closed under extensions. Let $R$ be a commutative noetherian ring and $M$ be an $R$-module with $N \subseteq M$. If $M/N$ is reduced, $M$ is Rad-supplemented if and only if $N$ and $M/N$ are Rad-supplemented. It follows that a ring $R$ is semilocal if and only if every left $R$-module with Rad-supplemented radical is Rad-supplemented. Over a Dedekind domain a radical module is amply Rad-supplemented and indecomposable if and only if the module is hollow radical. Every indecomposable, w-local and amply Rad-supplemented module over a Dedekind domain is local.

2. Rad-supplemented modules over any rings

Let $R$ be any ring and $M$ be an $R$-module. A submodule $N$ of $M$ is called radical if $N$ has no maximal submodules, i.e. $N = \text{Rad}(N)$. Note that radical modules are Rad-supplemented. This fact plays a key role in our study. By $P(M)$ we denote the sum of all radical submodule of a module $M$. It is clear that, for any module $M$, $P(M)$ is the largest radical submodule and so $P(M)$ is Rad-supplemented. Using the mentioned facts, we give examples of a module, which is Rad-supplemented but not supplemented. We see, for example, the left $\mathbb{Z}$-module $M = \mathbb{Z} \mathbb{Q}$. 


Firstly we have the following lemma.

**Lemma 1.** Let $M$ be a module and $N \subseteq U \subseteq M$. Then $U$ is Rad-supplemented if and only if $\frac{U}{P(N)}$ is Rad-supplemented.

**Proof.** ($\Rightarrow$) Let $U$ be Rad-supplemented. By [1, 2.2 (2)], $\frac{U}{P(N)}$ is Rad-supplemented as a factor module of $U$.

($\Leftarrow$) Let $U$ be any submodule of $U$. By the assumption, there exists a submodule $V$ of $U$ such that $\frac{U + P(N)}{P(N)} = \frac{U}{P(N)}$ and

$$
\left( \frac{U' + P(N)}{P(N)} \right) \cap \left( \frac{V}{P(N)} \right) \subseteq \text{Rad} \left( \frac{V}{P(N)} \right).
$$

Then $(U' + P(N)) + V = U$ and hence $U' + V = U$. Since $P(N) = \text{Rad}(P(N)) \subseteq \text{Rad}(V)$, it follows that $\frac{U' + P(N)}{P(N)} = \left( \frac{U' + P(N)}{P(N)} \right) \cap \left( \frac{V}{P(N)} \right) \subseteq \text{Rad} \left( \frac{V}{P(N)} \right)$, which means that $U' \cap V \subseteq \text{Rad}(V)$. So $V$ is a Rad-supplement of $U$ in $U$. Hence $U$ is Rad-supplemented. \(\square\)

**Corollary 1.** Let $M$ be a module and $N$ be a submodule of $M$. $M$ is Rad-supplemented if and only if $\frac{M}{P(N)}$ is Rad-supplemented. In particular, $M$ is Rad-supplemented if and only if $\frac{M}{P(M)}$ is Rad-supplemented.

**Proof.** It follows from Lemma 1. \(\square\)

Recall from [5, Corollary 4.2] that if a submodule $V$ of a module $M$ is a Rad-supplement in $M$, then $\text{Rad}(V) = V \cap \text{Rad}(M)$.

Now we shall show that the rings whose modules are Rad-supplement submodules in every extension are semisimple in the following theorem.

**Theorem 1.** Let $R$ be any ring. Then the following statements are equivalent.

1. $R$ is semisimple.
2. Every left $R$-module is a Rad-supplement in every extension.
3. Every left $R$-module is a Rad-supplement in every injective extension.
4. Every left ideal of $R$ is a Rad-supplement in every injective extension.

**Proof.** $(1) \Rightarrow (2)$ Let $N$ be an $R$-module and $M$ be any extension of $N$. By the hypothesis and [6, Corollary 8.2.2 (a)], $M$ is semisimple, and so $N$ is a direct summand of $M$. It follows that $N$ is a Rad-supplement in $M$.

$(2) \Rightarrow (3) \Rightarrow (4)$ Clear.

$(4) \Rightarrow (1)$ Let $I$ be any left ideal of $R$. By the hypothesis, $I$ is a Rad-supplement in its injective hull $E(I)$. Then we have $I + J = E(I)$ and $I \cap J \subseteq \text{Rad}(I)$ for some submodule $J \subseteq E(I)$. If $m \in I \cap J$, then $Rm \subseteq \text{Rad}(I) \subseteq \text{Rad}(E(I))$. By $(4)$, $Rm$ is a Rad-supplement in $E(I)$ and so $\text{Rad}(Rm) = Rm \cap \text{Rad}(E(I)) = Rm$. 

Therefore \( m = 0 \). This means that \( I \oplus J = E(I) \) and so \( I \) is injective, and hence a direct summand of \( R \). By [6, Corollary 8.2.2 (a)], \( R \) is semisimple.

A ring \( R \) is Rad-supplemented if \( R_R \) (or \( R_R \)) is a Rad-supplemented module. It is clear that semiperfect (i.e., supplemented) rings are Rad-supplemented. Characterizations of semiperfect rings have been studied extensively by many authors recently.  

Now we shall give a characterization of Rad-supplemented rings. Firstly, we need the following simple lemmas.

**Lemma 2.** Let \( R \) be any ring with identity. Then \( R \) is Rad-supplemented if and only if every cyclic \( R \)-module is Rad-supplemented.

**Proof.** Let \( R \) be a Rad-supplemented ring. Suppose that \( M \) is any cyclic \( R \)-module. Then there exists an element \( m \) of \( M \) such that \( M \cong Rm \). Note that \( \text{Ann}(m) = Rm \), where \( \text{Ann}(m) \) is the set of all elements \( r \) of \( R \) such that \( rm = 0 \). From [1, 2.2.(2)] the hypothesis implies that \( Rm \) is Rad-supplemented and so \( Rm \) is Rad-supplemented. The converse is clear.

**Lemma 3.** Let \( M \) be a module with \( U + V = M \) for submodules \( U, V \) of \( M \). If \( V \) contains a Rad-supplement of \( U \) in \( M \), then \( U \cap V \) has a Rad-supplement in \( V \).

**Proof.** Suppose that a submodule \( K \) of \( V \) is a Rad-supplement of \( U \) in \( M \). Then, we have \( U + K = M \) and \( U \cap K \subseteq \text{Rad}(K) \). From the modular law, \( U \cap V + K = V \). Since \( K \subseteq V \), then \( (U \cap V) \cap K = U \cap K \subseteq \text{Rad}(K) \). So \( K \) is a Rad-supplement of \( U \cap V \) in \( V \).

**Theorem 2.** The following statements are equivalent for any ring \( R \).

1. \( R \) is Rad-supplemented.
2. \( R \) has ample Rad-supplements in every finitely generated extension.
3. Every cyclic \( R \)-module has ample Rad-supplements in every finitely generated extension.

**Proof.** (1) \( \Rightarrow \) (3) Let \( N \) be any cyclic \( R \)-module and \( M \) be any finitely generated extension of \( N \). Since \( R \) is Rad-supplemented, by Lemma 2, every cyclic submodule of \( M \) is Rad-supplemented and so \( M \) is amply Rad-supplemented by [11, Corollary 3.6]. Therefore \( N \) has ample Rad-supplements in \( M \).

(3) \( \Rightarrow \) (2) It is obvious.

(2) \( \Rightarrow \) (1) For any left ideal \( I \) of \( R \), consider the finitely generated pushout \( R \)-module \( N = \frac{R \oplus R}{K} \), where \( K \) is the set of all elements \( k \) of \( R \oplus R \) such that \( k = (r, -r) \) for all \( r \in I \). Then there exist monomorphisms \( f, g : R \to N \) such that \( N = f(R) + g(R) \). The hypothesis implies that \( f(R) \) has a Rad-supplement \( V \) in \( N \) with \( V \subseteq g(R) \). So, by Lemma 3, \( V \) is a Rad-supplement of \( f(R) \cap g(R) \) in \( g(R) \). Note that \( I = g^{-1}(f(R) \cap g(R)) \). It follows that \( R = I + g^{-1}(V) \) and \( I \cap g^{-1}(V) \subseteq \text{Rad}(g^{-1}(V)) \). Hence \( R \) is Rad-supplemented. 

\( \square \)
We say that a module $M$ is \textit{w-local} if $\text{Rad}(M)$ is a maximal submodule of $M$ as in [4]. Every local module is w-local. It is well known that a commutative ring $R$ has the property that every $R$-module is a direct sum of local $R$-modules if and only if $R$ is an artinian principal ideal ring. Now, we prove that if $R$ is a commutative ring and every $R$-module is a direct sum of w-local $R$-modules, then $R$ is an artinian principal ideal ring in the following theorem.

**Theorem 3.** The following are equivalent for a commutative ring $R$.

1. Every left $R$-module is a direct sum of w-local $R$-modules.
2. $R$ is semilocal and every Rad-supplemented left $R$-module is a direct sum of w-local $R$-modules.
3. $R$ is an artinian principal ideal ring.

**Proof.** (1) $\Rightarrow$ (2) Write $\frac{R}{\text{Rad}(R)} = \bigoplus_{i \in I} N_i$, where each $N_i$ is w-local. Since $\text{Rad}(\frac{R}{\text{Rad}(R)}) = 0$, for all $i \in I$, $\text{Rad}(N_i) = 0$. So $N_i$ is simple. Thus $\frac{R}{\text{Rad}(R)}$ is semisimple and so $R$ is semilocal. The rest of the proof is clear.

(2) $\Rightarrow$ (3) Let $F = R^{(\Lambda)}$ any index set $\Lambda$. Suppose that $\text{Rad}(\frac{F}{N}) = \frac{F}{N}$ for some submodule $N$ of $F$. By the assumption, we can write $\frac{F}{N} = \bigoplus_{i \in I} M_i$ where $M_i$ is w-local for all $i \in I$. By [12, 21.6.(5)], $\text{Rad}(\frac{F}{N}) = \bigoplus_{i \in I} \text{Rad}(M_i)$ and so each $M_i$ is radical as a direct summand of $\frac{F}{N}$. Since $M_i$ is w-local, we obtain that, for all $i \in I$, $M_i = 0$. Therefore $\frac{F}{N} = 0$. This means that $\text{Rad}(F) << F$. It follows from [12, 43.9] that $R$ is left perfect. Applying [12, 43.9] again, we deduce that every left $R$-module is Rad-supplemented and so every left $R$-module is a direct sum of w-local $R$-modules. If $N$ is aw-local, then $N$ is local because $R$ is left perfect. Hence every left $R$-module is a direct sum of cyclic $R$-modules. By [9, Theorem 6.7], $R$ is an artinian principal ideal ring.

(3) $\Rightarrow$ (1) is clear. □

The following corollary is an immediate consequence of Theorem 3.

**Corollary 2.** Let $R$ be a commutative semilocal ring. Then, $R$ is an artinian principal ideal ring if and only if every Rad-supplemented left $R$-module is a direct sum of w-local $R$-modules.

Let $f : P \to M$ be an epimorphism. Xue [13] calls $f$ a \textit{(generalized) cover} if $(\text{Ker}(f) \subseteq \text{Rad}(P)) \text{Ker}(f) << P$, and calls a \textit{(generalized) cover} $f$ a \textit{(generalized) projective cover} if $P$ is a projective module. In the spirit of [13], a module $M$ is said to be \textit{(generalized) semiperfect} if every factor module of $M$ has a (generalized) projective cover. He [13, Theorem 2.2] proved that every generalized semiperfect module is Rad-supplemented. Now, we obtain the following result.

**Proposition 1.** Let $R$ be a semilocal ring. Every Rad-supplemented left $R$-module is \textit{(generalized) semiperfect} if and only if $R$ is left perfect.
Proof. (⇒) Let $M = \text{Rad}(M)$. Since $M$ is Rad-supplemented, it follows from the hypothesis that $M$ is generalized semiperfect. Then, there exists a generalized cover $f : F \to M$ with a projective module $F$. Since $\text{Ker}(f) \subseteq \text{Rad}(F) \neq F$, it follows that $M = 0$. By [12, 43.9], $R$ is left perfect.

(⇐) This is immediate. □

3. Rad-supplemented modules over commutative Noetherian rings

Throughout this section, unless otherwise stated, we shall consider commutative noetherian rings.

An $R$-module $M$ is called coatomic if every proper submodule of $M$ is contained in a maximal submodule of $M$, and it is called reduced if every submodule of $M$ contains a maximal submodule, that is, $P(M) = 0$. Note that $\text{Rad}(M)$ is small in $M$ for every coatomic $R$-module $M$.

**Lemma 4.** The following statements are equivalent for a Rad-supplemented module $M$.

1. $M$ is coatomic.
2. $M$ is reduced.
3. $\text{Rad}(M)$ is small in $M$.

If the module $M$ satisfies one of the equivalent conditions, then $M$ is supplemented.

Proof. (1) ⇒ (2) Let $M$ be a coatomic module. By [15, Lemma 1.1], every submodule of $M$ is coatomic and so $P(M) = 0$, which means that $M$ is reduced.

(2) ⇒ (3) Suppose that $M = \text{Rad}(M) + N$ for some submodule $N$ of $M$. Then we can write $\text{Rad}(\frac{M}{N}) = \frac{M}{N}$. Since $M$ is Rad-supplemented, $N$ has a Rad-supplement $V$ in $M$. From (2) it follows that $V$ has a maximal submodule $K$. So $\frac{K}{N \cap V}$ is a maximal submodule of $\frac{V}{N \cap V}$. Note that $\frac{M}{N} \cong \frac{V}{N \cap V}$ contains a maximal submodule and thus $\frac{M}{N} = 0$. Therefore $M = N$. This proves (3).

(3) ⇒ (1) The assumption implies that, for any proper submodule $U \subseteq M$, there exists a submodule $V$ of $M$ such that $U + V = M$ and $U \cap V \subseteq \text{Rad}(V)$. Since $\text{Rad}(M) << M$, $V$ is not contained in a maximal submodule $K$ of $M$. Then the submodule $U + V \cap K$ of $M$ is maximal. Thus $M$ is coatomic.

Suppose that Rad-supplemented module $M$ satisfies one of these conditions. Then $M$ is supplemented by [5, Proposition 7.3].

The following result follows from [5, Proposition 7.3]. We give this result as a consequence of Lemma 4.

**Corollary 3.** For a module $M$, $M$ is Rad-supplemented if and only if $\frac{M}{P(M)}$ is supplemented.
A submodule of a Rad-supplemented module need not be Rad-supplemented, in general. To see this actuality, we shall consider the left \( \mathbb{Z} \)-module \( M = \mathbb{Z} \mathbb{Q} \). It is well known that \( M \) is Rad-supplemented. On the other hand, the submodule \( \mathbb{Z} \mathbb{Z} \) of \( M \) is not semisimple.

Now, we show that a submodule of a Rad-supplemented module is Rad-supplemented under a certain condition.

**Proposition 2.** Let \( M \) be a module and \( N \subseteq M \). Suppose that \( \frac{M}{N} \) is reduced. If \( M \) is Rad-supplemented, then \( N \) is Rad-supplemented.

**Proof.** According to [1, 2.2.(2)], \( \frac{M}{N} \) is Rad-supplemented as a factor module of \( M \). Since \( \frac{M}{N} \) is reduced, \( P(\frac{M}{N}) = 0 \). Therefore \( \frac{M}{N} \) is supplemented by Lemma 4. Since \( M \) is Rad-supplemented, \( \frac{M}{P(N)} \) is Rad-supplemented by Corollary 1. Note that \( \frac{M}{P(N)} \approx \frac{M}{N} \) is reduced and thus \( \frac{M}{P(N)} \) is reduced by [14, Lemma 1.5 (a)]. It follows from Lemma 4 that \( \frac{N}{P(N)} \) is supplemented. Hence \( N \) is Rad-supplemented by Lemma 1.

Using Proposition 2, we obtain the following result.

**Corollary 4.** The following statements are equivalent for any module \( M \).

1. \( M \) is Rad-supplemented.
2. Every maximal submodule of \( M \) is Rad-supplemented.
3. Every cofinite submodule of \( M \) is Rad-supplemented.

**Proof.** (1) \( \Rightarrow \) (3) If \( N \) is a cofinite submodule of \( M \), then \( \frac{M}{N} \) is finitely generated and so \( \frac{M}{N} \) is reduced. From Proposition 2, the proof follows.

(3) \( \Rightarrow \) (2) is clear.

(2) \( \Rightarrow \) (1) Let \( M = M_1 + M_2 \), where \( M_1 \) and \( M_2 \) are maximal submodules of \( M \). Since \( M_1 \) and \( M_2 \) are Rad-supplemented modules, \( M \) is Rad-supplemented according to [1, 2.3.(3)]. If \( M \) is w-local, Rad\( (M) \) is maximal and so \( M = \text{Rad}(M) + U \) for every proper submodule \( U \) of \( M \) with \( U \not\subseteq \text{Rad}(M) \). By [1, 2.3.(1)], \( U \) has a Rad-supplement in \( M \) since \( \text{Rad}(M) \) is Rad-supplemented. Hence \( M \) is Rad-supplemented.

The following example shows that the class of Rad-supplemented modules is not closed under extensions, in general.

**Example 1.** Let \( A \) be a collection of maximal ideals of the noetherian commutative ring \( \mathbb{Z} \). Suppose that \( M \) is the left \( \mathbb{Z} \)-module \( \prod_{P \in A}(\mathbb{Z}/p) \). Then \( \text{Rad}(M) = 0 \). By [3, Lemma 2.9], for some submodule \( N \) of \( M \), we have \( \frac{N}{T} \approx \mathbb{Q} \), where \( T \) is the...
direct sum of simple $\mathbb{Z}/p\mathbb{Z}$-modules. Then $N$ is an extension of $T$ by $Q$. Since $T$ is semisimple, it is Rad-supplemented. On the other hand, the submodule $N$ is not Rad-supplemented.

Later we shall give another example of such modules (see Example 2).

**Theorem 4.** Let $0 \rightarrow N \rightarrow M \rightarrow K \rightarrow 0$ be a short exact sequence. Suppose that $K$ is reduced. Then $M$ is Rad-supplemented if and only if $N$ and $K$ are Rad-supplemented.

**Proof.** ($\Rightarrow$) It follows from Proposition 2 and [1, 2.2.(2)].

($\Leftarrow$) By Lemma 4, $K$ is supplemented. Since $N$ is Rad-supplemented, $\frac{N}{P(N)}$ is supplemented by Corollary 3. It follows from [8, Proposition 2.6] that $\frac{M}{P(M)}$ is Rad-supplemented. Hence $M$ is Rad-supplemented by Corollary 1.

**Corollary 5.** A module $M$ is Rad-supplemented if and only if it is an extension of a Rad-supplemented submodule by a reduced supplemented module.

**Proof.** If $M$ has no maximal submodules, the result is obvious as $\frac{M}{P(M)} = 0$. Suppose that $M \neq P(M)$. Then this gives the existence of a reduced factor module of $M$. Therefore the assertion follows from Theorem 4.

**Proposition 3.** Let $M$ be a module. $M$ is Rad-supplemented if and only if $M$ is semilocal and $\text{Rad}(M)$ is Rad-supplemented.

**Proof.** If $M$ is Rad-supplemented, then $M$ is semilocal. Thus $\frac{M}{\text{Rad}(M)}$ is reduced. By Proposition 2, $\text{Rad}(M)$ is Rad-supplemented. Conversely, suppose that $M$ is semilocal and $\text{Rad}(M)$ is Rad-supplemented. From Theorem 4 the assumption implies that $M$ is Rad-supplemented.

Using the above proposition we obtain the following characterization of semilocal rings.

**Corollary 6.** The following conditions on a ring $R$ is equivalent:

1. $R$ is semilocal.
2. Every left $R$-module with Rad-supplemented radical is Rad-supplemented.

**Proof.** (1) $\Rightarrow$ (2) If $R$ is semilocal, then every left $R$-module is semilocal by [7, Theorem 3.5]. The result follows from Proposition 3.

(2) $\Rightarrow$ (1) Since $\text{Rad}(\frac{R}{\text{Rad}(R)}) = 0$, it follows from the hypothesis that $\frac{R}{\text{Rad}(R)}$ is Rad-supplemented. So $\frac{R}{\text{Rad}(R)}$ is semisimple, i.e. $R$ is semilocal.

In [5], a module $M$ is said to be totally Rad-supplemented if every submodule of $M$ is Rad-supplemented. Every semisimple module is totally Rad-supplemented. It is easy to check that the class of totally Rad-supplemented modules is closed under factor modules and submodules. The following fact is a modification of Theorem 4.
Theorem 5. Let \( M \) be a module and \( \frac{M}{N} \) be reduced for some submodule \( N \) of \( M \). Then \( M \) is totally \( \text{Rad} \)-supplemented if and only if \( N \) and \( \frac{M}{N} \) are totally \( \text{Rad} \)-supplemented.

Proof. Suppose that \( N \) and \( \frac{M}{N} \) are totally \( \text{Rad} \)-supplemented. Let \( U \) be any submodule of \( M \). By the hypothesis, \( U \cap N \) and \( \frac{U + N}{N} \) are \( \text{Rad} \)-supplemented. Note that

\[
\frac{U + N}{N} \cong \frac{U}{U \cap N}
\]

is reduced because \( \frac{M}{N} \) is reduced. By Theorem 4, \( U \) is \( \text{Rad} \)-supplemented. Hence \( M \) is totally \( \text{Rad} \)-supplemented. □

Corollary 7. Let \( M \) be a \( \text{Rad} \)-supplemented module. Then, \( M \) is totally \( \text{Rad} \)-supplemented if and only if \( P(M) \) is totally \( \text{Rad} \)-supplemented.

Proof. Suppose that \( P(M) \) is totally \( \text{Rad} \)-supplemented. By the hypothesis, \( \frac{M}{P(M)} \) is supplemented. Applying [8, Proposition 2.6], we deduce that \( \frac{M}{P(M)} \) is totally supplemented. Therefore \( M \) is totally \( \text{Rad} \)-supplemented by Theorem 5. □

4. \( \text{Rad} \)-Supplemented Modules Over Commutative Domains

In this section a ring \( R \) will be a commutative domain. Let \( R \) be such a ring and \( M \) be an \( R \)-module. We denote by \( T(M) \) the set of all elements \( m \) of \( M \) for which there exists a non-zero element \( r \) of \( R \) such that \( rm = 0 \), i.e., \( \text{Ann}(m) \neq 0 \). Then \( T(M) \), which is a submodule of \( M \), called the torsion submodule of \( M \). If \( M = T(M) \), then \( M \) is called a torsion module and \( M \) is called torsion-free provided \( T(M) = 0 \).

Proposition 4. Let \( R \) be a non-semilocal commutative domain and \( M \) be an \( R \)-module. If \( M \) is totally \( \text{Rad} \)-supplemented, \( M \) is a torsion module.

Proof. Let \( 0 \neq m \in M \). Suppose that \( \text{Ann}(m) = 0 \), i.e. \( R \cong \text{Ann}(m) \). Since \( M \) is totally \( \text{Rad} \)-supplemented, the left \( R \)-submodule \( \text{Ann}(m) \) of \( M \) is \( \text{Rad} \)-supplemented. So \( R \) is \( \text{Rad} \)-supplemented. Therefore \( \frac{R}{\text{Rad}(R)} \) is semisimple, i.e. \( R \) is semilocal. This contradicts the assumption. Hence \( \text{Ann}(m) \neq 0 \), this implies that \( M \) is torsion. □

Corollary 8. Let \( R \) be a non-semilocal dedekind domain and \( M \) be a totally \( \text{Rad} \)-supplemented \( R \)-module. Then \( M \) is torsion.

Let \( R \) be a dedekind domain and \( M \) be an \( R \)-module. We denote by \( \Omega \) the set of all maximal (i.e., prime) ideals of \( R \). Suppose that \( \mathfrak{p} \) is any element of \( \Omega \). We denote by \( T_{\mathfrak{p}}(M) \), which is a submodule of \( M \), the set of all elements \( m \) of \( M \) for which there exists a positive integer \( n \) such that \( \mathfrak{p}^n m = 0 \). Then \( T_{\mathfrak{p}}(M) \) is called the \( \mathfrak{p} \)-primary part of \( M \). For a torsion module \( M \) over a dedekind domain, we have the decomposition \( M = \bigoplus_{\mathfrak{p} \in \Omega} T_{\mathfrak{p}}(M) \).
Lemma 5. Let \( R \) be a non-local dedekind domain and \( M \) be an \( R \)-module. Then \( M \) is \( \text{Rad} \)-supplemented if and only if \( \frac{M}{P(M)} \) is torsion and every \( p \)-primary part of \( \frac{M}{P(M)} \) is (\( \text{Rad} \)-)supplemented.

Proof. According to [14, Theorem 3.1] and [5, Theorem 7.4], the proof of the lemma is clear. \( \square \)

Let \( R \) be a dedekind domain and \( M \) be an \( R \)-module. By [2, Lemma 4.4], \( P(M) \) is injective and so there exists a direct summand \( N \) of \( M \) such that \( \frac{M}{P(M)} \cong N \). This fact and Lemma 5 give the following basic result for torsion-free modules.

Corollary 9. Let \( M \) be a torsion-free \( \text{Rad} \)-supplemented module over a non-local dedekind domain. Then \( M \) is radical.

Let \( M \) be a radical module. \( M \) is called simply radical if \( M \) has no proper radical submodules.

Proposition 5. Let \( R \) be a noetherian ring and \( M \) be a simply radical \( R \)-module. If \( M \) is amply \( \text{Rad} \)-supplemented, \( M \) is hollow radical. In particular, every \( \text{Rad} \)-supplemented proper submodule of \( M \) is supplemented.

Proof. Let \( U \) be any proper submodule of \( M \). Suppose that \( U + V = M \) for some submodule \( V \) of \( M \). By the hypothesis, there exists a submodule \( V' \) of \( V \) such that \( U + V' = M \) and \( U \cap V' \subseteq \text{Rad}(V') \). Since \( M \) is simply radical, it follows that \( \text{Rad}(V') = V' \cap \text{Rad}(M) = V' \cap M = V' \). So \( V' \) is radical. Therefore \( V' = M \) and so \( V = M \). Then we deduce that \( U \) is small in \( M \). Hence \( M \) is hollow radical. Suppose that a proper submodule \( N \) of \( M \) is \( \text{Rad} \)-supplemented. Since \( M \) is simply radical, every submodule of \( N \) contains a maximal submodule, i.e., \( P(N) = 0 \). By Lemma 4, \( N \) is supplemented. \( \square \)

Corollary 10. Let \( R \) be a dedekind domain and \( M \) be a radical \( R \)-module. Then \( M \) is amply \( \text{Rad} \)-supplemented and indecomposable if and only if the module is hollow radical.

Proof. Since indecomposable radical modules over dedekind domains is simply radical, \( M \) is hollow radical by Proposition 5. The converse is clear. \( \square \)

Proposition 6. Let \( M \) be a module over a Dedekind domain. Then the following statements are equivalent.

1. \( M \) is indecomposable, \( w \)-local and amply \( \text{Rad} \)-supplemented.
2. \( M \) is local.

Proof. (1) \( \Rightarrow \) (2) Let \( U \) be any proper submodule of \( M \). Suppose that \( U \) is not contained \( \text{Rad}(M) \). Since \( M \) is \( w \)-local, \( \text{Rad}(M) \) is maximal and so \( U + \text{Rad}(M) = M \). By the hypothesis, there exists a submodule \( V \) of \( \text{Rad}(M) \) such that \( U + V = M \) and \( U \cap V \subseteq \text{Rad}(V) \). It follows that \( \text{Rad}(V) = V \cap \text{Rad}(M) = V \), i.e. \( V \) is radical.
Then, by [2, Lemma 4.4], \( V \) is injective and so there exists a submodule \( L \) of \( M \) such that \( M = V \oplus L \). Since \( M \) is indecomposable and w-local, we get \( V = 0 \). Thus, \( U = M \), implying that \( M \) is local.

(2) \(\Rightarrow\) (1) is clear.

Now, we give an analogous characterization of [14, Theorem 3.1] for totally Rad-supplemented modules.

**Theorem 6.** Let \( M \) be a non-semilocal Dedekind domain and \( M \) be an \( R \)-module. Then \( M \) is totally Rad-supplemented if and only if \( M \) is torsion and every \( p \)-primary part of \( M \) is totally Rad-supplemented.

**Proof.** The necessity of the condition is obvious by Corollary 8. Conversely, suppose that \( M \) is torsion and every \( p \)-primary part of \( M \) is totally Rad-supplemented. Let \( N \subseteq U \subseteq M \). Since \( M = \bigoplus_{p \in \Omega} T_p(M) \), we have \( U = \bigoplus_{p \in \Omega} (U \cap T_p(M)) \) and \( N = \bigoplus_{p \in \Omega} (N \cap T_p(M)) \). By the hypothesis, \( N \cap T_p(M) \) has a Rad-supplement \( V_p \) in \( U \cap T_p(M) \). So \( U \cap T_p(M) = N \cap T_p(M) + V_p \) and \( N \cap V_p \subseteq \text{Rad}(V_p) \). Let \( V = \bigoplus_{p \in \Omega} V_p \). Then \( N + V = U \). Since \( N \cap V_p \subseteq \text{Rad}(V_p) \) for every \( p \in \Omega \), by [6, Corollaries 9.1.5 (c)], \( N \cap V = (\bigoplus_{p \in \Omega} (N \cap T_p(M))) \cap (\bigoplus_{p \in \Omega} V_p) \subseteq \text{Rad}(V) \). Hence \( U \) is Rad-supplemented. This completes the proof.

Finally, we give an example showing the class of (totally) Rad-supplemented modules is not closed under extensions, in general. For a module \( M \), \( \text{Soc}(M) \) will indicate the sum of all simple submodules of \( M \).

**Example 2.** (see [10, Example 2.3]) Consider the non-Noetherian commutative ring which is the direct product \( \prod_{i \geq 1} F_i \), where \( F_i = F \) is any field. Suppose that \( R \) is the subring of the ring consisting of all sequences \( (r_n)_{n \in \mathbb{N}} \) such that there exist \( r \in F, m \in \mathbb{N} \) with \( r_n = r \) for all \( n \geq m \). Let \( M =_R R \). Then \( M \) is a regular module which is not semisimple. Therefore \( \text{Soc}(M) \) is a maximal submodule of \( M \). This means that \( \text{Soc}(M) \) and \( M/\text{Soc}(M) \) are Rad-supplemented. On the other hand, \( M \) is not Rad-supplemented.

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